Microscopic theory of skyrmions in quantum Hall ferromagnets

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We present a microscopic theory of skyrmions in the monolayer quantum Hall ferromagnet. It is a peculiar feature of the system that the number density and the spin density are entangled intrinsically as dictated by the W_∞ algebra. The skyrmion and antiskyrmion states are constructed as W_∞-rotated states of the hole-excited and electron-excited states, respectively. They are spin textures accompanied with density modulation that decreases the Coulomb energy. We calculate their excitation energy as a function of the Zeeman gap and compared the result with experimental data.

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I. INTRODUCTION

Quantum coherence in the quantum Hall (QH) system has proved to be a fascinating subject during the past decade.[1,2] Electron spins are spontaneously polarized even in the absence of the Zeeman effect due to the exchange Coulomb interaction, leading to the QH ferromagnet. A prominent characteristic is that a charged excitation is a spin texture called skyrmion.[3] It is a coherent excitation of spins, and its excitation can be confirmed if the number of flipped spins is found to be more than that of a hole excitation.[4] A conventional way is to measure the increase of the activation energy by tilting samples,[5,6] which is roughly proportional to the number of flipped spins due to the Zeeman effect.

The skyrmion has been known[7] to be a classical solution to the nonlinear sigma model. Indeed, the concept of skyrmion was introduced[3] into QH ferromagnets first in this context. Subsequently a microscopic skyrmion state was considered to carry out a Hartree-Fock approximation.[8–11] In this paper we elaborate this idea and present a microscopic theory of skyrmions at the filling factor $\nu=1$, employing the framework of noncommutative geometry.[12] We also compare the result with the experimental data.[6]

The property of electrons becomes very peculiar when they are confined within the lowest Landau level.[13] The electron position is described solely by the guiding center $\boldsymbol{X}=(X,Y)$ subject to the noncommutative relation, $[X,Y]=-i\ell_B^2$, with ℓ_B the magnetic length. This noncommutativity is related with the so-called W_∞ algebra.[14,15] When the spin degree of freedom is introduced, the algebraic property becomes the SU(2) extension of the W_∞ algebra,[16] which we have named the $W_\infty(2)$ algebra.[12] It implies an intrinsic entanglement of the number density and the spin density of electrons, which amounts to a spin excitation accompanied with a density modulation.

This paper is organized as follows: In Sec. II we formulate the number density and the spin density of electrons confined within the lowest Landau level. They form the $W_\infty(2)$ algebra. In Sec. III we address the question whether there are states having the same electron number but a lower excitation energy compared with the hole state. In Sec. IV we propose the skyrmion and antiskyrmion states as $W_\infty(2)$-rotated states of the hole-excited and electron-excited states, respectively. They agree with the ansatz[8] made in a Hartree-Fock approximation. Since a $W_\infty(2)$ rotation modulates not only the spin density but also the number density, it decreases the Coulomb energy of the excitation. In Sec. V we derive the wave function of the skyrmion state. In particular we examine the case where the wave function is factorizable in the electron coordinates. We call such a skyrmion the factorizable skyrmion. In Sec. VI we study the system governed by the hard-core interaction instead of the Coulomb interaction, where the factorizable skyrmion is an eigenstate of the Hamiltonian. However, it is shown that skyrmion excitations are energetically unfavored once the Zeeman effect is taken into account. In Sec. VII we study the realistic system with the Coulomb interaction together with the Zeeman interaction. The factorizable skyrmion cannot be physical since its Zeeman energy diverges. We propose a trial skyrmion which interpolates the hole and the factorizable skyrmion. By minimizing the total energy we determine the state as well as the excitation energy as a function of the Zeeman gap. The result is compared successfully with the experimental data[6] in Sec. VIII. In Sec. IX we re-examine the semiclassical approximation[17] in the present context of the microscopic theory. It is shown that, though strictly speaking the factorizable skyrmion cannot be physical, it still presents a reasonable approximation with an appropriate cutoff of the divergent Zeeman energy. Section X is devoted for the conclusion.

II. DENSITY OPERATORS

Electrons in a plane perform cyclotron motion under strong magnetic field $B_\perp$ and create Landau levels. The coordinate $\boldsymbol{x}=(x,y)$ of the electron is decomposed into the guiding center $\boldsymbol{X}=(X,Y)$ and the relative coordinate $\boldsymbol{R}=(R_x,R_y)$, $\boldsymbol{x}=\boldsymbol{X}+\boldsymbol{R}$, where $R_x=-P_y/eB_\perp$ and $R_y=P_x/eB_\perp$ with $\boldsymbol{P}=(P_x,P_y)$ the covariant momentum. In the symmetric gauge the decomposition implies

$$X=\frac{1}{2}x-i\ell_B^2\frac{\partial}{\partial y},\quad Y=\frac{1}{2}y+i\ell_B^2\frac{\partial}{\partial x},\tag{2.1}$$

where $\ell_B=\sqrt{\hbar/eB_\perp}$ is the magnetic length. When the cyclotron energy gap is large, thermal excitations across Landau

levels are practically suppressed at sufficiently low temperature. In such a system the electron position is specified solely by the guiding center subject to the noncommutative relation,

$$[X,Y] = -i\ell_B^2. \tag{2.2}$$

Due to this noncommutative relation an electron cannot be localized and occupies an area $2\pi\ell_B^2$ called the Landau site. There are $N_\Phi = S/2\pi\ell_B^2$ Landau sites per one Landau level, where S is the area of the system. It is equal to the number $N_\Phi = B_\perp S/\Phi_D$ of flux quanta passing through the system, where $\Phi_D = 2\pi\hbar/e$ is the Dirac flux quantum. One Landau site may accommodate two electrons with up and down spins according to the Pauli exclusion principle. The filling factor is $\nu = N/N_\Phi = 2\pi\ell_B^2\rho_0$, where N is the total number of electrons and $\rho_0 = N/S$ is the electron density. The system becomes incompressible, leading to the integer QH effect, when the filling factor ν takes an integer value.

In this paper we study the QH system at $\nu = 1$. Thus all electrons are assumed to be confined within the lowest Landau level. We define operators

$$b = \frac{1}{\sqrt{2}\ell_B}(X - iY), \quad b^\dagger = \frac{1}{\sqrt{2}\ell_B}(X + iY) \tag{2.3}$$

with Eq. (2.1). They obey $[b, b^\dagger] = 1$. The Landau site $|n\rangle$ may be identified with the holomorphic basis $|n\rangle = (n!)^{-1/2}(b^\dagger)^n|0\rangle$. Its wave function is easily calculable,

$$\langle \boldsymbol{x}|n\rangle = \sqrt{\frac{1}{2^{n+1}\pi\ell_B^2 n!}} z^n e^{-r^2/4\ell_B^2}, \tag{2.4}$$

where $z = (x+iy)/\ell_B$ and $r = |\boldsymbol{x}|$. We now construct the Fock space $\mathbb{H}_{LLL}$ with the use of the creation operator $c_\mu^\dagger(n)$ on this Landau site $|n\rangle$, satisfying $\{c_\mu(n), c_\nu^\dagger(m)\} = \delta_{\mu\nu}\delta_{nm}$, where μ denotes the spin index, $\mu = \uparrow, \downarrow$.

We expand the two-component electron field $\Psi = (\psi_\uparrow, \psi_\downarrow)$ in terms of wave functions,

$$\psi_\mu(\boldsymbol{x}) = \sum_n \langle \boldsymbol{x}|n\rangle c_\mu(n). \tag{2.5}$$

The physical variables are the number density $\rho(\boldsymbol{x}) = \Psi^\dagger(\boldsymbol{x})\Psi(\boldsymbol{x})$ and the spin density $S_a(\boldsymbol{x}) = \frac{1}{2}\Psi^\dagger(\boldsymbol{x})\tau_a\Psi(\boldsymbol{x})$ with the Pauli matrix τ_a. Substituting Eq. (2.5) into these density operators, we obtain

$$\rho(\boldsymbol{x}) = \sum \langle m|\boldsymbol{x}\rangle\langle \boldsymbol{x}|n\rangle\rho(n,m), \tag{2.6a}$$

$$S_a(\boldsymbol{x}) = \sum \langle m|\boldsymbol{x}\rangle\langle \boldsymbol{x}|n\rangle S_a(n,m) \tag{2.6b}$$

with

$$\rho(n,m) = \sum_\mu c_\mu^\dagger(m)c_\mu(n), \tag{2.7a}$$

$$S_a(n,m) = \frac{1}{2}\sum_{\mu\nu} c_\mu^\dagger(m)(\tau_a)_{\mu\nu}c_\nu(n). \tag{2.7b}$$

We employ the relation

$$\int d^2x e^{-i\boldsymbol{q}\boldsymbol{x}}\langle m|\boldsymbol{x}\rangle\langle \boldsymbol{x}|n\rangle = e^{-\ell_B^2\boldsymbol{q}^2/4}\langle m|e^{-i\boldsymbol{q}\boldsymbol{X}}|n\rangle, \tag{2.8}$$

and transform them in the momentum space,

$$\rho(\boldsymbol{q}) = e^{-\ell_B^2\boldsymbol{q}^2/4}\hat{\rho}(\boldsymbol{q}), \quad S_a(\boldsymbol{q}) = e^{-\ell_B^2\boldsymbol{q}^2/4}\hat{S}_a(\boldsymbol{q}), \tag{2.9}$$

where

$$\hat{\rho}(\boldsymbol{q}) = \frac{1}{2\pi}\sum_{mn}\langle m|e^{-i\boldsymbol{q}\boldsymbol{X}}|n\rangle\rho(n,m), \tag{2.10a}$$

$$\hat{S}_a(\boldsymbol{q}) = \frac{1}{2\pi}\sum_{mn}\langle m|e^{-i\boldsymbol{q}\boldsymbol{X}}|n\rangle S_a(n,m). \tag{2.10b}$$

We call $\hat{\rho}(\boldsymbol{q})$ and $\hat{S}_a(\boldsymbol{q})$ the bare densities. In the coordinate space we have the relation

$$\rho(\boldsymbol{x}) = \frac{1}{\pi\ell_B^2}\int d^2x' e^{-|\boldsymbol{x}-\boldsymbol{x}'|^2/\ell_B^2}\hat{\rho}(\boldsymbol{x}') \tag{2.11}$$

between the physical density $\rho(\boldsymbol{x})$ and the bare density $\hat{\rho}(\boldsymbol{x})$.

The density operators generate the SU(2) extension of the W_∞ algebra,

$$[\rho(m,n), \rho(i,j)] = \delta_{mj}\rho(i,n) - \delta_{in}\rho(m,j),$$

$$[\rho(m,n), S_a(i,j)] = \delta_{mj}S_a(i,n) - \delta_{in}S_a(m,j),$$

$$[S_a(m,n), S_b(i,j)] = \frac{i}{2}\varepsilon_{abc}[\delta_{mj}S_c(i,n) + \delta_{in}S_c(m,j)] + \frac{1}{4}\delta_{ab}[\delta_{mj}\rho(i,n) - \delta_{in}\rho(m,j)], \tag{2.12}$$

which we have named[12] the $W_\infty(2)$ algebra. The electron density and the spin density are intrinsically entangled in the algebra, as implies that a spin rotation induces a density modulation.

III. HAMILTONIAN

The Hamiltonian of the QH system consists of a four-fermion interaction term and the Zeeman term, $H = H_V + H_Z$ with

$$H_V = \frac{1}{2}\int d^2x d^2y V(\boldsymbol{x}-\boldsymbol{y})\delta\rho(\boldsymbol{x})\delta\rho(\boldsymbol{y}), \tag{3.1a}$$

$$H_Z = -\Delta_Z\int d^2x S_z(\boldsymbol{x}), \tag{3.1b}$$

where $\delta\rho(\boldsymbol{x})$ is the density excitation operator,

$$\delta\rho(\boldsymbol{x}) = \rho(\boldsymbol{x}) - \rho_0, \tag{3.2}$$

and $\Delta_Z = |g|\mu_B B_\perp$ is the Zeeman gap with μ_B the Bohr magneton. In the actual system $V(\boldsymbol{x})$ is given by the Coulomb potential, but here we only assume that it represents a repulsive interaction.

