Superfluid turbulence in rotating containers: Phenomenological description of the influence of the wall

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In this paper a previous equation for the evolution of vortex line density L in counterflow superfluid turbulence in rotating containers is generalized, in order to take into account the influence of the walls. This model incorporates the effects of counterflow velocity V and of angular velocity Ω of the container, and introduces corrective terms depending on δ/d, δ being the intervortex spacing, of the order $L^{-1/2}$, and d the diameter of the channel. The stability of the solutions for L, for several regimes of averaged counterflow velocity V and angular velocity Ω, is analyzed. Our mathematical analysis reveals that qualitative consistency allows us to reduce the four coefficients characterizing the dependence on δ/d to only one additional independent coefficient, linked to the critical angular velocity Ω_c needed for the appearance of vortex lines in a rotating superfluid.

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I. INTRODUCTION

In recent years there has been growing interest in superfluid turbulence,[1–3] because of its similarity with classical turbulence.[4,5] Among the aspects receiving much attention, there are the effects of the walls as, for instance, on boundary layers. Besides its theoretical interest, the influence of walls on superfluid turbulence may have a practical incidence in refrigeration of small devices by means of flow of superfluid helium along narrow tubes: when the tubes become narrower, the relevance of wall effects will increase. In the analysis we will perform, and following previous literature on this problem, we emphasize the spatial average of the heat flux, rather than its detailed local features.

Superfluid turbulence has been much studied in two physical situations: counterflow experiments and rotating containers. As is known, thermal counterflow induces a quasi-isotropic disordered tangle, while rotation creates an ordered polarized vortex array. In both cases the vortex tangle is described by introducing a scalar quantity L, the average vortex line length per unit volume, briefly called the vortex line density. In pure rotation, the vortex lines are aligned along the rotation axis, and L depends on the angular velocity Ω of the sample as[6]

$$L = L_R \simeq \frac{2\Omega}{\kappa}, \tag{1.1}$$

where κ is the quantum of vorticity ($\kappa = h/m$, with h the Planck constant, and m the mass of the helium atom: $\kappa \simeq 9.97 \times 10^{-4}$ cm^2/s). In counterflow experiments, there is a disordered tangle of vortex lines, with a line density L (in fully developed turbulence) given by[6]

$$L = L_H \simeq \mathcal{A} V^2, \tag{1.2}$$

where $V = |\langle \mathbf{V} \rangle|$ is the modulus of the spatial average of the counterflow velocity $\mathbf{V} = \mathbf{v}_n - \mathbf{v}_s$, $\mathbf{v}_n$ and $\mathbf{v}_s$ being the velocities of normal and superfluid components, respectively. This averaged quantity is related to the absolute value of the heat flux by $q = \rho T S V$, $\mathbf{q}$, ρ, T, and S being, respectively, the heat flux, density, temperature, and entropy of liquid helium II. Here, following most of the references on this topic, we consider V as homogeneous; one may consider it as an average value of V over the cross section of the tube. This is so, because this average value is the easiest one to measure. Note that in our analysis for slow rotation the wall effects are not restricted to a narrow zone near the walls but they have an effect over the whole cross section, because in this case the average separation between vortices, $L^{-1/2}$, is comparable to the diameter of the channel. Then, it is compatible, in this situation, to talk about wall effects and consider a homogeneous counterflow velocity across the channel.

The evolution of L in superfluid counterflow turbulence is described by Vinen's equation,[7] which, in its original form, states that

$$\frac{dL}{dt} = \alpha_1 V L^{3/2} - \eta \kappa L^2, \tag{1.3}$$

with α_1 and η dimensionless constants. The steady-state solution of Eq. (1.3) is $L_H = (\alpha_1/\eta\kappa)^2 V^2$, which has the form mentioned in Eq. (1.2).

A microscopic derivation of Vinen's equation has been obtained by Schwarz[8] using the vortex filament model and assuming homogeneous and isotropic turbulence. Recently Lipniacki[9] has modified the Vinen-Schwarz equation, introducing in it the effects of the anisotropy. He characterizes the anisotropy by means of a vector $\mathbf{I}$, related to the vortex tangle structure by $\mathbf{I} = \langle \mathbf{s}' \times \mathbf{s}'' \rangle / \langle |s''| \rangle$, where $\mathbf{s}(\xi, t)$ describes the vortex lines, with ξ the length along the vortices; the primes indicate differentiation with respect to ξ. Angular brackets stand for averages over the total vortex length of the tangle.

When $\mathbf{I}$ is parallel to $\mathbf{V}$, Vinen's equation is obtained. If $\mathbf{I}$ is not directly parallel to $\mathbf{V}$, Lipniacki proposes to modify Vinen's equation by writing[9]

$$\frac{dL}{dt} = \alpha \mathbf{V} \cdot \mathbf{I} L^{3/2} - \eta \kappa L^2. \tag{1.4}$$

The origin of the Lipniacki proposal (1.4) may be found in the microscopic analysis of vortex dynamics by Schwarz,[8] where an equation analogous to (1.4) is derived. Usually, the term in $\mathbf{I}$ is included in the α_1 coefficient of Vinen's equation (1.3); indeed comparing (1.4) with (1.3) one finds

$$\alpha_1 = \alpha |\mathbf{I}| \cos(\mathbf{V} \cdot \mathbf{I}). \tag{1.5}$$

In Refs. 10 and 11, by analogy with the modifications of transport coefficients in kinetic theory of gases and in generalized hydrodynamics, nonlocal terms in Vinen's equation were introduced, in order to take into account the influence of a nonvanishing ratio between the average separation δ ($\simeq L^{-1/2}$) between vortex lines and the diameter d of the channel. In this way the two transitions from the laminar to the turbulent regime and from type-I (TI) to type-II (TII) turbulence have been described.

