Scaling of the superfluid density in high-temperature superconductors

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A scaling relation $N_c \simeq 4.4\sigma_{dc}T_c$ has been observed in the copper-oxide superconductors, where N_c is the spectral weight associated with the formation of the superconducting condensate $\rho_s = 8N_c$, T_c is the critical temperature, and σ_{dc} is the normal-state dc conductivity close to T_c. This scaling relation is examined within the context of a clean and dirty-limit BCS superconductor. These limits are well established for an isotropic BCS gap 2Δ and a normal-state scattering rate $1/\tau$; in the clean limit $1/\tau \ll 2\Delta$, and in the dirty limit $1/\tau > 2\Delta$. The dirty limit may also be defined operationally as the regime where ρ_s varies with $1/\tau$. It is shown that the scaling relation N_c or $\rho_s \propto \sigma_{dc}T_c$, which follows directly from the Ferell-Glover-Tinkham sum rule, is the hallmark of a BCS system in the dirty-limit. While the gap in the copper-oxide superconductors is considered to be d wave with nodes and a gap maximum Δ_0, if $1/\tau > 2\Delta_0$ then the dirty-limit case is preserved. The scaling relation implies that the copper-oxide superconductors are likely to be in the dirty limit and, as a result, that the energy scale associated with the formation of the condensate scales linearly with T_c. The a-b planes and the c axis also follow the same scaling relation. It is observed that the scaling behavior for the dirty limit and the Josephson effect (assuming a BCS formalism) are essentially identical, suggesting that in some regime these two pictures may be viewed as equivalent.

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I. INTRODUCTION

Scaling laws express a systematic and universal simplicity among complex systems in nature. For example, such laws are of enormous significance in biology, where the scaling relation between body mass and metabolic rate spans 21 orders of magnitude.[1,2] Scaling relations are equally important in the physical sciences. Since the discovery of superconductivity at elevated temperatures in copper-oxide materials,[3] there has been considerable effort to find trends and correlations between the physical quantities, as a clue to the origin of the superconductivity.[4] One of the earliest patterns that emerged was the linear scaling of the superfluid density $\rho_s \propto 1/\lambda^2$ (where λ is the superconducting penetration depth) in the copper-oxygen planes of the hole-doped materials with the superconducting transition temperature T_c. This is referred to as the Uemura relation,[5,6] and it works reasonably well for the underdoped materials. However, it does not describe very underdoped,[7] optimally doped (i.e., T_c is a maximum), overdoped,[8,9] or electron-doped materials;[10,11] a similar attempt to scale ρ_s with the dc conductivity σ_{dc} was only partially successful.[12] In contrast, we have recently demonstrated that the scaling relation $\rho_s \propto \sigma_{dc}T_c$ may be applied to a large number of high-temperature superconductors, regardless of doping level or type, nature of disorder, crystal structure, or direction (parallel or perpendicular to the copper-oxygen planes).[13] The optical values of the superconducting plasma frequency $\omega_{ps}(T \ll T_c)$ and $\sigma_{dc}(T \gtrsim T_c)$ within the a-b planes have been determined for a large number of copper-oxide superconductors, as well as the bismuth-oxide material $Ba_{1-x}K_xBiO_3$. The results are expressed as a linear plot of the spectral weight of the condensate N_c versus $\sigma_{dc}T_c$ in Fig. 1 (the precise definition of $\omega_{ps}^2 = 8N_c$ and $\rho_s \equiv \omega_{ps}^2$ will be discussed in more detail in the following section). In this representation, the underdoped points near the origin tend to lie rather close together, and the c-axis points which are not shown would be visible only as a single point slightly below the underdoped data. While there is some scatter in the data, a linear trend is clearly visible, although it has been suggested that there is some deviation from this behavior in the extremely overdoped materials.[14] When plotted as a log-log plot in Fig. 2, the linear trend is more apparent and indicates