Substituting the density operators (2.9) into these we obtain

$$H_V = \sum_{mnij} V_{mnij} \sum_{\mu\nu} c_\mu^\dagger(m) c_\nu^\dagger(i) c_\nu(j) c_\mu(n)$$

$$+ (N_\Phi + \delta N)\epsilon_X - (N_\Phi + 2\delta N)\epsilon_D, \tag{3.3a}$$

$$H_Z = -\Delta_Z \sum_n S_z(n,n), \tag{3.3b}$$

where

$$V_{mnij} = \frac{1}{4\pi} \int d^2k V(\boldsymbol{k}) e^{-\ell_B^2 \boldsymbol{k}^2/2} \langle m|e^{i\boldsymbol{X}\boldsymbol{k}}|n\rangle\langle i|e^{-i\boldsymbol{X}\boldsymbol{k}}|j\rangle, \tag{3.4}$$

with $V(\boldsymbol{k})$ the Fourier transformation of the potential $V(\boldsymbol{x})$. We have used the notation

$$\delta N = \int d^2x \delta\rho(\boldsymbol{x}), \tag{3.5a}$$

$$\epsilon_D = \sum_j V_{nnjj} = \frac{\rho_0}{2} \int d^2x V(\boldsymbol{x}), \tag{3.5b}$$

$$\epsilon_X = \sum_j V_{njjn} = \frac{\rho_0}{2} \ell_B^2 \int d^2k V(\boldsymbol{k}) e^{-\ell_B^2 \boldsymbol{k}^2/2}, \tag{3.5c}$$

where ϵ_D and ϵ_X are the direct and exchange energy parameters, respectively.

In the QH state at $\nu=1$, the number of electrons is equal to the number of Landau sites. One Landau site contains one electron due to the repulsive interaction. The system without the Zeeman effect ($\Delta_Z=0$) is most interesting. An intriguing feature is that the SU(2) symmetry is spontaneously broken by the repulsive interaction and all spins are spontaneously polarized, as demonstrated in Appendix A. Thus the QH system is a ferromagnet, where quantum coherence develops spontaneously.

Without loss of generality we may take

$$|g\rangle = \prod_n c_\uparrow^\dagger(n)|0\rangle \tag{3.6}$$

as the ground state. It satisfies

$$\rho(m,n)|g\rangle = \delta_{mn}|g\rangle, \quad S_z(m,n)|g\rangle = \tfrac{1}{2}\delta_{mn}|g\rangle, \tag{3.7}$$

and

$$H_V|g\rangle = 0. \tag{3.8}$$

A hole-excited and electron-excited states are given by

$$|h\rangle = c_\uparrow(0)|g\rangle, \quad |e\rangle = c_\downarrow^\dagger(0)|g\rangle, \tag{3.9}$$

with their energies

$$\langle h|H_V|h\rangle = \langle e|H_V|e\rangle = \epsilon_X. \tag{3.10}$$

The main question is whether there are states having the same electron number but a lower excitation energy compared with the hole-excited or the electron-excited state.

IV. MICROSCOPIC SKYRMION STATES

We consider a $W_\infty(2)$-rotated state of the hole-excited state $|h\rangle$ and the electron-excited state $|e\rangle$,

$$|\mathfrak{S}_{\rm sky}^-\rangle = e^{iW^-} c_\uparrow(0)|g\rangle, \tag{4.1a}$$

$$|\mathfrak{S}_{\rm sky}^+\rangle = e^{iW^+} c_\downarrow^\dagger(0)|g\rangle, \tag{4.1b}$$

where $W^\pm$ are elements of the $W_\infty(2)$ algebra (2.12). The Coulomb energy of the state depends on $W^\pm$, since the $W_\infty(2)$ rotation modulates not only the spin texture but also the electron density around the hole or electron excitation. The element $W^\pm$ is to be determined by requiring the excitation energy to be minimized. As we shall see soon, $|\mathfrak{S}_{\rm sky}^-\rangle$ and $|\mathfrak{S}_{\rm sky}^+\rangle$ describe the skyrmion and antiskyrmion states. We use the index $(-)$ for the skyrmion and $(+)$ for the antiskyrmion.

Similarly,

$$|\mathfrak{S}_{\rm sky}^-;k\rangle = e^{iW} c_\uparrow(0)\cdots c_\uparrow(k-1)|g\rangle, \tag{4.2}$$

$$|\mathfrak{S}_{\rm sky}^+;k\rangle = e^{i\tilde{W}} c_\downarrow^\dagger(0)\cdots c_\downarrow^\dagger(k-1)|g\rangle \tag{4.3}$$

describe a multiskyrmion and a multiantiskyrmion as $W_\infty(2)$-rotated states of a multihole state and a multielectron state.

More generally we may consider a wide class of states presented by

$$|\mathfrak{S}\rangle = e^{iW}|\mathfrak{S}_0\rangle, \tag{4.4}$$

where $|\mathfrak{S}_0\rangle$ is a state of the form

$$|\mathfrak{S}_0\rangle = \prod_{\mu n} [c_\mu^\dagger(n)]^{\nu_\mu(n)}|0\rangle, \tag{4.5}$$

where $\nu_\mu(n)$ may take the value either 0 or 1 depending whether the spin state μ at a site n is occupied or not. The class of states (4.4) is quite general though it may not embrace all possible ones. Nevertheless all physically relevant states seem to fall in this category. Indeed, as far as we know, perturbative excitations are spin waves and nonperturbative excitations are skyrmions in QH systems. The corresponding states belong surely to this category.

The electron number of the $W_\infty(2)$-rotated state (4.4) is easily calculable,

$$\langle\mathfrak{S}|N|\mathfrak{S}\rangle = \langle\mathfrak{S}_0|e^{-iW}Ne^{+iW}|\mathfrak{S}_0\rangle = \langle\mathfrak{S}_0|N|\mathfrak{S}_0\rangle, \tag{4.6}$$

since $N=\Sigma_n\Sigma_\mu c_\mu^\dagger(n)c_\mu(n)$ is a Casimir operator. We set

$$\delta N^{\rm cl} = \langle\mathfrak{S}|N|\mathfrak{S}\rangle - \langle g|N|g\rangle. \tag{4.7}$$

This is the electron number carried by the excitation described by the state $|\mathfrak{S}\rangle$. It is an integer since the electron number of the state $|\mathfrak{S}\rangle$ is the same as that of the state $|\mathfrak{S}_0\rangle$.

We examine more in detail the state (4.1) with $\delta N^{\rm cl} = \mp 1$. A simplest $W_\infty(2)$ rotation mixes only neighboring two sites and is given by the choice of $W^- = \Sigma_{n=0}^\infty W_n^-$ with

$$iW_n^- = \alpha_n[c_\downarrow^\dagger(n)c_\uparrow(n+1) - c_\uparrow^\dagger(n+1)c_\downarrow(n)], \tag{4.8}$$

where α_n is a real parameter. Note that W_n^- is a Hermitian operator belonging to the $W_\infty(2)$ algebra, and $[W_n^-, W_m^-]=0$. We find

$$e^{+iW^-}c_\uparrow^\dagger(n+1)e^{-iW^-} = e^{+iW_n^-}c_\uparrow^\dagger(n+1)e^{-iW_n^-} \equiv \xi^\dagger(n), \tag{4.9}$$

since the spin-up operator $c_\uparrow^\dagger(n+1)$ is contained only in W_n^-. We calculate $\xi^\dagger(n)$ using the standard technique of deriving the differential equation with respect to α_n. Since it satisfies

$$\frac{d^2\xi^\dagger(n)}{d\alpha_n^2} = -\xi^\dagger(n) \tag{4.10}$$

together with the initial condition $d\xi^\dagger/d\alpha_n = c_\downarrow^\dagger(n)$ at $\alpha_n=0$, we integrate it as

$$\xi^\dagger(n) = u_-(n)c_\downarrow^\dagger(n) + v_-(n)c_\uparrow^\dagger(n+1), \tag{4.11}$$

where we have set $u_-(n)=\sin\alpha_n$ and $v_-(n)=\cos\alpha_n$. Thus the constraint

$$u_-^2(n) + v_-^2(n) = 1 \tag{4.12}$$

is imposed on $u_-(n)$ and $v_-(n)$. We now find

$$|\mathfrak{S}_{\text{sky}}^-\rangle = e^{iW^-}\prod_{n=0} c_\uparrow^\dagger(n+1)|0\rangle = \prod_{n=0}\xi^\dagger(n)|0\rangle, \tag{4.13}$$

where we have used (4.9) and $e^{-iW_n^-}|0\rangle=0$.

Similarly we choose $W^+=\Sigma_n W_n^+$ in Eq. (4.1b) with

$$iW_n^+ = \sum_n \beta_n[c_\downarrow^\dagger(n+1)c_\uparrow(n) - c_\uparrow^\dagger(n)c_\downarrow(n+1)]. \tag{4.14}$$

We find

$$|\mathfrak{S}_{\text{sky}}^+\rangle = c_\downarrow^\dagger(0)\prod_{n=0}\zeta^\dagger(n)|0\rangle, \tag{4.15}$$

where

$$\zeta^\dagger(n) = u_+(n)c_\uparrow^\dagger(n) + v_+(n)c_\downarrow^\dagger(n+1) \tag{4.16}$$

together with

$$u_+^2(n) + v_+^2(n) = 1, \tag{4.17}$$

since $u_+(n)=\cos\beta_n$ and $v_+(n)=\sin\beta_n$. The states (4.13) and (4.15) agree with the skyrmion state and the antiskyrmion state proposed by Fertig *et al.*,[8] respectively.

The operators $\xi(m)$ and $\zeta(n)$ satisfy the standard canonical anticommutation relations,

$$\{\xi(m),\xi^\dagger(n)\} = \delta_{mn}, \quad \{\xi(m),\xi(n)\} = 0,$$

$$\{\zeta(m),\zeta^\dagger(n)\} = \delta_{mn}, \quad \{\zeta(m),\zeta(n)\} = 0, \tag{4.18}$$

with the use of the constrains (4.12) and (4.17). Since these states should approach the ground state asymptotically it is necessary that

$$\lim_{n\to\infty} u_-(n) = 0, \quad \lim_{n\to\infty} v_-(n) = 1,$$

$$\lim_{n\to\infty} u_+(n) = 1, \quad \lim_{n\to\infty} v_+(n) = 0. \tag{4.19}$$

For later convenience we define

$$u_-(-1) = 1, \quad v_-(-1) = 0,$$

$$u_+(-1) = 0, \quad v_+(-1) = 1 \tag{4.20}$$

in accordance with Eqs. (4.12) and (4.17).

In what follows we calculate explicitly the classical densities

$$\rho^{\pm\text{cl}}(m,n) \equiv \langle\mathfrak{S}_{\text{sky}}^\pm|\rho^\pm(m,n)|\mathfrak{S}_{\text{sky}}^\pm\rangle, \tag{4.21}$$

$$S_a^{\pm\text{cl}}(m,n) \equiv \langle\mathfrak{S}_{\text{sky}}^\pm|S_a^\pm(m,n)|\mathfrak{S}_{\text{sky}}^\pm\rangle. \tag{4.22}$$

The basic relations for the skyrmion state are

$$c_\uparrow(n)|\mathfrak{S}_{\text{sky}}^-\rangle = v_-(n-1)\xi(n-1)|\mathfrak{S}_{\text{sky}}^-\rangle,$$

$$c_\downarrow(n)|\mathfrak{S}_{\text{sky}}^-\rangle = u_-(n)\xi(n)|\mathfrak{S}_{\text{sky}}^-\rangle. \tag{4.23}$$

They reduce the action of c operators to that of ξ, thus allowing to carry out exact calculus. We employ $\langle\mathfrak{S}_{\text{sky}}^-|\xi^\dagger(m)\xi(n)|\mathfrak{S}_{\text{sky}}^-\rangle = \delta_{mn}$ together with (4.23) and its conjugate. In this way we come to

$$\rho^\pm(n,n) = u_\pm^2(n) + v_\pm^2(n-1),$$

$$S_z^\pm(n,n) = \pm\tfrac{1}{2}[u_\pm^2(n) - v_\pm^2(n-1)],$$

$$S_x^\pm(n+1,n) = S_x^\pm(n,n+1) = \tfrac{1}{2}u_\pm(n)v_\pm(n),$$

$$S_y^\pm(n+1,n) = -S_y^\pm(n,n+1) = \frac{i}{2}u_\pm(n)v_\pm(n). \tag{4.24}$$

All other components, $\rho^\pm(n,m)$, etc., vanish. Here and hereafter we omit the superscript "cl." By substituting these into the physical density (2.6), we obtain

$$\frac{\rho^\pm(\boldsymbol{x})}{\rho_0} = e^{-|z|^2/2}\sum_{n=0}[u_\pm^2(n) + v_\pm^2(n-1)]\left(\frac{|z|^2}{2}\right)^n,$$

$$\frac{S_z^\pm(\boldsymbol{x})}{\rho_0} = \pm\frac{1}{2}e^{-|z|^2/2}\sum_{n=0}\frac{u_\pm^2(n) - v_\pm^2(n-1)}{n!}\left(\frac{|z|^2}{2}\right)^n,$$

$$\frac{S_x^\pm(\boldsymbol{x})}{\rho_0} = \frac{x}{\sqrt{2}\ell_B}e^{-|z|^2/2}\sum_{n=0}\frac{u_\pm(n)v_\pm(n)}{n!\sqrt{n+1}}\left(\frac{|z|^2}{2}\right)^n,$$

$$\frac{S_y^\pm(\boldsymbol{x})}{\rho_0} = \pm\frac{y}{\sqrt{2}\ell_B}e^{-|z|^2/2}\sum_{n=0}\frac{u_\pm(n)v_\pm(n)}{n!\sqrt{n+1}}\left(\frac{|z|^2}{2}\right)^n. \tag{4.25}$$

We have calculated the classical densities of the $W_\infty(2)$-rotated states of a hole-excited and electron-excited states, which contain infinitely many variables $u_\pm(n)$ and $v_\pm(n)$. We estimate the energies of these states and minimize them in later sections.

V. *N*-BODY WAVE FUNCTIONS

We consider the N-electron system over $N+1$ sites. The wave function of a many-body state $|\mathfrak{S}\rangle$ is defined by

$$\mathfrak{S}_{\mu_1\mu_2\cdots\mu_N}[\boldsymbol{x}] = \langle 0|\psi_{\mu_1}(\boldsymbol{x}_1)\psi_{\mu_2}(\boldsymbol{x}_2)\cdots\psi_{\mu_N}(\boldsymbol{x}_N)|\mathfrak{S}\rangle, \tag{5.1}$$

where $\psi_\mu(\boldsymbol{x})$ is given by Eq. (2.5), or

$$\psi_\mu(\boldsymbol{x}) = \rho_0^{1/2} e^{-|z|^2/4} \sum_{n=0}^{N} \alpha(n) z^n c_\mu(n) \tag{5.2}$$

with

$$\alpha(n) = \sqrt{\frac{1}{2^n n!}}. \tag{5.3}$$

For the hole-excited state $|h\rangle = c_\uparrow(0)|g\rangle$ it is easy to see

$$\mathfrak{S}_h[\boldsymbol{x}] = \prod_r^N \begin{pmatrix} z_r \\ 0 \end{pmatrix} \mathfrak{S}_{\mathrm{LN}}[\boldsymbol{x}], \tag{5.4}$$

where $\mathfrak{S}_{\mathrm{LN}}[\boldsymbol{x}]$ is the Slater determinant of the one-body wave functions,

$$\mathfrak{S}_{\mathrm{LN}}[\boldsymbol{x}] = \rho_0^{N/2} \begin{vmatrix} 1 & z_1 & \cdots & z_1^{N-1} \\ 1 & z_2 & \cdots & z_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & z_N & \cdots & z_N^{N-1} \end{vmatrix} e^{-\Sigma_{r=1}^N |z_r|^2/4}. \tag{5.5}$$

The wave function of the state $|h\rangle$ is factorizable in the electron coordinates apart from the factor $\mathfrak{S}_{\mathrm{LN}}[\boldsymbol{x}]$.

We proceed to derive the wave function of the skyrmion state (4.13). First we examine the component with all spins up, to which only the term $v_-(n)c_\uparrow^\dagger(n+1)$ in $\xi^\dagger(n)$ contributes,

$$\mathfrak{S}^-_{\uparrow\uparrow\cdots\uparrow}[\boldsymbol{x}] = \prod_{n=0}^{N-1} v_-(n) \langle 0| \prod_{i=1}^{N} \psi_\uparrow(\boldsymbol{x}_i) \prod_{n=1}^{N} c_\uparrow^\dagger(n)|0\rangle. \tag{5.6}$$

Apart from the factor $\Pi_{n=0}^{N-1} v_-(n)$ this is nothing but the wave function of a hole. Thus

$$\mathfrak{S}^-_{\uparrow\uparrow\cdots\uparrow}[\boldsymbol{x}] = C_N \prod_{n=1}^{N} z_n \mathfrak{S}_{\mathrm{LN}}[\boldsymbol{x}], \tag{5.7}$$

where $C_N = \Pi_{n=1}^N \alpha(n) v_-(n-1)$. Next we consider the component with all spins down, which arises only from the term $u_-(n)c_\downarrow^\dagger(n)$ in $\xi^\dagger(n)$,

$$\begin{aligned} \mathfrak{S}^-_{\downarrow\downarrow\cdots\downarrow}[\boldsymbol{x}] &= \prod_{n=1}^{N} u_-(n) \langle 0| \prod_{i=1}^{N} \psi_\downarrow(\boldsymbol{x}_i) \prod_{n=1}^{N} c_\downarrow^\dagger(n)|0\rangle \\ &= C_N \prod_{n=0}^{N-1} \beta_n \mathfrak{S}_{\mathrm{LN}}[\boldsymbol{x}]. \end{aligned} \tag{5.8}$$

Here we have set

$$\beta_n = \frac{\alpha(n)u_-(n)}{\alpha(n+1)v_-(n)} = \frac{u_-(n)}{v_-(n)} \sqrt{2(n+1)} \tag{5.9}$$

with Eq. (5.3). Comparing (5.7) and (5.8) we remark that the wave function $\mathfrak{S}^{\mathrm{sky}}_{\downarrow\downarrow\cdots\downarrow}[\boldsymbol{x}]$ is obtained by replacing z_n with β_{n-1} within the factor Πz_n of the wave function $\mathfrak{S}_{\uparrow\uparrow\cdots\uparrow}[\boldsymbol{x}]$ for all n. The wave function with mixed spin components is similarly derived, where z_n is replaced with β_{n-1} for certain indices n within the factor Πz_n. In general we derive

$$\begin{aligned} &\mathfrak{S}^-_{\mu_1\mu_2\cdots\mu_N}[\boldsymbol{x}] \\ &= C_N e^{-\Sigma_{r=1}^N |z_r|^2/4} \\ &\quad \times \begin{vmatrix} \begin{pmatrix} z_1 \\ \beta_0 \end{pmatrix}_{\mu_1} & z_1 \begin{pmatrix} z_1 \\ \beta_1 \end{pmatrix}_{\mu_1} & \cdots & z_1^{N-1} \begin{pmatrix} z_1 \\ \beta_{N-1} \end{pmatrix}_{\mu_1} \\ \begin{pmatrix} z_2 \\ \beta_0 \end{pmatrix}_{\mu_2} & z_2 \begin{pmatrix} z_2 \\ \beta_1 \end{pmatrix}_{\mu_2} & \cdots & z_2^{N-1} \begin{pmatrix} z_2 \\ \beta_{N-1} \end{pmatrix}_{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} z_N \\ \beta_0 \end{pmatrix}_{\mu_N} & z_N \begin{pmatrix} z_N \\ \beta_1 \end{pmatrix}_{\mu_N} & \cdots & z_N^{N-1} \begin{pmatrix} z_N \\ \beta_{N-1} \end{pmatrix}_{\mu_N} \end{vmatrix}. \end{aligned} \tag{5.10}$$

This is the wave function of the skyrmion.

It is notable that, when all β_n are equal ($\beta_n = \sqrt{2}\omega$), or

$$\frac{u_-(n)}{v_-(n)} \sqrt{2(n+1)} = \beta_n = \sqrt{2}\omega, \tag{5.11}$$

it is reduced to

$$\mathfrak{S}^-_{\mathrm{sky}}[\boldsymbol{x}] = \prod_r^N \begin{pmatrix} z_r \\ \sqrt{2}\omega \end{pmatrix} \mathfrak{S}_{\mathrm{LN}}[\boldsymbol{x}]. \tag{5.12}$$

The wave function is factorizable as in the case of a hole. We call such a skyrmion the factorizable skyrmion. As we shall see in the following section, this is the wave function of the skyrmion in the hard-core model. It is quite difficult to write down the wave function of an antiskyrmion in terms of the analytic variable z_r. We discuss it in Appendix B.

VI. HARD-CORE INTERACTION

We first investigate a detailed structure of skyrmions in the system governed by the hard-core interaction,[18]

$$V(\boldsymbol{x}-\boldsymbol{y}) = \delta^2(\boldsymbol{x}-\boldsymbol{y}), \tag{6.1}$$

instead of the Coulomb interaction. The Hamiltonian (3.1a) reads

$$H_{\mathrm{hc}} = \frac{1}{2} \int d^2x \, \delta\rho(\boldsymbol{x}) \, \delta\rho(\boldsymbol{x}). \tag{6.2}$$

All previous formulas hold with

$$\begin{aligned} V_{mnij} &= \frac{1}{8\pi^2} \int d^2k e^{-\ell_B^2 \boldsymbol{k}^2/2} \langle m|e^{i\boldsymbol{X}\boldsymbol{k}}|n\rangle \langle i|e^{-i\boldsymbol{X}\boldsymbol{k}}|j\rangle \\ &= \frac{1}{8\pi\ell_B^2} \frac{\sqrt{(m+i)!(n+j)!}}{\sqrt{m!i!n!j!}} \frac{\delta_{m+i,n+j}}{\sqrt{2^{m+i+n+j}}}, \end{aligned} \tag{6.3}$$

and

$$\epsilon_D = \epsilon_X = \frac{1}{4\pi\ell_B^2}. \tag{6.4}$$

We rewrite Eq. (6.2) into the normal ordered form,

$$H_{\rm hc} = \int d^2x \psi_\uparrow^\dagger(\boldsymbol{x})\psi_\downarrow^\dagger(\boldsymbol{x})\psi_\downarrow(\boldsymbol{x})\psi_\uparrow(\boldsymbol{x}) - \frac{1}{4\pi\ell_B^2}\int d^2x \delta\rho(\boldsymbol{x}). \tag{6.5}$$

The ground state $|g\rangle$ is given by Eq. (3.6) as an eigenstate of the Hamiltonian, $H_{\rm hc}|g\rangle=0$. A hole-excited and electron-excited states are given by $|h\rangle=c_\uparrow(0)|g\rangle$ and $|e\rangle=c_\downarrow^\dagger(0)|g\rangle$.

We determine the skyrmion and antiskyrmion states by minimizing the energy of the $W_\infty(2)$-rotated state (4.1). We deal with the skyrmion state (4.13) explicitly, but similar formulas follow also for the antiskyrmion state (4.15).

We define the state

$$|\mathfrak{S}'\rangle = \psi_\downarrow(\boldsymbol{x})\psi_\uparrow(\boldsymbol{x})|\mathfrak{S}_{\rm sky}^-\rangle = \sum_{mn}\varphi_m(\boldsymbol{x})\varphi_n(\boldsymbol{x})c_\downarrow(m)c_\uparrow(n)|\mathfrak{S}_{\rm sky}^-\rangle. \tag{6.6}$$

Using Eq. (4.23) we come to

$$|\mathfrak{S}'\rangle = \sum_{m=0}\sum_{n=0}\varphi_m(\boldsymbol{x})\varphi_{n+1}(\boldsymbol{x})u_-(m)v_-(n)\xi(m)\xi(n)|\mathfrak{S}_{\rm sky}^-\rangle, \tag{6.7}$$

where we have used $v(-1)=0$. Hence

$$\int d^2x\langle\mathfrak{S}'|\mathfrak{S}'\rangle = 2\sum_{k,l=0} v_-(k)u_-(l)v_-(l)u_-(k)V_{k+1,k,l,l+1}. \tag{6.8}$$

Using Eq. (6.3) we obtain

$$\int d^2x\langle\mathfrak{S}'|\mathfrak{S}'\rangle \tag{6.9}$$

$$= \frac{1}{8\pi\ell_B^2}\sum_{k,l=0}\frac{(k+l+1)!}{k!l!2^{k+l+1}}\left[\frac{v_-(k)u_-(l)}{\sqrt{k+1}} - \frac{v_-(l)u_-(k)}{\sqrt{l+1}}\right]^2 \geqslant 0. \tag{6.10}$$

The skyrmion energy $\langle\mathfrak{S}_{\rm sky}^-|H_{\rm hc}|\mathfrak{S}_{\rm sky}^-\rangle$ is minimized when the equality is achieved, as occurs when the factorizable-skyrmion condition (5.11) is satisfied, or

$$u_-^2(n) = \frac{\omega^2}{n+1+\omega^2}, \quad v_-^2(n) = \frac{n+1}{n+1+\omega^2} \tag{6.11}$$

due to the normalization (4.12).