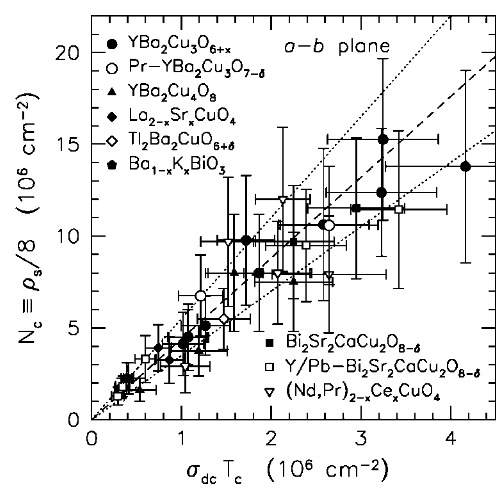

FIG. 1. The spectral weight of the condensate N_c vs $\sigma_{dc}T_c$ for the a-b planes of the hole-doped copper-oxide superconductors for pure and Pr-doped $YBa_2Cu_3O_{6+x}$ (Refs. 12, 36, and 60–62); pure and Zn-doped $YBa_2Cu_4O_8$ (Refs. 60 and 63); pure and Y/Pb-doped $Bi_2Sr_2CaCu_2O_{8+\delta}$ (Refs. 62, 64, and 65); underdoped $La_{2-x}Sr_xCuO_4$ (Ref. 69); $Tl_2Ba_2CuO_{6+\delta}$ (Ref. 66); electron-doped $(Nd,Pr)_{2-x}Ce_xCuO_4$ (Refs. 10, 13, 67, and 68) and the bismate material $Bi_{1-x}K_xBiO_3$ (Ref. 70). Within error, all the points may be described by a single (dashed) line, $N_c \simeq 4.4\sigma_{dc}T_c$; the upper and lower dotted lines, $N_c \simeq 5.5\sigma_{dc}T_c$ and $3.5\sigma_{dc}T_c$, respectively, represent approximately the spread of the data.

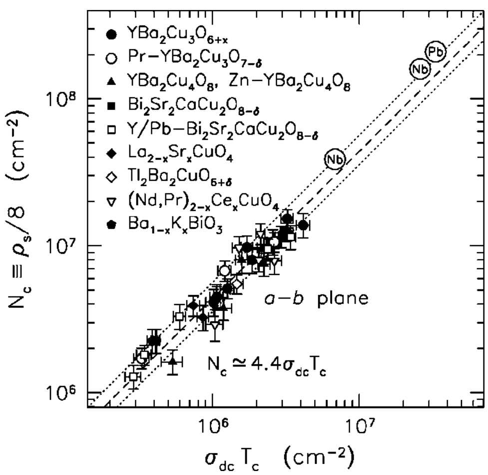

FIG. 2. The log-log plot of the spectral weight of the condensate N_c vs $\sigma_{dc}T_c$ for the *a-b* planes of the hole-doped copper-oxide superconductors shown in Fig. 1. The dashed and dotted lines shown in this figure are the same lines that were shown in Fig. 1. The points for Nb and Pb, indicated by the atomic symbols, also fall close to the dotted line, $N_c \simeq 4.4\sigma_{dc}T_c$ (Refs. 45 and 46).

that within the error the points may be described by the scaling relation $N_c \simeq 4.4\sigma_{dc}T_c$. (In this instance, both sides of the equation possess the same units, so that the constant is dimensionless. The dimensionless constant and the description of the scaling in terms of N_c rather than ρ_s results in a prefactor which is smaller than observed in our previous work.[13]) In addition, the elemental BCS superconductors Nb and Pb (without any special regards to preparation) are also observed to follow this scaling relation reasonably well.

II. EXPERIMENT

The values for σ_{dc} and ω_{ps} shown in Table I have been obtained almost exclusively from reflectance measurements from which the complex optical properties have been determined through a Kramers-Kronig analysis.[15] The dc conductivity has been extrapolated from the real part of the optical conductivity $\sigma_{dc}=\sigma_1(\omega\to 0)$ at $T \gtrsim T_c$. For $T \ll T_c$, the response of the dielectric function to the formation of a condensate is expressed purely by the real part of the dielectric function $\epsilon_1(\omega)=\epsilon_\infty-\omega_{ps}^2/\omega^2$, which allows the strength of the condensate to be calculated from $\omega_{ps}^2=-\omega^2\epsilon_1(\omega)$ in the $\omega\to 0$ limit. Here, $\omega_{ps}^2=4\pi n_s e^2/m^*$ is the square of the superconducting plasma frequency, n_s is the superconducting carrier concentration, m^* is an effective mass, and ϵ_∞ is the high-frequency contribution to the real part of the dielectric function. The strength of the condensate is simply $\rho_s \equiv \omega_{ps}^2$, which is proportional to n_s/m^*. The value of ρ_s may also be estimated by examining the changes in the optical conductivity just above and well below T_c. The f-sum rule for the conductivity[16] has the form $\int_0^\infty \sigma_1(\omega)d\omega=\omega_p^2/8$, where $\omega_p^2=4\pi ne^2/m$ is the classical plasma frequency. The spectral weight is defined here as