It follows $|\mathfrak{S}'\rangle=0$ when the equality holds in (6.8). Hence, $|\mathfrak{S}_{\rm sky}^-\rangle$ is an eigenstate of the Hamiltonian with $\delta N^{\rm cl}=-1$,

$$H_{\rm hc}|\mathfrak{S}_{\rm sky}^-\rangle = \frac{1}{4\pi}\int d^2x\delta\rho(\boldsymbol{x})|\mathfrak{S}_{\rm sky}^-\rangle = -\frac{1}{4\pi}\delta N^{\rm cl}|\mathfrak{S}_{\rm sky}^-\rangle. \tag{6.12}$$

Similarly, when

$$u_+^2(n) = \frac{n+1}{n+1+\omega^2}, \quad v_+^2(n) = \frac{\omega^2}{n+1+\omega^2} \tag{6.13}$$

the antiskyrmion state is an eigenstate of the Hamiltonian with $\delta N^{\rm cl}=1$,

$$H_{\rm hc}|\mathfrak{S}_{\rm sky}^+\rangle = \frac{1}{4\pi}\int d^2x\delta\rho(\boldsymbol{x})|\mathfrak{S}_{\rm sky}^+\rangle = \frac{1}{4\pi}\delta N^{\rm cl}|\mathfrak{S}_{\rm sky}^+\rangle. \tag{6.14}$$

We may summarize them as $H_{\rm hc}|\mathfrak{S}_{\rm sky}^\pm\rangle=E_{\rm hc}|\mathfrak{S}_{\rm sky}^\pm\rangle$ with $E_{\rm hc}=|\delta N^{\rm cl}|/4\pi$. The skyrmion state and the antiskyrmion state are the lowest energy states possessing the electron numbers $\delta N^{\rm cl}=-1$ and $\delta N^{\rm cl}=1$, respectively. In conclusion, we have determined the $W_\infty(2)$-rotated states (4.1) by minimizing the energy, though their energy is independent of the scale ω and degenerates with the hole-excited and electron-excited state.

The physical densities can be expressed in terms of the Kummer function $M(a;b;x)$,

$$M(a;a+1;x) = a\sum_{n=0}^{\infty}\frac{x^n}{(n+a)n!}, \tag{6.15}$$

as

$$\begin{aligned}
\frac{\delta\rho^\pm(\boldsymbol{x})}{\rho_0} &= \pm e^{-(1/2)|z|^2}M(\omega^2;\omega^2+1;|z|^2/2) \\
&\quad \mp \frac{\omega^2}{\omega^2+1}e^{-(1/2)z^2}M(\omega^2+1;\omega^2+2;|z|^2/2), \\
\frac{S_z^\pm(\boldsymbol{x})}{\rho_0} &= \frac{1}{2} - \frac{1}{2}e^{-(1/2)|z|^2}M(\omega^2;\omega^2+1;|z|^2/2) \\
&\quad - \frac{1}{2}\frac{\omega^2}{\omega^2+1}e^{-(1/2)|z|^2}M(\omega^2+1;\omega^2+2;|z|^2/2), \\
\frac{S_x^\pm(\boldsymbol{x})}{\rho_0} &= \frac{1}{\sqrt{2}}\frac{\omega x/\ell_B}{\omega^2+1}e^{-(1/2)|z|^2}M(\omega^2+1;\omega^2+2;|z|^2/2), \\
\frac{S_y^\pm(\boldsymbol{x})}{\rho_0} &= \frac{\pm 1}{\sqrt{2}}\frac{\omega y/\ell_B}{\omega^2+1}e^{-(1/2)|z|^2}M(\omega^2+1;\omega^2+2;|z|^2/2),
\end{aligned} \tag{6.16}$$

with $|z|^2=r^2/\ell_B^2$. We have illustrated the density $\rho^-(r)$ for typical values of the parameters ω in Fig. 1.

Using the relations

$$M(1;b;x) = 1 + \frac{x}{b}M(1;b+1;x),$$

$$M(a;b;z) = e^z M(b-a;b;-z), \tag{6.17}$$

we are able to summarize the densities as

$$\delta\rho^+(\boldsymbol{x}) = -\delta\rho^-(\boldsymbol{x}), \quad S_a^\pm(\boldsymbol{x}) = \rho^-(\boldsymbol{x})\mathcal{S}_a^\pm(\boldsymbol{x}), \tag{6.18}$$

with

$$\mathcal{S}_x(\boldsymbol{x}) = \frac{2\lambda x}{r^2+\lambda^2}, \quad \mathcal{S}_y(\boldsymbol{x}) = \frac{\mp 2\lambda y}{r^2+\lambda^2},$$

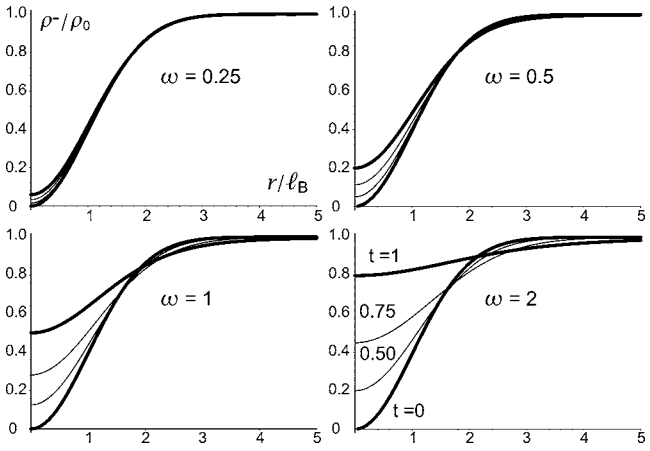

FIG. 1. The density modulation (ρ^-/ρ_0) accompanied with the skyrmion excitation is plotted as a function of the radius (r/ℓ_B). The heavy curve $(t=1)$ is for the hard-core model based on Eq. (6.16) for the choice of $\omega=0.25,0.5,1,2$, while the heavy curve $(t=0)$ represents the density of a hole. The thin curves represent the interpolating formula (7.8) for $t=0.50,0.75$, where the parameter t is defined by Eq. (7.5).

$$\mathcal{S}_z(\boldsymbol{x}) = \frac{r^2-\lambda^2}{r^2+\lambda^2}, \tag{6.19}$$

and

$$\rho^-(\boldsymbol{x}) = \frac{r^2+\lambda^2}{2\ell_B^2+\lambda^2} M(1;\omega^2+2;-|z|^2/2)\rho_0, \tag{6.20}$$

where we have set $\lambda=\sqrt{2}\omega\ell$. It is interesting that the physical spin densities $S_a^{\pm}(\boldsymbol{x})$ are factorizable into the number density $\rho^-(\boldsymbol{x})$ and the normalized spin field $\mathcal{S}_a^{\pm}(\boldsymbol{x})$ as in Eq. (6.18). The normalized spin field (6.19) agrees with the nonlinear-sigma-model skyrmion configuration.[7]

We explore the properties of the density modulation. The skyrmion is reduced to a hole for $\lambda=0$, where $\rho^-(\boldsymbol{x})$ is given by

$$\rho^-(\boldsymbol{x}) = (1-e^{-r^2/2\ell_B^2})\rho_0 \tag{6.21}$$

and vanishes at $r=0$. On the other hand, $\rho^-(\boldsymbol{x})>0$ for all $\lambda \neq 0$ with

$$\rho^-(0) = \frac{\lambda^2}{\lambda^2+2\ell_B^2}\rho_0. \tag{6.22}$$

We can expand $\rho^-(\boldsymbol{x})$ in a power series of ℓ_B^2 as follows. Let us set

$$\rho^-(\boldsymbol{x}) = \left(1-\frac{2\ell_B^2}{\lambda^2+2\ell_B^2}f(r^2)\right)\rho_0. \tag{6.23}$$

By comparing (6.23) with (6.20), $f(x)$ is found to be the Kummer function of the form

$$f(x) = M\left(2;\frac{\lambda^2}{2\ell_B^2}+2;-\frac{x}{\ell_B^2}\right) \tag{6.24}$$

with $x=r^2$, and satisfies the Kummer equation,

$$2\ell_B^2 x\frac{d^2f}{dx^2} + (\lambda^2+x+4\ell_B^2)\frac{df}{dx} + 2f = 0. \tag{6.25}$$

This can be solved by expanding f in a power series of ℓ_B^2, $f=\Sigma_{n=0} f_n \ell_B^{2n}$. In particular, the lowest order term is given by setting $\ell_B^2=0$, whose result is

$$f_0(x) = \frac{\lambda^4}{(x+\lambda^2)^2}. \tag{6.26}$$

Hence we have

$$\rho^-(\boldsymbol{x}) = \left(1-\frac{2\lambda^2\ell_B^2}{(r^2+\lambda^2)^2}+O(\ell_B^4)\right)\rho_0, \tag{6.27}$$

or

$$\frac{\delta\rho^{\pm}(\boldsymbol{x})}{\rho_0} = \pm\frac{2\lambda^2\ell_B^2}{(r^2+\lambda^2)^2}+O(\ell_B^4), \tag{6.28a}$$

$$\frac{S_z^{\pm}(\boldsymbol{x})}{\rho_0} = \frac{1}{2}-\frac{\lambda^2}{r^2+\lambda^2}+O(\ell_B^2). \tag{6.28b}$$

The densities $\rho^{\pm}(\boldsymbol{x})$ as well as $S_z^{\pm}(\boldsymbol{x})$ approach the ground-state values only polynomially.

Finally we examine what happens when the Zeeman interaction is taken into account. The number of spins flipped around a skyrmion is given by

$$N_{\text{spin}} = \int d^2x\left\{S_z^{\text{cl}}(\boldsymbol{x})-\frac{1}{2}\rho_0\right\}, \tag{6.29}$$

which we call the skyrmion spin. Substituting Eq. (6.18) into this, unless $\lambda=0$, we find $N_{\text{spin}}^{\text{sky}}$ to diverge logarithmically due to the asymptotic behavior,

$$\lim_{r\to\infty} S_z^{\text{cl}}(\boldsymbol{x}) = \frac{\rho_0}{2}\left(1-2\frac{\lambda^2}{r^2}\right). \tag{6.30}$$

The Zeeman energy $H_Z^{\text{sky}}=-\Delta_Z N_{\text{spin}}$ is divergent, except for the hole, from the infrared contribution however small the Zeeman effect is. The factorizable skyrmion (6.18) is no longer valid. There exists a skyrmion state which has a finite Zeeman energy; see an example of (7.5) we use for the Coulomb interaction. Nevertheless, we can show that the hole state has the lowest energy. The reason reads as follows. The factorizable skyrmion is an eigenstate of the hard-core Hamiltonian, $H_{\text{hc}}|\mathfrak{S}_{\text{sky}}^{\pm}\rangle = E_{\text{hc}}|\mathfrak{S}_{\text{sky}}^{\pm}\rangle$ with $E_{\text{hc}}=|\delta N^{\text{cl}}|/4\pi$, as in Eq. (6.12). Accordingly any spin texture $|\mathfrak{S}\rangle$ possessing the same electron number δN^{cl} has a higher energy, $\langle\mathfrak{S}|H_{\text{hc}}|\mathfrak{S}\rangle \geqslant E_{\text{hc}}$. Furthermore its Zeeman energy is larger than that of the hole, $\langle\mathfrak{S}|H_Z|\mathfrak{S}\rangle \geqslant \frac{1}{2}\Delta_Z$. Hence,

$$\langle\mathfrak{S}|(H_{\text{hc}}+H_Z)|\mathfrak{S}\rangle \geqslant E_{\text{hc}}+\tfrac{1}{2}\Delta_Z, \tag{6.31}$$

where the equality holds for the hole state. Consequently there are no skyrmions in the presence of the Zeeman interaction in the system with the hard-core interaction.

VII. COULOMB INTERACTION

We next investigate the realistic system (3.1a) governed by the Coulomb Hamiltonian H_C with the potential

$$V(\boldsymbol{x}-\boldsymbol{y}) = \frac{e^2}{4\pi\varepsilon|\boldsymbol{x}-\boldsymbol{y}|}, \tag{7.1}$$

for which the exchange energy parameter reads

$$\epsilon_X = \frac{\sqrt{2\pi}}{4}\epsilon_C \tag{7.2}$$

with $\epsilon_C = e^2/4\pi\varepsilon\ell_B$ the Coulomb energy unit. It is hard to construct the skyrmion state explicitly as an eigenstate. We are satisfied to estimate the excitation energy of a skyrmion by minimizing the expectation value of the Hamiltonian.

Before so doing, it is instructive to estimate the Coulomb energy by using the factorizable skyrmion (6.18) obtained in the hard-core model. The result is very different from the one in the hard-core model. The Coulomb energy is a monotonously decreasing function of the scale parameter λ, and

$$\lim_{\lambda\to\infty}\langle H_C\rangle = \pm 8\pi J_s \tag{7.3}$$

with

$$J_s = \frac{1}{16\sqrt{2\pi}}\epsilon_C. \tag{7.4}$$

Consequently, an infinitely large skyrmion is necessarily excited. However, when the Zeeman interaction is introduced, the Zeeman energy of the factorizable skyrmion diverges due to the asymptotic behavior (6.30). Namely the factorizable skyrmion cannot be a quasiparticle in the realistic Coulomb system.

It is necessary to consider a skyrmion not factorizable as in (6.18). Making a slight generalization of the parametrization (6.11) we search for a skyrmion possessing a finite energy even in the presence of the Zeeman effect. We propose an ansatz,

$$u_-^2(n) = v_+^2(n) = \frac{\omega^2 t^{2n+2}}{n+1+\omega^2}. \tag{7.5}$$

The parameter t presents a smooth interpolation between the hole ($t=0$) and the factorizable skyrmion ($t=1$). This anzats automatically satisfies the condition (4.20) for $n=-1$.