$$N(\omega,T) = \int_{0^+}^{\omega} \sigma_1(\omega',T)d\omega', \qquad (1)$$

which is simply the area under the conductivity curve. The copper-oxide materials, and superconductors in general, show a dramatic suppression of the low-frequency conductivity upon entering the superconducting state; this difference between the $T \simeq T_c$ to $T \ll T_c$ conductivities is often referred to as the "missing area." The spectral weight associated with the formation of the superconducting condensate is then $N_c = N_n - N_s$, where $N_n \equiv N(\omega, T \simeq T_c)$, and $N_s \equiv N(\omega, T \ll T_c)$. Here, N_c is simply the spectral weight associated with the missing area in the conductivity, which is related to the square of the superconducting plasma frequency

$$\omega_{ps}^2 = 8N_c, \qquad (2)$$

or $\rho_s = 8N_c$. This expression is the well-known Ferrell-Glover-Tinkham sum rule.[17,18] These two different techniques typically yield nearly identical values for ρ_s; an exception exists in the underdoped materials along the c axis, where it has been suggested that there is missing spectral weight.[19] Typically, the conductivity is expressed in units of $\Omega^{-1}\,cm^{-1}$, which allowing for the change of units would yield $\omega_{ps}^2=(120/\pi)N_c$. However, with the exception of the values for σ_{dc} listed in Table I, the conductivity will be expressed throughout the text in units of cm^{-1} (Ref. 20).

III. DISCUSSION

A deeper understanding of the scaling relation as it relates to both the elemental superconductors and the copper-oxide materials may be obtained from an examination of the spectral weight above and below T_c in relation to the normal-state scattering rate and the superconducting energy gap. When Nb is in the dirty limit, it follows the $\rho_s \propto \sigma_{dc}T_c$ relation, but in the clean limit there is a deviation from this linear behavior. (This result will be explored in more detail shortly.) The terms "clean" and "dirty" originate from the comparison of the isotropic BCS energy gap 2Δ with the normal-state scattering rate $1/\tau$; the clean limit is taken as $1/\tau \ll 2\Delta$, while the dirty limit is $1/\tau > 2\Delta$. The clean and dirty limits may also be expressed as $l \gg \xi_0$ and $l < \xi_0$, respectively, where l is the quasiparticle mean-free path and ξ_0 is the BCS coherence length; because $l \propto \tau$ and $\xi_0 \propto 1/\Delta$, this is equivalent to the previous statement.[21] The use of these definitions depends on having accurate values for $1/\tau$ and Δ.

In general, BCS superconductors have relatively low values for T_c, thus $1/\tau$ is assumed to have little temperature dependence close to the superconducting transition. This assumption may be tested by suppressing T_c through the application of a magnetic field in excess of the upper critical field (H_{c2}) and examining the transport properties, which typically reveal little temperature dependence in $1/\tau$ below the zero-field value of T_c. The application of the clean and dirty-limit picture to the copper-oxide superconductors is complicated by both the high critical temperature and the superconducting energy gap, which is thought to be d wave in nature and momentum dependent (Δ_k) and contains nodes.[22,23] The high value for T_c suggests that $1/\tau$ may still have a significant temperature dependence close to T_c. In the normal state, the scattering rate is often observed to be rather large, scaling linearly with temperature,[24] and is presumed to be dominated by inelastic processes. Indeed, below T_c the quasiparticle

TABLE I. The critical temperature T_c, dc conductivity $\sigma_{dc} \equiv \sigma_1(\omega \to 0)$ just above T_c, plasma frequency of the condensate ω_{ps}, and penetration depth λ_{ab} for $T \ll T_c$, for light polarized in the *a-b* planes for a variety of single-layer and double-layer copper-oxygen high-temperature superconductors. Values for $Ba_{1-x}K_xBiO_3$ as well as several elemental superconductors have also been included.