Substituting (7.5) into (4.25) we obtain

$$\frac{\delta\rho^{\pm}(\boldsymbol{x})}{\rho_0} = \pm e^{-(1/2)|z|^2} M\left(\omega^2;\omega^2+1;\frac{t|z|^2}{2}\right) \mp \frac{t^2\omega^2}{\omega^2+1} e^{-(1/2)|z|^2} M\left(\omega^2+1;\omega^2+2;\frac{t|z|^2}{2}\right),$$

$$\frac{S_z^{\pm}(\boldsymbol{x})}{\rho_0} = \frac{1}{2} - \frac{1}{2} e^{-(1/2)|z|^2} M\left(\omega^2;\omega^2+1;\frac{t|z|^2}{2}\right) - \frac{1}{2}\frac{t^2\omega^2}{\omega^2+1} e^{-(1/2)|z|^2} M\left(\omega^2+1;\omega^2+2;\frac{t|z|^2}{2}\right),$$

$$\frac{S_x^{\pm}(\boldsymbol{x})}{\rho_0} = \frac{t\omega x/\ell_B}{\sqrt{2}} e^{-(1/2)|z|^2} \sum_n \frac{1}{n!}\frac{\vartheta_n(w,t)}{n+1+\omega^2}\left(\frac{t|z|^2}{2}\right)^n,$$

$$\frac{S_y^{\pm}(\boldsymbol{x})}{\rho_0} = \frac{\pm t\omega y/\ell_B}{\sqrt{2}} e^{-(1/2)|z|^2} \sum_n \frac{1}{n!}\frac{\vartheta_n(w,t)}{n+1+\omega^2}\left(\frac{t|z|^2}{2}\right)^n, \tag{7.6}$$

where

$$\vartheta_n(w,t) = \sqrt{1+\frac{1-t^{2n+2}}{n+1}\omega^2}. \tag{7.7}$$

It is easy to see that various densities approach the ground-state values exponentially fast for all $t\neq 1$. Using Eq. (6.17), we can rewrite these as

$$\frac{\delta\rho^{\pm}(\boldsymbol{x})}{\rho_0} = \pm e^{-(1/2)(1-t^2)|z|^2} M(1;\omega^2+1;-t^2|z|^2/2) \mp \frac{t^2\omega^2}{\omega^2+1} e^{-(1/2)(1-t^2)|z|^2} M(1;\omega^2+2;-t^2|z|^2/2),$$

$$\frac{S_z^{\pm}(\boldsymbol{x})}{\rho_0} = \frac{1}{2} - \frac{1}{2} e^{-(1/2)(1-t^2)|z|^2} M(1;\omega^2+1;-t^2|z|^2/2) - \frac{1}{2}\frac{t^2\omega^2}{\omega^2+1} e^{-(1/2)(1-t^2)|z|^2} M(1;\omega^2+2;-t^2|z|^2/2). \tag{7.8}$$

We have illustrated the density $\rho^-(r)$ for typical values of the parameters ω and t in Fig. 1.

It is instructive to expand them in power series of ℓ_B^2. Setting

$$(1-t^2)|z|^2 = (1-t^2)r^2/\ell_B^2 = r^2/(\beta^2+\ell_B^2), \tag{7.9}$$

and writing down similar equations to (6.25), we obtain the lowest order term as

$$\frac{\delta\rho^{\pm}(\boldsymbol{x})}{\rho_0} = \pm\left[\frac{2\lambda^2\ell_B^2}{(r^2+\lambda^2)^2} + \frac{2\lambda^2\ell_B^2}{(r^2+\lambda^2)\beta^2} + O(\ell_B^4)\right] e^{-r^2/\beta^2}, \tag{7.10a}$$

$$\frac{S_z^{\pm}(\boldsymbol{x})}{\rho_0} = \frac{1}{2} - \left[\frac{\lambda^2}{r^2+\lambda^2} + O(\ell_B^2)\right] e^{-r^2/\beta^2}. \tag{7.10b}$$

The Zeeman energy remains finite due to a rapid decrease to the ground-state value.

In calculating the expectation value of the Coulomb Hamiltonian we appeal to the decomposition formula,

$$\langle H_C\rangle = H_D^{\mathrm{cl}} + H_X^{\mathrm{cl}}, \tag{7.11}$$

dictating that the Coulomb energy consists of the direct energy H_D^{cl} and the exchange energy H_X^{cl}. We have already used it on a case-by-case basis in our previous papers.[12,19,20] In Appendix C we derive the formula for the class of states (4.4). Here the direct and exchange energies read

$$H_D^{\mathrm{cl}} = \pi\int d^2k V(\boldsymbol{k}) e^{-(1/2)\ell_B^2 k^2} \delta\hat{\rho}^{\mathrm{cl}}(-\boldsymbol{k})\delta\hat{\rho}^{\mathrm{cl}}(\boldsymbol{k}), \tag{7.12a}$$

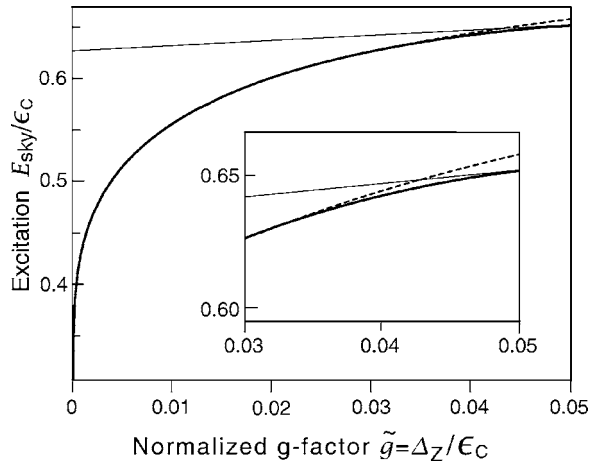

FIG. 2. The skyrmion excitation energy is plotted as a function of the normalized Zeeman gap $\tilde{g}=\Delta_Z/\epsilon_C$. The heavy solid curve is obtained by the numerical analysis based on the density formulas (7.6). The thin line is the hole excitation energy. It is seen that a skyrmion is excited for $\tilde{g}<0.045$ but a hole is practically excited for $\tilde{g}>0.045$. We have also given the excitation energy calculated analytically based on the simplified formula (9.14), which is represented by the dotted curve. The two curves are practically identical for $\tilde{g}<0.038$. The inset is an enlargement of the figure near $\tilde{g}=0.04$, where three curves meet.

$$H_X^{\mathrm{cl}} = \pi \int d^2k\, \delta V_X(\boldsymbol{k}) \left[\hat{S}_a^{\mathrm{cl}}(-\boldsymbol{k}) \hat{S}_a^{\mathrm{cl}}(\boldsymbol{k}) + \tfrac{1}{4} \hat{\rho}^{\mathrm{cl}}(-\boldsymbol{k}) \hat{\rho}^{\mathrm{cl}}(\boldsymbol{k}) \right], \tag{7.12b}$$

in terms of the bare densities, where $\delta V_X(\boldsymbol{k}) = V_X(0) - V_X(\boldsymbol{k})$ with

$$V_X(\boldsymbol{k}) \equiv \frac{\ell_B^2}{\pi} \int d^2k e^{-i\ell_B^2 \boldsymbol{k}\wedge\boldsymbol{k}'} e^{-(1/2)\ell_B^2 k'^2} V(\boldsymbol{k}') = 4\varepsilon_X \ell_B^2 e^{-(1/4)\ell_B^2 k^2} I_0\left(\frac{k^2}{4}\right), \tag{7.13}$$

and $I_0(z)$ is the modified Bessel function.

We are able to determine the parameters ω and t so as to minimize the sum of the Coulomb and Zeeman energies, $\langle H\rangle_{\mathrm{sky}} = \langle H_C\rangle_{\mathrm{sky}} + \langle H_Z\rangle_{\mathrm{sky}}$, as a function of the Zeeman gap Δ_Z. Calculating numerically $\langle H\rangle_{\mathrm{sky}}$ as a function of ω and t for a given value of Δ_Z, we determine the values of ω and t which minimizes $\langle H\rangle_{\mathrm{sky}}$. In this way we obtain the skyrmion excitation energy $\langle H\rangle_{\mathrm{sky}}$ as a function of Δ_Z. We have plotted the excitation energy $\langle H\rangle_{\mathrm{sky}}$ as a function of the normalized Zeeman gap $\tilde{g}=\Delta_Z/\epsilon_C$ in Fig. 2, where ϵ_C is the Coulomb unit.

VIII. EXPERIMENTAL DATA

A skyrmion excitation is characterized by a coherent excitation of spins. Hence an evidence of the skyrmion excitation is given if $N_{\mathrm{spin}} > 1/2$, where $N_{\mathrm{spin}} = 1/2$ for a hole or an electron excitation. It is a remarkable fact that skyrmions have already been observed experimentally in QH systems.[4–6]

Let us show the number of flipped spins is given by

$$N_{\mathrm{spin}} = \frac{d\langle H\rangle_{\mathrm{sky}}}{d\Delta_Z}. \tag{8.1}$$

The excitation energy $E_{\mathrm{sky}} = \langle H\rangle_{\mathrm{sky}}$ is the sum of the Coulomb energy $E_C = \langle H_C\rangle_{\mathrm{sky}}$ and Zeeman energy $\Delta_Z N_{\mathrm{spin}}$, which depend on a set of parameters (t, ω) denoted collectively by t for simplicity. The quantity to minimize with respect to t is

$$E_{\mathrm{sky}}(t) = E_C(t) + \Delta_Z N_{\mathrm{spin}}(t). \tag{8.2}$$

At the minimum we obtain

$$\frac{\partial E_C}{\partial t} + \Delta_Z \frac{\partial N_{\mathrm{spin}}}{\partial t} = 0, \tag{8.3}$$

from which we solve out $t = t_0(\Delta_Z)$ and substituting it back into Eq. (8.2),

$$E_{\mathrm{sky}}(\Delta_Z) = E_C[t_0(\Delta_Z)] + \Delta_Z N_{\mathrm{spin}}[t_0(\Delta_Z)], \tag{8.4a}$$

$$N_{\mathrm{spin}}(\Delta_Z) = N[t_0(\Delta_Z)]. \tag{8.4b}$$

Using Eq. (8.3) it is easy to verify that

$$N_{\mathrm{spin}}(\Delta_Z) = \frac{dE_{\mathrm{sky}}(\Delta_Z)}{d\Delta_Z}, \tag{8.5}$$

which is Eq. (8.1).

To compare our theoretical result with experimental data, it is necessary to take into account two points so far neglected.

First, what we observe experimentally is the thermal activation energy of a skyrmion-antiskyrmion pair. But this activation takes place in the presence of charged impurities. The existence of charged impurities reduces the activation energy considerably. We include an offset parameter Γ_{offset} to treat this effect phenomenologically.[21]

Second, we have so far assumed an ideal two-dimensional space for electrons. This is not the case. Electrons are confined within a quantum well of a finite width. This modifies the Coulomb energy considerably. The Coulomb energy becomes smaller than what we have assumed. It is quite difficult to make a rigorous analysis of the Coulomb energy in an actual quantum well. We simulate the effect by including the reduction factor γ.