Material	Note	(Ref.)	T_c (K)	σ_{dc} (Ω^{-1} cm^{-1})	ω_{ps} (cm^{-1})	λ_{ab} (μm)
$YBa_2Cu_3O_{6.95}$	a,b	36	70	4400±500	5750±600	0.276
$YBa_2Cu_3O_{6.95}$	a,b	36	80	6500±600	8840±800	0.180
$YBa_2Cu_3O_{6.95}$	a,b	36	85	9200±900	9220±900	0.172
$YBa_2Cu_3O_{6.95}$	a,b	36	93.5	10 500±1000	11 050±800	0.144
$YBa_2Cu_3O_{6.60}$	c	60 and 61	59	6500±600	6400±500	0.248
$YBa_2Cu_3O_{6.95}$	c	60 and 61	93.2	10 500±900	9950±700	0.159
$YBa_2Cu_3O_{7-\delta}$	b	62	92	8700±900	9200±700	0.172
Pr-$YBa_2Cu_3O_{7-\delta}$	b	62	40	2500±300	3700±300	0.430
Pr-$YBa_2Cu_3O_{7-\delta}$	b	62	75	4900±500	7350±600	0.216
$YBa_2Cu_4O_8$	c	60 and 63	80	6000±600	8000±800	0.198
$Bi_2Ca_2SrCu_2O_{8+\delta}$	c	64	90	11 500±900	9565±900	0.166
$Bi_2Ca_2SrCu_2O_{8+\delta}$	b	62	91	9800±800	9600±800	0.165
$Bi_2Ca_2SrCu_2O_{8+\delta}$	b	62	85	8500±800	8710±700	0.182
Y/Pb-$Bi_2Ca_2SrCu_2O_{8+\delta}$	b	62	35	2500±300	3200±300	0.497
Y-$Bi_2Ca_2SrCu_2O_{8+\delta}$	b	62	40	2600±300	3800±300	0.418
Y-$Bi_2Ca_2SrCu_2O_{8+\delta}$	b	65	43	4200±400	5140±500	0.309
$Tl_2Ba_2CuO_{6+\delta}$	b	66	88	5000±500	6630±500	0.240
$Nd_{1.85}Ce_{0.15}CuO_4$	b	10 and 67	23	28 000±2000	10 300±900	0.154
$Pr_{1.85}Ce_{0.15}CuO_4$	b	13	19	33 000±3000	8000±800	0.198
$Pr_{1.85}Ce_{0.15}CuO_4$	d	68	21	15 000±2000	4820±600	0.330
$Pr_{1.87}Ce_{0.15}CuO_4$	d	68	16	50 000±6000	7960±800	0.207
$La_{1.87}Sr_{0.13}CuO_4$	b	69	32	7000±700	5600±450	0.284
$La_{1.86}Sr_{0.14}CuO_4$	b	69	36	9000±900	6000±500	0.265
$Ba_{0.62}K_{0.38}BiO_3$	b	70	31	3800±300	4240±400	0.375
$Ba_{0.60}K_{0.40}BiO_3$	b	70	28	4400±300	4240±400	0.375
$Ba_{0.54}K_{0.46}BiO_3$	b	70	21	6000±300	4240±400	0.375
Nb	d	46	8.3	2.5e5	17 600	0.090
Nb	d	45	9.3	8.5e5	35 800	0.044
Pb	d	45	7.2	1.4e6	41 000	0.038

[a]Radiation damaged, twinned single crystal.
[b]Light polarized in the *a-b* plane of a single crystal.
[c]Light polarized along the *a* axis of a twin-free single crystal.
[d]Thin film or oriented thin film.

scattering rate in the cuprates is observed to decrease by nearly two orders of magnitude at low temperatures.[25] This rapid decrease in $1/\tau$ is also observed optically, but not to the same extent.[26] A gap with $d_{x^2-y^2}$ symmetry may be written as $\Delta_k = \Delta_0[\cos(k_x a) - \cos(k_y a)]$; the gap reaches a maximum at the $(0,\pi)$ and $(\pi,0)$ points and vanishes along the nodal (π,π) directions. The fact that the scattering rate of the quasiparticles restricted to the nodal regions of the Fermi surface for $T \ll T_c$ is quite small has been taken as evidence that these materials are in the clean limit.[27–29] While it is certainly true that for $T \ll T_c$ the scattering rate is small and the nodal quasiparticles have very long mean-free paths, it is problematic to assert that, therefore, the superconductor is in the clean limit. In a normal BCS superconductor, $1/\tau$ is also observed to decrease dramatically below T_c, regardless of the normal-state value of $1/\tau$, due to the formation of a condensate.[30] Thus, the criteria of a small value of the quasiparticle scattering rate for $T \ll T_c$ is not necessarily a good measure of whether or not the superconductivity is in the clean or dirty limit. As with BCS materials, it is desirable to suppress T_c in the copper-oxide materials through the application of a magnetic field to determine the low-temperature behavior of $1/\tau$. While H_{c2} is quite large in the cuprates, experiments using pulsed magnetic fields can suppress T_c; in these experiments, the resistivity of the optimally doped materials matches the zero-field values at high temperatures due to the low magnetoresistance, and the trend of slowly decreasing resistivity continues smoothly to low temperatures,[31–35] often saturating at a value close to that observed at T_c. The implication of these experiments is that the normal-state value of $1/\tau$ is a good measure of the scattering rate in those systems in which T_c has been suppressed and, therefore, is the value that should be considered when determining whether a system is in the clean or dirty limit. In addition to this explicit approach, a simpler method is to adopt an operational definition which states that if ρ_s changes with respect to the normal-state value of $1/\tau$ then the material is in the dirty limit; when ρ_s is no longer sensitive to the value of $1/\tau$ then the material is in the clean limit. Most of the materials in Fig. 2 are studied as a function of carrier doping, but it is also important to note that the introduction of disorder for fixed doping levels has also been studied.[36] The fact that all the observed results follow this linear scaling relation strongly suggests that many of the copper-oxide superconductors are close to or in the dirty limit (i.e., the superfluid density changes in response to variations in $1/\tau$).

A. Clean limit

The BCS model is used to describe the superconductivity of simple metals and alloys. If the normal-state properties may be described by the simple Drude model in which the complex dielectric function is written as $\tilde{\epsilon}(\omega)=\epsilon_\infty-\omega_p^2/[\omega(\omega+i\gamma)]$, where the plasma frequency ω_p^2 has been previously defined, $\gamma=1/\tau$ is the scattering rate, and ϵ_∞ is a high-frequency contribution. The dielectric function and the conductivity are related through $\tilde{\sigma}=\sigma_1+i\sigma_2=-i\omega(\tilde{\epsilon}-\epsilon_\infty)/4\pi$, thus the real part of the frequency-dependent conductivity has the form $\sigma_1(\omega)=\sigma_{dc}/(1+\omega^2\tau^2)$ and $\sigma_{dc}=\omega_p^2\tau/4\pi$ (in units of cm^{-1}), which has the shape of a Lorentzian centered at zero frequency with a width at half maximum given by $1/\tau$. The optical conductivity below T_c has been calculated from an isotropic (s wave) energy gap 2Δ that considers an arbitrary purity level.[37] The clean limit case ($1/\tau\ll 2\Delta$) is illustrated in Fig. 3(a) for the choice $1/\tau=0.2\Delta$. An aspect of clean-limit systems is that nearly all of the spectral weight associated with the condensate lies below 2Δ. As a result, the normalized spectral weight of the condensate $8N_c/\rho_s$ (Ref. 19), the difference in the area under the two curves indicated by the hatched region, shown in the inset of Fig. 3(a), approaches unity at frequencies closer to $1/\tau$ rather than 2Δ. The spectral weight for the condensate may be estimated simply as $N_c\simeq\sigma_{dc}/\tau$. If $1/\tau\propto T_c$ for $T\simeq T_c$ in the copper-oxide materials,[38] then $N_c\propto\sigma_{dc}T_c$ is in agreement with the observed scaling relation. It is interesting to note that $1/\tau\propto T_c$ may yield rather large values for the normal-state scattering rate, and it has been suggested that the copper-oxide materials are close to the maximum level of dissipation allowed for these systems.[39] Furthermore, even though a d wave system complicates the interpretation of the clean and dirty limits, large normal-state values of $1/\tau$ and relatively short normal-state mean-free paths[40] are problematic for a clean-limit picture; to achieve the clean limit it is not only necessary that $1/\tau\ll 2\Delta_0$, but also that $1/\tau\lesssim 2\Delta_k$ in the nodal regions. In fact, the clean-limit requirement is much more stringent for a d wave system than it is for a material with an isotropic energy gap, and it is not clear that it will ever be satisfied in the copper-oxide superconductors. This suggests that a dirty-limit view may be more appropriate.