We consider the excitation energy of a skyrmion-antiskyrmion pair since it is an observable quantity. It is simply twice of the skyrmion excitation energy. Taking into account these two points, instead of Eq. (8.2) we set the activation energy as

$$\Delta_{\mathrm{gap}}(t) = 2\gamma E_C(t) + 2\Delta_Z N(t) - \Gamma_{\mathrm{offset}}, \tag{8.6}$$

where $0 < \gamma < 1$. Repeating the same steps as for Eq. (8.4) we come to

$$E_\gamma(\Delta_Z) = \gamma E_{\mathrm{sky}}\left(\frac{\Delta_Z}{\gamma}\right) - \Gamma_{\mathrm{offset}}, \tag{8.7a}$$

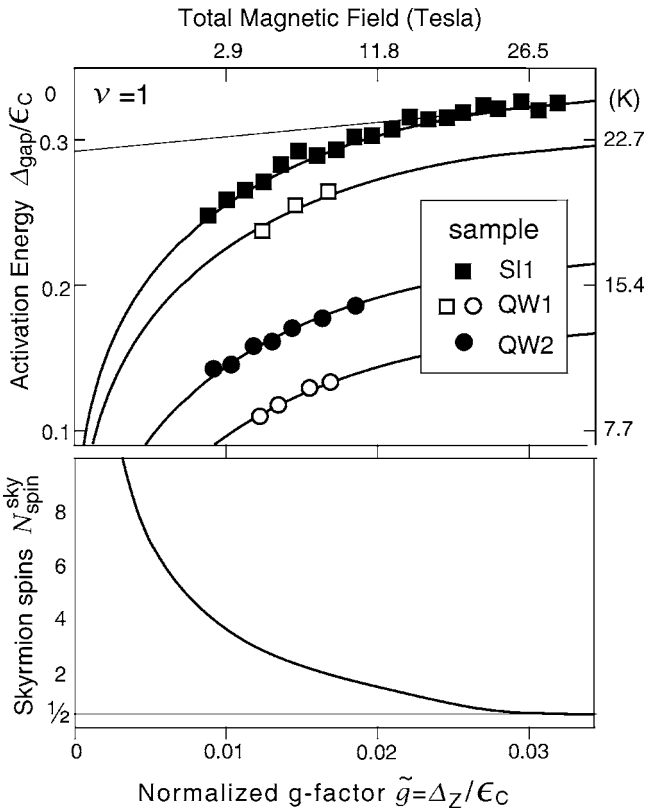

FIG. 3. A theoretical result on the activation energy $\Delta_{\rm gap}$ of a skyrmion-antiskyrmion pair is compared with experimental data at $\nu=1$. The data are taken from Schmeller *et al.* (Ref. 6). The theoretical curve is based on the formula (8.6) with $\gamma=0.56$. We have taken $\Gamma_{\rm offset}=0.41$ for sample SI1. The offset $\Gamma_{\rm offset}$ increases as the mobility decreases. The skyrmion spin is one half of the slope of the activation-energy curve, $N_{\rm spin}=\frac{1}{2}\partial\Delta_{\rm gap}/\partial\tilde{g}$, where $\Delta_{\rm gap}$ is taken in units of ϵ_C. The number $N_{\rm spin}$ depends sensitively on the normalized Zeeman gap $\tilde{g}$ for small $\tilde{g}$. The thin line is for the hole excitation energy. It is seen that a hole is practically excited for $\tilde{g}>0.03$.

$$N_\gamma(\Delta_Z)=N_{\rm spin}\left(\frac{\Delta_Z}{\gamma}\right)=\frac{\partial E_\gamma(\Delta_Z)}{\partial\Delta_Z}, \tag{8.7b}$$

with the use of $E_{\rm sky}(\Delta_Z)$ and $N_{\rm spin}(\Delta_Z)$ derived in Eq. (8.4).

Experimental data[6] were obtained in three samples (Fig. 3): a single heterointerface (SI1) and two GaSa single quantum wells (QW1 and QW2) with widths of 20 nm and 14 nm. The sample SI1 has a much wider thickness of the two-dimensional sheet. In Fig. 3, from top to bottom the sample mobilities are 3.4, 0.52, 0.18, and 0.16×10^6 cm^{-2}, respectively. We have compared our theoretical result (8.6) successfully with the experimental data by making appropriate choices of the reduction factor γ and the offset parameter $\Gamma_{\rm offset}$ in Fig. 3. We have used $\gamma=0.6$ for all samples. It would imply the thickness independence of the excitation energy, suggesting that a skyrmion has a certain fixed size in the third direction. However this problem is yet to be explored. On the other hand we have used different values of $\Gamma_{\rm offset}$ for each. We have set $\Gamma_{\rm offset}=0.4$ for sample SI1, and have found that $\Gamma_{\rm offset}$ increases as the mobility decreases. This is reasonable since both the mobility and the offset $\Gamma_{\rm offset}$ represent impurity effects.

IX. SEMICLASSICAL APPROXIMATION

The formulas we have used to estimate the skyrmion excitation energy are quite complicated. The modulations of the electron density and the spin density, being expressed only in infinite power series as in Eq. (7.6), are very difficult to handle with. Skyrmion excitations have been observed also in bilayer QH systems,[22–24] where it is practically impossible to repeat the calculations carried out here because too many dynamical variables are involved. We have already applied a semiclassical approximation to those systems.[19] In this section we compare the semiclassical result with the microscopic result.

In the semiclassical approximation we can make the spin-charge separation of the electron field based on the composite-boson picture,[17]

$$\psi_\mu(\boldsymbol{x})=\psi(\boldsymbol{x})n_\mu(\boldsymbol{x}), \tag{9.1}$$

where the U(1) filed $\psi(\boldsymbol{x})$ carries the electric charge while the CP^1 field $n_\mu(\boldsymbol{x})$ carries the spin. It leads to the factorization of the spin field $S_a(\boldsymbol{x})$ into the number density $\rho(\boldsymbol{x})$ and the normalized spin field $\mathcal{S}_a(\boldsymbol{x})$,

$$S_a(\boldsymbol{x})=\rho(\boldsymbol{x})\mathcal{S}_a(\boldsymbol{x}) \tag{9.2}$$

with

$$\mathcal{S}_a(\boldsymbol{x})=\tfrac{1}{2}n_\mu^\dagger(\boldsymbol{x})(\tau_a)_{\mu\nu}n_\nu(\boldsymbol{x}). \tag{9.3}$$

The CP^1 field of the skyrmion is given by[17]

$$n_\nu(\boldsymbol{x})=\frac{1}{\sqrt{|z|^2+\lambda^2}}\begin{pmatrix}z\\ \lambda\end{pmatrix} \tag{9.4}$$

together with the normalized spin field (6.19), and the wave function is factorizable as in (5.12). We may also derive the soliton equation[17]

$$\frac{1}{4\pi}\nabla^2\ln\rho^{\rm cl}(\boldsymbol{x})-\rho^{\rm cl}(\boldsymbol{x})+\rho_0=\frac{2\lambda^2\ell_B^2}{(r^2+\lambda^2)^2}\rho_0, \tag{9.5}$$

which determines the density modulation around the charge excitation.

We evaluate the excitation energy in the lowest order of ℓ_B^2 by assuming a large-scale skyrmion. It corresponds to the commutative limit in the microscopic scheme.

The Coulomb energy is given by Eq. (7.12). To calculate the direct energy (7.12a) we take the leading term in the solution of the soliton equation (9.5),

$$\delta\rho^{\rm cl}(\boldsymbol{x})\simeq\frac{2\lambda^2\ell_B^2}{(r^2+\lambda^2)^2}\rho_0, \tag{9.6}$$

which agrees with the leading term in the density modulation (6.28a) in the microscopic theory. The direct Coulomb energy (7.12a) reads

$$H_D^{\rm cl} = \pi \int d^2k V(\boldsymbol{k}) e^{-\ell_B^2 k^2/2} \delta\rho^{\rm cl}(-\boldsymbol{k}) \delta\rho^{\rm cl}(\boldsymbol{k}) = \frac{3\pi^2}{128\kappa}\epsilon_C, \tag{9.7}$$

where $\kappa = \lambda/2\ell_B$.

In calculating the exchange energy (7.12b), since the exchange potential $V_X(\boldsymbol{x})$ is short-ranged, we make the derivative expansion and take the lowest order term. It is to make an approximation in Eq. (7.12b),

$$\delta V_X(\boldsymbol{k}) \simeq -\frac{2J_s}{\pi\rho_0^2}\boldsymbol{k}^2 \tag{9.8}$$

with Eq. (7.4). Furthermore we approximate $\rho^{\rm cl}(\boldsymbol{x}) \simeq \rho_0$ in the exchange energy in the lowest order of ℓ_B^2. Then the exchange energy turns out to be the nonlinear sigma model,

$$H_X^{\rm cl} = \frac{J_s}{2}\sum_a \int d^2x [\partial_k \mathcal{S}_a^{\rm cl}(\boldsymbol{x})]^2. \tag{9.9}$$

This is bounded from below,

$$H_X^{\rm cl} \geqslant \pm 8\pi J_s Q_{\rm sky}. \tag{9.10}$$

The lower bound, $H_X^{\rm cl} = 8\pi J_s$, is saturated by the skyrmion configuration (6.19). We note that the Hamiltonian $H_X^{\rm cl}$ describes a spin wave. The coherence length ξ is given by

$$\xi^2 = \frac{2J_s}{\rho_0\Delta_Z} = \frac{\sqrt{2\pi}}{8\tilde{g}}\ell_B^2 \tag{9.11}$$

in the presence of the Zeeman effect.

We cut off the divergence of the Zeeman energy. First, the skyrmion excitation occurs within the coherent domain with the coherence length ξ. Second, the skyrmion size is proportional to the scale parameter κ. The Zeeman energy is approximated as

$$\langle H_Z\rangle_{\rm sky} = \frac{\lambda^2\tilde{g}}{2\ell_B^2}\ln\left(\frac{\alpha^2\xi^2}{4\ell_B^2}+1\right), \tag{9.12}$$

where we have cut off the upper limit of the integration at $r = \alpha\kappa\xi$ with a phenomenological parameter α.

Substituting these into Eq. (9.7), we obtain

$$\langle H\rangle_{\rm sky} = \left[\frac{1}{4}\sqrt{\frac{\pi}{2}}\beta + \frac{3\pi^2}{128\kappa} + 2\kappa^2\tilde{g}\ln\left(\alpha^2\frac{\sqrt{2\pi}}{32\tilde{g}}+1\right)\right]\epsilon_C, \tag{9.13}$$

where we have introduced another phenomenological parameter β to reduce the exchange energy, since the true exchange energy is smaller than $8\pi J_s$. Minimizing this with respect to κ analytically we obtain

$$\langle H\rangle_{\rm sky} \simeq \left\{\frac{1}{4}\sqrt{\frac{\pi}{2}}\beta + \frac{9\pi^2}{256\kappa}\right\}\epsilon_C, \tag{9.14}$$

with

$$\kappa \simeq \frac{1}{2}\left(\frac{3\pi^2}{64}\right)^{1/3}\left\{\tilde{g}\ln\left(\alpha^2\frac{\sqrt{2\pi}}{32\tilde{g}}+1\right)\right\}^{-1/3}. \tag{9.15}$$

We have plotted the result in Fig. 2. It reproduces excellently the previous numerical result for $\tilde{g} < 0.038$ with the choice of $\alpha = 1.4$ and $\beta = 0.9$. We may practically use it even in the region where a hole is excited. We have thus confirmed the validity of the semiclassical approximation.

X. CONCLUSION

In this paper we have presented a microscopic theory of skyrmions in QH ferromagnets. We have shown that the skyrmion is a $W_\infty(2)$-rotated state of a hole-excited state. Because of an intrinsic entanglement between the electron density and the spin density implied by the $W_\infty(2)$ algebra, a $W_\infty(2)$ rotation modulates not only the spin configuration but also the electron density around a hole, thus decreasing the Coulomb energy. Similarly, the antiskyrmion is a $W_\infty(2)$-rotated state of an electron-excited state. There is a simple type of skyrmion state characterized by the fact that its wave function is factorizable in the electron coordinates. We call it the factorizable skyrmion. It corresponds to the nonlinear-sigma-model skyrmion previously derived in the semiclassical approximation. We have analyzed the skyrmion state in the realistic Coulomb system with the Zeeman interaction. By minimizing the excitation energy we have estimated the activation energy of a skyrmion-antiskyrmion pair. The result is found to explain the experimental data[6] remarkably well.

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APPENDIX A: SPONTANEOUS SYMMETRY BREAKING

We analyze the problem of spontaneous symmetry breaking due to a repulsive interaction between electrons in the $\nu = 1$ QH system. We assume that every Landau site is occupied by one electron. The problem is to show that the spin polarized state is the lowest-energy state though the Hamiltonian (3.1a) involves no spin variables.