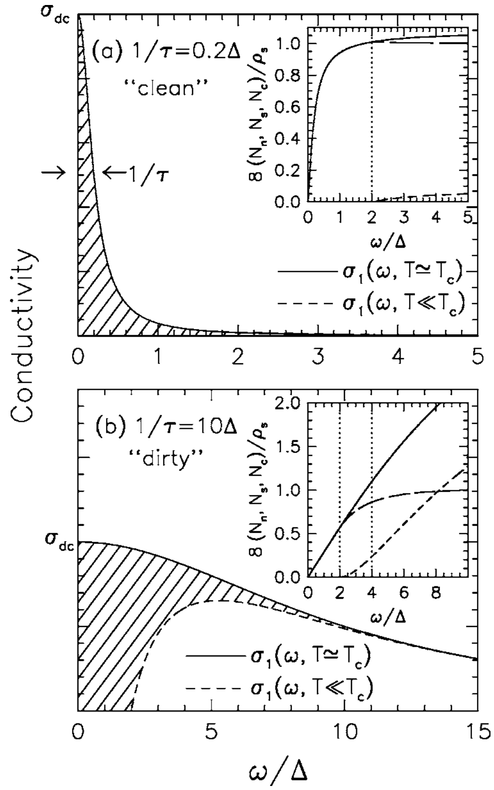

FIG. 3. The optical conductivity for the BCS model in the normal (solid line) and superconducting states (dashed line) for a material in (a) the clean limit ($1/\tau\ll 2\Delta$) and (b) the dirty limit ($1/\tau>2\Delta$). The normal-state conductivity is a Lorentzian centered at zero frequency with a full width at half maximum of $1/\tau$ for $T\simeq T_c$. The spectral weight associated with the formation of a superconducting condensate is indicated by the hatched area. Insets: $N_n\equiv N(\omega,T\simeq T_c)$ (solid line), $N_s\equiv N(\omega,T\ll T_c)$ (dashed line), and difference between the two $N_c=N_n-N_s$ (long-dashed line) normalized with respect to $\rho_s/8$; in the clean limit $8N_c/\rho_s$ converges rapidly to unity and is fully formed at energies comparable to $1/\tau$, while in the dirty limit, convergence occurs at energies comparable to 4Δ.

B. Dirty limit

In the BCS dirty limit, $1/\tau>2\Delta$, this is illustrated in Fig. 3(b) for the choice of $1/\tau=10\Delta$. In this case, the normal-state conductivity is a considerably broader Lorentzian, and much of the spectral weight has been pushed out above 2Δ. As a result, the normalized spectral weight of the condensate, shown in the inset, converges much more slowly than in the clean-limit case. However, a majority of the spectral weight is captured by 2Δ and N_c is almost fully formed above 4Δ (Ref. 19). In the dirty-limit case, the spectral weight of the condensate [the hatched area in Fig. 3(b)] may be estimated as $N_c \simeq \sigma_{dc} 2\Delta$. In the BCS model, the energy gap 2Δ scales linearly with T_c, yielding $N_c \propto \sigma_{dc} T_c$, which is in agreement with the observed scaling relation. This result necessarily implies that the energy scale for the condensate is proportional to T_c. As in the clean-limit case, the nature of the gap is important. If $1/\tau > 2\Delta_0$, the spirit of the dirty-limit case is preserved for all Δ_k. While many of the points in Figs. 1 and 2 are doping-dependent studies and do not track systematic changes in $1/\tau$, some of these points are for the same chemical doping with different scattering rates resulting from disorder that have either been deliberately introduced[36] or that exist simply as a byproduct of synthesis (Table I).[41,42] The observation that all the points obey a linear scaling relation satisfies the operational definition of the dirty limit, suggesting that the examined materials are either close to or in the dirty limit.

C. Behavior of Nb

It was noted in Fig. 2 that the points for Nb and Pb agreed reasonably well with the scaling relation used to describe the copper-oxide superconductors. It is important to determine if these values represent clean-limit or dirty-limit results. The expected behavior of Nb has been modeled using the BCS model[37] for an arbitrary purity level with a critical temperature of $T_c = 9.2$ K and a gap of $2\Delta = 22.3$ cm^{-1} (the BCS weak-coupling limit $2\Delta = 3.5 k_B T_c$). The normal-state is described using the Drude model with a classical plasma frequency of $\omega_p = 56\,000$ cm^{-1} (Ref. 43) and a range of scattering rates $1/\tau = 0.05\Delta \rightarrow 50\Delta$; from the Drude model the dc conductivity is $\sigma_{dc} = \omega_p^2 \tau / 4\pi$ (in units of cm^{-1} when the plasma frequency and the scattering rate also have units of cm^{-1}). The spectral weight of the condensate N_c has been determined by integrating to $\omega \simeq 200\Delta$, where N_c is observed to converge for all the values of $1/\tau$ examined. The result of this calculation is shown as the solid line in Fig. 4, and the vertical dashed line indicates where $1/\tau = 2\Delta$. The point to the right of the dashed line is for Nb recrystallized in an ultrahigh vacuum[44] to achieve clean-limit conditions in which the residual resistivity ratios $[\rho(\mathrm{RT})/\rho(T \gtrsim T_c)]$ are well in excess of 100, and where $N_c \rightarrow \omega_p^2/8$ (or $\rho_s \rightarrow \omega_p^2$) for $T \ll T_c$. As the scattering rate increases and the material becomes progressively more "dirty," the strength of the condensate begins to decrease until it adopts the linear scaling behavior $N_c \simeq 8.1 \sigma_{dc} T_c$ observed in Fig. 4. (It should be noted that the BCS model yields the same asymptotic behavior in the dirty limit, regardless of the choice of ω_p or Δ; the constant is only sensitive upon the ratio of Δ to T_c.) The two points for Nb shown in Fig. 2 (reproduced in Fig. 4) fall close to this line[45,46] and are clearly in the dirty limit. Thus, the scaling relation N_c or $\rho_s \propto \sigma_{dc} T_c$ is the hallmark of a BCS dirty-limit system.[47]

We previously noted that the spectral weight of the condensate could be estimated by $N_c \simeq 2\Delta \sigma_{dc}$; assuming weak-coupling BCS then $2\Delta = 3.5 k_B T_c$ then $N_c \simeq 2.4 \sigma_{dc} T_c$. However, this is less than the observed asymptotic behavior observed for the weak-coupling BCS model of $N_c \simeq 8.1 \sigma_{dc} T_c$ observed in Fig. 4. An examination of Fig. 3(b) indicates that the discrepancy arises from the fact that $N_c \simeq \sigma_{dc} 2\Delta$ underestimates the spectral weight by more than a factor of two. The results from Fig. 4 suggest that a more realistic estimate of the area is $N_c \simeq 3.3(\sigma_{dc} 2\Delta)$. In this regard, the observed scaling relation in the cuprates of $N_c \simeq 4.4 \sigma_{dc} T_c$ would imply that $2\Delta / k_B T_c \simeq 2$. Of course, this statement is valid only within a BCS formalism for an isotropic s wave gap. Another, perhaps more reasonable, explanation for the smaller than expected value for the numerical constant may arise from the fact that copper-oxide superconductors have nodes in the energy gap, and, as a consequence, there is still a substantial amount of low-frequency residual conductivity at low temperature.[48] An important continuation of this work will be to calculate the optical conductivity of d-wave systems in the normal and superconducting states[49,50] and determine if the spectral weight obeys the same scaling relation. In a d-wave system, if $2\Delta_0 \simeq 4 k_B T_c$, then $N_c \simeq 2\Delta \sigma_{dc}$ may indeed be a reasonable estimate of N_c. Regardless of these differences, the empirical scaling relation N_c or $\rho_s \propto \sigma_{dc} T_c$ is observed in both the copper oxide and disordered elemental superconductors. If it is true in general that $\rho_s \propto \sigma_{dc} 2\Delta$, then this necessarily implies that $\Delta \propto T_c$. In the optimally doped and overdoped materials, there is some evidence that $\Delta_0 \propto T_c$ (Refs. 51 and 52). In the underdoped materials, large gaps are observed to develop in the normal state[53] well above T_c. While it has been noted that the energy

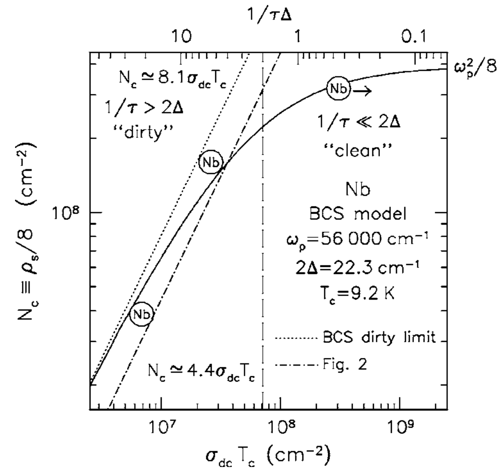

FIG. 4. The log-log plot of the predicted behavior from the BCS model of the spectral weight of the condensate N_c in Nb for a wide range of scattering rates $1/\tau = 0.05\Delta \rightarrow 50\Delta$ and assuming a plasma frequency $\omega_p = 56\,000$ cm^{-1}, critical temperature $T_c = 9.2$ K, and an energy gap of $2\Delta = 3.5 k_B T_c$ (solid line). The dashed line indicates $1/\tau = 2\Delta$. To the right of this line the material approaches the clean limit with a residual resistance ratio (RRR) of $\gtrsim 100$; the right arrow indicates that for larger RRRs, σ_{dc} close to T_c increases, but N_c has saturated to $\omega_p^2/8$ (or $\rho_s \rightarrow \omega_p^2$; the data point for Nb in this regime is from Ref. 44). As the scattering rate increases, the spectral weight of the condensate adopts a linear scaling behavior (dotted line); the two points for Nb (Refs. 45 and 46) shown in Fig. 2 lie close to this line, indicating that they are in the dirty limit. The scaling relation shown in Fig. 2 (dash-dot line) is slightly offset from the BCS dirty-limit result.

scale over which spectral weight is transferred into the condensate is much larger in the underdoped materials than it is for the optimally doped materials,[54,55] the majority of the spectral weight is still captured at energies comparable to T_c. This would tend to support the view that the energy scale relevant to phase coherence and the formation of the condensate is proportional to T_c.