We start with a proof, which makes the physical mechanism of spontaneous symmetry breaking clear. According to the decomposition formula (7.11) the energy of the four-fermion interaction Hamiltonian (3.1a) is a sum of the direct and exchange energies (7.12), or $\langle \mathfrak{S}|H_V|\mathfrak{S}\rangle = H_D^{\rm cl} + H_X^{\rm cl}$ with

$$H_D^{\rm cl} = \pi \int d^2k V(\boldsymbol{k}) e^{-(1/2)\ell_B^2 k^2} |\delta\hat{\rho}^{\rm cl}(\boldsymbol{k})|^2, \tag{A1a}$$

$$H_X^{\rm cl} = \pi \int d^2k\, \delta V_X(\boldsymbol{k}) \left[|\hat{S}_a^{\rm cl}(\boldsymbol{k})|^2 + \tfrac{1}{4} |\hat{\rho}^{\rm cl}(\boldsymbol{k})|^2 \right], \quad \text{(A1b)}$$

where $V(\boldsymbol{k}) > 0$ and $\delta V_X(\boldsymbol{k}) > 0$ for a repulsive interaction. Here, $\hat{\rho}^{\rm cl}(\boldsymbol{k}) = \langle \mathfrak{S} | \hat{\rho}(\boldsymbol{k}) | \mathfrak{S} \rangle$ and $\hat{S}_a^{\rm cl}(\boldsymbol{k}) = \langle \mathfrak{S} | \hat{S}_a(\boldsymbol{k}) | \mathfrak{S} \rangle$. The key observation is that, though the Hamiltonian (3.1a) involves no spin variables, the energy of a state does. It is important that both the energies are positive semidefinite. The direct energy $H_D^{\rm cl}$ is insensitive to spin orientations, and it vanishes for the homogeneous electron distribution since $\delta\hat{\rho}^{\rm cl}(\boldsymbol{k}) = 0$. The exchange energy $H_X^{\rm cl}$ depends on spin orientations. The spin texture is homogeneous when the spin is completely polarized, where $\hat{S}_a^{\rm cl}(\boldsymbol{k}) \propto \delta(\boldsymbol{k})$. Furthermore, $\hat{\rho}^{\rm cl}(\boldsymbol{k}) \propto \delta(\boldsymbol{k})$ due to the homogeneous electron distribution. For such a spin orientation the exchange energy also vanishes since $\delta V_X(\boldsymbol{k}) = V_X(0) - V_X(k) = 0$ in Eq. (A1a). On the other hand, $H_X^{\rm cl} > 0$ if the spin is not polarized completely since $\hat{S}_a^{\rm cl}(\boldsymbol{k})$ contains nonzero momentum components. Consequently the spin-polarized state has the lowest energy, which is zero. Hence the exchange interaction is the driving force of spontaneous symmetry breaking.

We may present another proof, which is mathematically more formal. First of all, the state

$$|g\rangle = \prod_n c_\uparrow^\dagger(n) |0\rangle \quad \text{(A2)}$$

is an eigenstate of the Hamiltonian (3.1a) with the zero energy, $H_V|g\rangle = 0$. It is one of the ground states. We shall prove the following: (a) any globally $W_\infty(2)$-rotated state is degenerate with $|g\rangle$; (b) any locally $W_\infty(2)$-rotated state has a positive energy.

A spin rotated state of $|g\rangle$ is $|\Omega\rangle = e^{-i\Omega}|g\rangle$ with

$$\Omega = \sum_a \sum_{mn} \xi_a(m,n) S_a(n,m), \quad \text{(A3)}$$

where $\xi_a(m,n)$ is a Hermitian matrix, $[\xi_a(m,n)]^* = \xi_a(n,m)$. We calculate how the rotation $e^{i\Omega}$ affects the polarized spin for an infinitesimal parameter ξ_a. Using

$$2\langle g | S_a(m,n) | g \rangle = \delta_{az}\delta_{mn}, \quad \text{(A4)}$$

we get

$$2\langle \Omega | S_a(m,n) | \Omega \rangle = \delta_{az}\delta_{mn} + \sum_{bc} \epsilon_{abc}\delta_{cz}\xi_b(m,n). \quad \text{(A5)}$$

Only relevant transformations are generated by $\xi_x(m,n)$ and $\xi_y(m,n)$, since $\xi_z(m,n)$ does not affect the spin polarization. Besides we are interested in transformations rotating spins without moving electrons from site to site. Then the parameter is reduced to $\xi_a(m,n) = \xi_a(m)\,\delta_{mn}$.

The energy induced by an infinitesimal transformation is

$$2\langle \Omega | H_V | \Omega \rangle = \sum_{a=x,y} \sum_{mn} [\epsilon_X \delta_{mn} - V_{mnnm}] \xi_a(m) \xi_a(n), \quad \text{(A6)}$$

where we have set $\langle g | H_V | g \rangle = 0$. The question is the positive definiteness of the quantity

$$E(\xi) = [\varepsilon_X \delta_{kn} - V_{kn}] \xi_k \xi_n, \quad \text{(A7)}$$

where $V_{kn} \equiv V_{knnk}$ is the symmetric matrix.

Without loss of generality we assume the site index to run up to a finite value N_Φ, which will eventually be taken to infinity. The analysis of Eq. (A7) is reduced to the analysis of the eigenvalues of V_{kn}. Introducing the complex valued function

$$\varphi(\boldsymbol{x},\boldsymbol{y};\xi) = \sum_k \xi_k \varphi_k^*(\boldsymbol{x}) \varphi_k(\boldsymbol{y}), \quad \text{(A8)}$$

we write

$$\sum_{kn} V_{kn}\xi_k\xi_n = \frac{1}{2} \int d^2x d^2y V(\boldsymbol{x}-\boldsymbol{y}) |\varphi(\boldsymbol{x},\boldsymbol{y};\xi)|^2 > 0. \quad \text{(A9)}$$

Hence, V_{kn} is positive definite, and all of its eigenvalues are positive.

Now we refer to Gerschgorin's theorem,[25] which originally deals with complex matrices. Formulating for a real matrix with positive elements $a_{kn} > 0$ the theorem reads as follows. Let Λ_k be the numbers defined as

$$\Lambda_k = \sum_{n \neq k} a_{kn} > 0. \quad \text{(A10)}$$

Then all of the eigenvalues of a_{kn} lie in the union of the segments $[a_{kk} - \Lambda_k, a_{kk} + \Lambda_k]$.

In our case we have

$$\Lambda_k = \sum_{n \neq k} V_{kn} = \sum_n V_{kn} - V_{kk} = \varepsilon_X - V_{kk} \quad \text{(A11)}$$

so that the segments appear as $[2V_{kk} - \varepsilon_X, \varepsilon_X]$. On the other hand, we have shown that the eigenvalues of V_{kn} are positive. Consequently, irrespective of the value of $2V_{kk} - \varepsilon_X$, we conclude that they lie within the segment $[0, \varepsilon_X]$, meaning that the eigenvalues E_k of the quadratic form (A7) satisfy the condition

$$0 \leqslant E_k \leqslant \varepsilon_X. \quad \text{(A12)}$$

The eigenstate of the lowest eigenvalue $E = 0$ is given by $\xi_k = \xi$,

$$\sum_n [\varepsilon_X \delta_{kn} - V_{kn}] \xi_n = \xi \sum_n [\varepsilon_X \delta_{kn} - V_{kn}] = 0, \quad \text{(A13)}$$

which corresponds to a global rotation of electron spins, while for an arbitrary local rotation we have $E(\xi) > 0$. It implies the degeneracy of the ground states only under a global rotation, leading to a spontaneous symmetry breaking of the rotational symmetry.

APPENDIX B: ANTISKYRMIONS

An antiskyrmion is a $W_\infty(2)$-rotated state of the electron-excited state. We consider the antiskyrmion state (4.15),

$$|\mathfrak{S}_{\rm sky}^+\rangle = c_{\downarrow,0}^\dagger [u_0^+ c_{\uparrow,0}^\dagger + v_0^+ c_{\downarrow,1}^\dagger][u_1^+ c_{\uparrow,1}^\dagger + v_1^+ c_{\downarrow,2}^\dagger] \cdots [u_{N-1}^+ c_{\uparrow,N-1}^\dagger + v_{N-1}^+ c_{\downarrow,N}^\dagger] c_{\uparrow,N}^\dagger |0\rangle \quad \text{(B1)}$$

with $(u_n^+)^2 + (v_n^+)^2 = 1$. We note that the state comprises $N+2$

electrons over $N+1$ sites. Here and hereafter, for notational simplicity, we set $c_{\mu n} \equiv c_\mu(n)$, $u_n^+ \equiv u^+(n)$, and so on.

In contrast with the skyrmion state (4.13), the antiskyrmion state $|\mathfrak{S}^+_{\rm sky}\rangle$ does not lead to a wave function with any reasonable structure like (5.10). The expression for general N is technically difficult to write down. As an example we present the one for $N=2$. Up to multiplicative factors it appears as

$$\mathfrak{S}^+_{\uparrow\uparrow\uparrow} = 0, \tag{B2a}$$

$$\mathfrak{S}^+_{\downarrow\uparrow\uparrow} = (z_2 - z_3)(z_2 - z_4)(z_3 - z_4), \tag{B2b}$$

$$\mathfrak{S}^+_{\downarrow\downarrow\uparrow} = (z_1 - z_2)(z_3 - z_4) \times [z_3 z_4 - (z_1 + z_2)(z_3 + z_4)], \tag{B2c}$$

$$\mathfrak{S}^+_{\downarrow\downarrow\downarrow} = (z_1 - z_2)(z_1 - z_3)(z_2 - z_3) z_4^2, \tag{B2d}$$

$$\mathfrak{S}^+_{\downarrow\downarrow\downarrow} = 0, \tag{B2e}$$

where no reasonable order is seen.

In this respect skyrmions and antiskyrmions might be regarded to possess rather different properties. Actually, this is not so since the above scheme of dealing with antiskyrmions is not quite satisfactory because of the following observation.

The skyrmion state (4.13) comprises N electrons, and the wave function (5.10) describes these electrons in terms of its N complex arguments. On the other hand, the antiskyrmion wave function comprises $N+2$ complex arguments, which at first sight seems to match the number of electrons in the corresponding Fock state. However, since two electrons out of those $N+2$ are fixed at $n=0$ and $n=N$, they cannot be transferred to the neighboring sites. Therefore, though constructed of $N+2$ electrons, the antiskyrmion configuration comprises only $N-1$ site-transferrable electrons, which is exactly the same as in the skyrmion state. In other word, the skyrmion and antiskyrmion Fock states comprise the equal number of the degrees of freedom, since the two fixed electrons carry no degrees of freedom at all. Nevertheless, the antiskyrmion wave function involves $N+2$ complex arguments, which does not match the number of site-transferable electrons. There is nothing wrong there, and the answer to this mismatch is that the components of antiskyrmion wave function are not independent, but subject to certain functional relations. The most transparent manifestation of this statement are the relations (B2a) and (B2e).

Though the scheme is not wrong by itself, it is inconvenient to work with. All these difficulties disappear if we develop a dual picture based on the electron-hole symmetry. So far the antiskyrmion Fock state is built up via creating electrons in the vacuum state. The dual picture deals with the same state via removing electrons from the completely filled system of $N+1$ sites. Namely, up to the overall sign, the state (B1) can be rewritten as

$$|\mathfrak{S}^+_{\rm sky}\rangle = [v_0^+ c_{\uparrow,0} - u_0^+ c_{\downarrow,1}][v_1^+ c_{\uparrow,1} - u_1^+ c_{\downarrow,2}] \cdots [v_{N-1}^+ c_{\uparrow,N-1} - u_{N-1}^+ c_{\downarrow,N}] \prod_{\mu n} c^\dagger_{\mu n} |0\rangle. \tag{B3}$$

We now introduce the hole annihilation operators $\tilde{c}_{\mu n}$ and the hole vacuum $|\tilde{0}\rangle$,

$$\tilde{c}_{\uparrow\downarrow,n} = c^\dagger_{\downarrow\uparrow,n} \quad |\tilde{0}\rangle = \prod_{\mu n} c^\dagger_{\mu n} |0\rangle$$

with $\tilde{c}_{\mu n}|\tilde{0}\rangle = 0$ and $\{\tilde{c}_{\mu m}, \tilde{c}^\dagger_{\nu n}\} = \delta_{\mu\nu}\delta_{mn}$.

Now the antiskyrmion Fock state can be presented as

$$|\mathfrak{S}^+_{\rm sky}\rangle = [v_0^+ \tilde{c}^\dagger_{\downarrow,0} - u_0^+ \tilde{c}^\dagger_{\uparrow,1}][v_1^+ \tilde{c}^\dagger_{\downarrow,1} - u_1^+ \tilde{c}^\dagger_{\uparrow,2}] \cdots [v_{N-1}^+ \tilde{c}^\dagger_{\downarrow,N-1} - u_{N-1}^+ \tilde{c}^\dagger_{\uparrow,N}]|\tilde{0}\rangle, \tag{B4}$$

where the analogy with the skyrmion state is manifest.

In the same way we introduce the hole field operator as $\tilde{\psi}_{\uparrow\downarrow}(\mathbf{r}) = \psi^\dagger_{\downarrow\uparrow}(\mathbf{r})$, or equivalently

$$\tilde{\psi}_\mu(\boldsymbol{x}) = \rho_0^{1/2} e^{-|\bar{z}|^2/4} \sum_{n=0} \alpha(n) \bar{z}^n \tilde{c}_\mu(n) \tag{B5}$$

where $\bar{z} = (x - iy)/\ell_B$.

Now the wave function of the antiskyrmion state comprising $N+2$ electrons contains N variables $\bar{z}_1, \ldots, \bar{z}_N$. So the unphysical degrees of freedom associated with two nontransferable electrons of Eq. (B1) do not appear at all, since those two electrons are accounted within the ground state $|\tilde{0}\rangle$.

Using the analogy with (5.10) it is straightforward to write down the N-body wave function of antiskyrmion configuration. We obtain

$$\mathfrak{S}^+_{\mu_1\mu_2\cdots\mu_N}[\boldsymbol{x}] = C_N e^{-\Sigma_{r=1}^{N-1}|z_r|^2/4} \times \begin{vmatrix} \bar{z}_1^0 \begin{pmatrix} \bar{z}_1 \\ \tilde{\kappa}_0 \end{pmatrix}_{\mu_1} & \bar{z}_1^1 \begin{pmatrix} \bar{z}_1 \\ \tilde{\kappa}_1 \end{pmatrix}_{\mu_1} & \cdots & \bar{z}_1^{N-1} \begin{pmatrix} \bar{z}_1 \\ \tilde{\kappa}_{N-1} \end{pmatrix}_{\mu_1} \\ \bar{z}_2^0 \begin{pmatrix} \bar{z}_2 \\ \tilde{\kappa}_0 \end{pmatrix}_{\mu_2} & \bar{z}_2^1 \begin{pmatrix} \bar{z}_2 \\ \tilde{\kappa}_1 \end{pmatrix}_{\mu_2} & \cdots & \bar{z}_2^{N-1} \begin{pmatrix} \bar{z}_2 \\ \tilde{\kappa}_{N-1} \end{pmatrix}_{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{z}_N^0 \begin{pmatrix} \bar{z}_N \\ \tilde{\kappa}_0 \end{pmatrix}_{\mu_N} & \bar{z}_N^1 \begin{pmatrix} \bar{z}_N \\ \tilde{\kappa}_1 \end{pmatrix}_{\mu_N} & \cdots & \bar{z}_N^{N-1} \begin{pmatrix} \bar{z}_N \\ \tilde{\kappa}_{N-1} \end{pmatrix}_{\mu_N} \end{vmatrix} \tag{B6}$$

as in Eq. (5.10).

APPENDIX C: DECOMPOSITION FORMULA

We prove the decomposition formula (7.11), according to which the four-fermion interaction energy consists of the direct energy and the exchange energy. We evaluate the energy of a state $|\mathfrak{S}\rangle$ in the class of states (4.4),

$$\langle H_V\rangle = \langle \mathfrak{S}|H_V|\mathfrak{S}\rangle, \tag{C1}$$

where the Hamiltonian is given by Eq. (3.1a). We assume that the electron field carries the SU(N_f) isospin index, $\mu = 1,2,\ldots,N_f$. For the sake of accuracy we first consider a system with a finite number of Landau sites ($m = 0,1,\ldots,N_\Phi - 1$) and take the limit $N_\Phi \to \infty$ in final expressions. It is convenient to combine the isospin and site indices into a multi-index $M \equiv (\mu, m)$, where the multi-index runs over the values $M = 1,2,\ldots,N_f N_\Phi$.

The $W_\infty(N_f)$ algebra is identical to the algebra $U(N_f N_\Phi)$ in the limit $N_\Phi \to \infty$, and the transformation rules for the fermion operators appear as

$$e^{-iW} c_M e^{iW} = (U)_{MM'} c_{M'},$$

$$e^{-iW} c_M^\dagger e^{iW} = c_{M'}^\dagger (U^\dagger)_{M'M}, \tag{C2}$$

where U is an $(N_f N_\Phi)\times(N_f N_\Phi)$ unitary matrix, $UU^\dagger = U^\dagger U = \mathbb{I}_{(N_f N_\Phi)\times(N_f N_\Phi)}$. Here and hereafter the repeated index implies the summation over it.

We first calculate the two-point averages by the state $|\mathfrak{S}\rangle = e^{iW}|\mathfrak{S}_0\rangle$ in Eq. (4.4). Using Eq. (C2) we get

$$\langle \mathfrak{S}|c_M^\dagger c_N|\mathfrak{S}\rangle = (U^\dagger)_{KM}(U)_{NL}\langle \mathfrak{S}_0|c_K^\dagger c_L|\mathfrak{S}_0\rangle. \tag{C3}$$

For the state $|\mathfrak{S}_0\rangle$ given by Eq. (4.5) we have

$$\langle \mathfrak{S}_0|c_K^\dagger c_L|\mathfrak{S}_0\rangle = \nu_K \delta_{KL}, \tag{C4}$$

which eventually leads to

$$\langle \mathfrak{S}|c_M^\dagger c_N|\mathfrak{S}\rangle = \nu_K (U)_{NK}(U^\dagger)_{KM}. \tag{C5}$$

Carrying out analogous manipulations in four-point averages we get

$$\langle \mathfrak{S}|c_M^\dagger c_S^\dagger c_T c_N|\mathfrak{S}\rangle = (U^\dagger)_{KM}(U^\dagger)_{IS}(U)_{TJ}(U)_{NL} \times \langle \mathfrak{S}_0|c_K^\dagger c_I^\dagger c_J c_L|\mathfrak{S}_0\rangle. \tag{C6}$$

We can use

$$\langle \mathfrak{S}_0|c_K^\dagger c_I^\dagger c_J c_L|\mathfrak{S}_0\rangle = \nu_J \nu_L (\delta_{IJ}\delta_{KL} - \delta_{IL}\delta_{KJ}) \tag{C7}$$

for the state $|\mathfrak{S}_0\rangle$. Substituting (C7) into (C6) and accounting (C5) we summarize as

$$\langle \mathfrak{S}|c_M^\dagger c_S^\dagger c_T c_N|\mathfrak{S}\rangle = \langle \mathfrak{S}|c_M^\dagger c_N|\mathfrak{S}\rangle\langle \mathfrak{S}|c_S^\dagger c_T|\mathfrak{S}\rangle - \langle \mathfrak{S}|c_M^\dagger c_T|\mathfrak{S}\rangle \times \langle \mathfrak{S}|c_S^\dagger c_N|\mathfrak{S}\rangle, \tag{C8}$$

where the direct and exchange terms are easily recognized. Here, we have

$$\langle \mathfrak{S}|c_\mu^\dagger(m) c_\nu(n)|\mathfrak{S}\rangle = \frac{\delta_{\nu\mu}}{N_f}\rho^{\rm cl}(n,m) + (\lambda_A)_{\nu\mu} S_A^{\rm cl}(n,m), \tag{C9}$$

in terms of decoupled spin and site indices, where $\rho^{\rm cl}(m,n)$ and $S_A^{\rm cl}(m,n)$ are defined by (4.21) with λ_A the Gell-Mann matrix, $A = 1,\ldots,N_f^2 - 1$.

In this way we achieve at the decomposition formula,

$$\langle H_V\rangle = H_D^{\rm cl} + H_X^{\rm cl}, \tag{C10}$$

where the direct and the exchange parts are given by

$$H_D^{\rm cl} = V_{mnij}\rho^{\rm cl}(n,m)\rho^{\rm cl}(j,i) - (N_\Phi + 2\Delta N)\epsilon_D, \tag{C11}$$

$$H_X^{\rm cl} = -2V_{mnij}S_A^{\rm cl}(j,m)S_A^{\rm cl}(n,i) - \frac{1}{N_f}V_{mnij}\rho^{\rm cl}(j,m)\rho^{\rm cl}(n,i) + (N_\Phi + \delta N)\epsilon_X. \tag{C12}$$

They read (A1) in the momentum representation.

In deriving the formula for $H_X^{\rm cl}$ we have accounted $V_X(0) = 4\ell^2\epsilon_X$ and also the relation

$$\frac{N_\Phi + \delta N}{4\pi\ell_B^2} = \frac{1}{4\pi\ell_B^2}\int d^2x\hat{\rho}^{\rm cl}(\boldsymbol{x}) = \int d^2k\left[\hat{S}_A^{\rm cl}(-\boldsymbol{k})\hat{S}_A^{\rm cl}(\boldsymbol{k}) + \frac{1}{2N_f}\hat{\rho}^{\rm cl}(-\boldsymbol{k})\hat{\rho}^{\rm cl}(\boldsymbol{k})\right]. \tag{C13}$$

This relation is derived as follows. We deal with the quantity

$$\langle \mathfrak{S}|c_M^\dagger c_K|\mathfrak{S}\rangle\langle \mathfrak{S}|c_K^\dagger c_N|\mathfrak{S}\rangle = \nu_K (U)_{NK}(U^\dagger)_{KM}, \tag{C14}$$

where we have used (C5) and $\nu_K^2 = \nu_K$ since $\nu_K = 0$ or 1. Comparing this with (C5) we conclude

$$\langle \mathfrak{S}|c_M^\dagger c_K|\mathfrak{S}\rangle\langle \mathfrak{S}|c_K^\dagger c_N|\mathfrak{S}\rangle = \langle \mathfrak{S}|c_M^\dagger c_N|\mathfrak{S}\rangle, \tag{C15}$$

which is

$$\langle \mathfrak{S}|c_\mu^\dagger(m) c_\kappa(k)|\mathfrak{S}\rangle\langle \mathfrak{S}|c_\kappa^\dagger(k) c_\nu(n)|\mathfrak{S}\rangle = \langle \mathfrak{S}|c_\mu^\dagger(m) c_\nu(n)|\mathfrak{S}\rangle. \tag{C16}$$

Substituting Eq. (C9) into this we find

$$\hat{S}_A^{\rm cl}(n,k)\hat{S}_A^{\rm cl}(k,m) + \frac{1}{4N_f}\hat{\rho}^{\rm cl}(n,k)\hat{\rho}^{\rm cl}(k,m) = \frac{1}{2}\hat{\rho}^{\rm cl}(n,m). \tag{C17}$$

This amounts to Eq. (C13) in the momentum space.

[1] Z. F. Ezawa, *Quantum Hall Effects: Field Theoretical Approach and Related Topics* (World Scientific, Singapore, 2000).
[2] *Perspectives in Quantum Hall Effects*, edited by S. Das Sarma and A. Pinczuk (Wiley, New York, 1997).
[3] S. L. Sondhi, A. Karlhede, S. A. Kivelson, and E. H. Rezayi, Phys. Rev. B **47**, 16419 (1993).
[4] S. E. Barrett, G. Dabbagh, L. N. Pfeiffer, K. W. West, and R. Tycko, Phys. Rev. Lett. **74**, 5112 (1995).
[5] E. H. Aifer, B. B. Goldberg, and D. A. Broido, Phys. Rev. Lett. **76**, 680 (1996).
[6] A. Schmeller, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **75**, 4290 (1995).
[7] R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).
[8] H. A. Fertig, L. Brey, R. Cote, and A. H. MacDonald, Phys. Rev. B **50**, 11018 (1994).
[9] M. Abolfath, J. J. Palacios, H. A. Fertig, S. M. Girvin, and A. H. MacDonald, Phys. Rev. B **56**, 6795 (1997).
[10] A. H. MacDonald, H. A. Fertig, and L. Brey, Phys. Rev. Lett. **76**, 2153 (1996).
[11] H. A. Fertig, L. Brey, R. Cote, A. H. MacDonald, A. Karlhede, and S. L. Sondhi, Phys. Rev. B **55**, 10671 (1997).
[12] Z. F. Ezawa, G. Tsitsishvili, and K. Hasebe, Phys. Rev. B **67**, 125314 (2003).
[13] S. M. Girvin, A. H. MacDonald, and P. M. Platzman, Phys. Rev. B **33**, 2481 (1986).
[14] S. Iso, D. Karabali, and B. Sakita, Phys. Lett. B **196**, 143 (1992).
[15] A. Cappelli, C. Trugenberger, and G. Zemba, Nucl. Phys. B **396**, 465 (1993).
[16] Z. F. Ezawa, Phys. Rev. B **55**, 7771 (1997).
[17] Z. F. Ezawa, Phys. Rev. Lett. **82**, 3512 (1999).
[18] B-H. Lee, K. Moon, and C. Rim, Phys. Rev. D **64**, 085014 (2001).
[19] Z. F. Ezawa and G. Tsitsishvili, Phys. Rev. B **70**, 125304 (2004).
[20] Z. F. Ezawa, M. Eliashvili, and G. Tsitsishvili, Phys. Rev. B **71**, 125318 (2005).
[21] K. Sasaki and Z. F. Ezawa, Phys. Rev. B **60**, 8811 (1999).
[22] S. Q. Murphy, J. P. Eisenstein, G. S. Boebinger, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **72**, 728 (1994).
[23] A. Sawada, D. Terasawa, N. Kumada, M. Morino, K. Tagashira, Z. F. Ezawa, K. Muraki, T. Saku, and Y. Hirayama, Physica E (Amsterdam) **18**, 118 (2003).
[24] D. Terasawa, M. Morino, K. Nakada, S. Kozumi, A. Sawada, Z. F. Ezawa, N. Kumada, K. Muraki, T. Saku, and Y. Hirayama, Physica E (Amsterdam) **22**, 52 (2004).
[25] For example, [15.814], in *Table of Integrals, Series, and Products*, 6th ed., edited by I. S. Gradshteyn and I. M. Ryzhik (Academic, New York, 2000).