It is of some interest at this point to compare the empirical relation, that ρ_s is proportional to $\sigma_{dc}T_c$, with the expression for the penetration depth that is given by the Ginzburg-Landau theory modified for the dirty limit. In general, the expression for the London penetration depth is given by $\lambda_L(T\rightarrow 0)=\sqrt{mc^2/(4\pi n_s e^2)}$, where $n_s \equiv n$ is the superconducting carrier concentration. In the dirty limit, one can show that $\rho_s(\text{dirty})/\rho_s(\text{clean})=l/\xi_0$ (Ref. 21). An increase in $1/\tau$ reduces the amount of superfluid and the penetration depth increases and can be written as $\lambda^2=(\xi_0/l)\lambda_L^2$. Since $\lambda^2 \propto 1/\rho_s$, $\xi_0 \propto 1/T_c$, and $\sigma_{dc}\propto l$, then one can recover the result that $\rho_s \propto \sigma_{dc}T_c$. It is possible that in a d-wave system the presence of nodal regions with a small superfluid density and $\Delta_k \ll \Delta_0$, that the coherence length in the above expression for λ^2 now involves some average including the nodal regions.

D. The c axis

It was previously observed[13] that the scaling relation $N_c \simeq 4.4\sigma_{dc}T_c$ is a universal result that describes not only the the a-b planes, but the c axis as well, as shown in Fig. 5. While a description of the scaling based on scattering rates within the context of clean and dirty limits may be appropriate for the a-b planes where the transport is coherent, it is inappropriate along the c axis, where the activated nature of the temperature dependence of the resistivity indicates that the transport in this direction is incoherent and governed by hopping.[56] In this case, the superconductivity along the c axis may be described by the Josephson effect, which for the BCS weak-coupling case ($2\Delta=3.5k_BT_c$) yields $N_c \simeq 8.1\sigma_{dc}T_c$ (Ref. 57). Surprisingly, this is precisely the result that was obtained in the a-b planes for the BCS weak-coupling case in the dirty limit in Fig. 4, indicating that from a functional point of view the scaling behavior of the dirty limit and the Josephson effect are nearly identical. One interpretation of this result is that the Josephson effect may arise naturally out of systems with an increasing amount of disorder and, as a result, any crossover from coherent to incoherent behavior still results in the same overall scaling relation. Another somewhat more speculative possibility is that the copper-oxide superconductors may be so electronically inhomogeneous that it may be possible to view the Josephson effect as appropriate not only for the c axis, but for the a-b planes as well.[58,59]

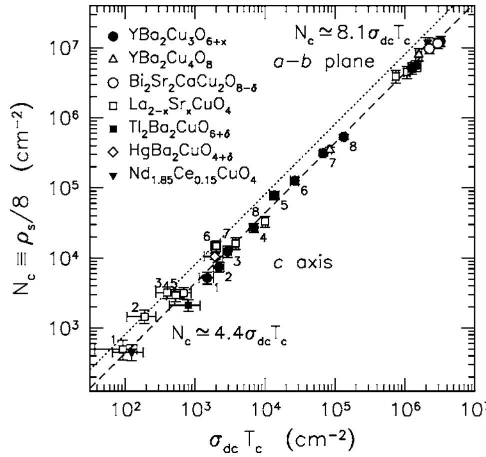

FIG. 5. The log-log plot of the spectral weight of the condensate N_c vs $\sigma_{dc}T_c$ for the a-b plane and the c axis for a variety of cuprates. Within error, all of the points fall on the same universal (dashed) line defined by $N_c \simeq 4.4\sigma_{dc}T_c$; the dotted line is the dirty-limit result $N_c \simeq 8.1\sigma_{dc}T_c$ for the BCS weak-coupling case ($2\Delta = 3.5k_BT_c$) from Fig. 4, and also represents the Josephson result for the BCS weak-coupling case, used to describe the scaling along the c axis (Ref. 57). (Values for the c-axis points are listed in the supplemental information of Ref. 13.)

IV. CONCLUSIONS

The implications of the linear scaling relation N_c or $\rho_s \propto \sigma_{dc}T_c$ in the copper-oxide superconductors has been examined within the context of clean-limit and dirty-limit systems. In the conventional BCS superconductors (such as Nb), this linear scaling is the hallmark of a dirty-limit superconductor. The copper-oxide materials are thought to be d-wave superconductors, in which the clean limit is difficult to achieve. The observed linear scaling strongly suggests that the copper-oxide superconductors are either close to or in the dirty limit. Estimates of N_c (or ρ_s) based on geometric arguments imply that the energy scale below which the majority of the spectral weight is transferred into the condensate scales linearly with T_c. The a-b planes and the c axis follow the same scaling relation.[13] The scaling behavior for the dirty limit and the Josephson effect (assuming a BCS formalism) is essentially identical from a functional point of view, suggesting that in some regime the dirty limit and the Josephson effect may be viewed as equivalent.

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