Combined rotation and heat flux recently have been the object of investigations.[3,12–15] Experimental observations[16] and numerical simulations[13–15] have shown that, in this situation, the effects of rotation and counterflow are not merely additive, but they exhibit some subtle nonlinear interplay, and the vortex tangle appears to be polarized, to accomplish the rotation. In a previous paper,[17] we dealt with the combined situation in the limit of fast rotation. In that work, neglecting the dependence on $L^{-1/2}/d$, the following equation for evolution of L in counterflow superfluid turbulence in the presence of rotation was proposed:

$$\frac{dL}{dt} = -\eta \kappa L^2 + [\alpha_1 V + \beta_2 (\kappa \Omega)^{1/2}] L^{3/2} - [\beta_1 \Omega + \beta_4 V (\Omega/\kappa)^{1/2}] L. \tag{1.6}$$

Here α_i and β_i are parameters (which may be functions of temperature, pressure, and the anisotropy $\mathbf{I}$ of the tangle). The first two terms in Eq. (1.6) are those appearing in Vinen's equation (1.3); in particular, the coefficient α_1 is given by Eq. (1.5). Equation (1.6) describes some of the most relevant observed features of counterflow-rotational superfluid turbulence in the limit of high rotation.[17]

Here, we shall study from a more general perspective the situation of combined rotation and counterflow, incorporating the effects of the walls. These effects are especially relevant in the limit of slow rotation. In this limit, if also the counterflow velocity is small, the vortex tangle is nonhomogeneous and the influence of walls on vortex formation and destruction is important, as is shown in the simulations of Refs. 13,15. Following the lines of thought outlined in Refs. 10,11, we modify the model proposed in Ref. 17 introducing in it corrective terms depending on δ/d, to take into account the influence of channel walls on the evolution of vortex line density L. The approach used in this paper is phenomenological. Our aim is to give some insight into the combination of parameters that are more relevant in the macroscopic description of this phenomenon, with the hope that our study can stimulate research in this interesting field.

The plan of the paper is as follows. Section II provides the basis for an equation describing the evolution of vortex line density, which includes the influence of V (counterflow), Ω (rotation), and d (walls) on the evolution of L [Eq. (2.5)]. In Sec. III, the situations $V \neq 0$, $\Omega = 0$ (Sec. III A) and $V = 0$, $\Omega \neq 0$ (Sec. III B) are examined. The flow of superfluid helium between two concentric rotating cylinder is also studied in Sec. III C. Section IV deals with situations with coupled $V \neq 0$ and $\Omega \neq 0$.

II. DERIVATION OF AN EVOLUTION EQUATION FOR L IN THE PRESENCE OF COUNTERFLOW AND ROTATION INCORPORATING THE EFFECTS OF THE WALLS

Besides lacking the influence of walls, Eq. (1.3) does not incorporate the effect of angular velocity Ω, which would be necessary to have a joint description of coupled rotation and counterflow. The purpose of this paper is to propose an equation for L incorporating V, Ω, and d, which will reduce, in the suitable limit, to the previous generalization in Ref. 17.

First, we briefly recall Vinen's original derivation of Eq. (1.3). Vinen assumes that

$$\frac{dL}{dt} = \left(\frac{dL}{dt}\right)_f - \left(\frac{dL}{dt}\right)_d, \tag{2.1}$$

and, assuming that the growth of L depends on L, V, and κ, he writes

$$\left(\frac{dL}{dt}\right)_f = \kappa L^2 \phi_f\left(\frac{V}{\kappa L^{1/2}}\right), \tag{2.2}$$

where ϕ_f is a dimensionless function which, by analogy with the growth of a vortex ring, he supposes linearly dependent on its argument; further, he determines the form of the $(dL/dt)_d$ term, responsible for the vortex decay, in analogy with classical turbulence, obtaining

$$\left(\frac{dL}{dt}\right)_f = \alpha_1 V L^{3/2}, \quad \left(\frac{dL}{dt}\right)_d = -\eta \kappa L^2. \tag{2.3}$$

Substituting Eq. (2.3) in Eq. (2.1) one obtains immediately Vinen's equation (1.3).

To derive an evolution equation for L in the presence of counterflow and rotation, motivated by the fact that the formation of vortex lines is now due to V and Ω, and taking into account of the presence of channel walls, following the lines of thought outlined in Refs. 10, 11, and 17, we model the formation term in Eq. (2.1) as

$$\left(\frac{dL}{dt}\right)_f = \kappa L^2 \phi_f\left(\frac{V}{\kappa L^{1/2}}, \frac{\Omega^{1/2}}{(\kappa L)^{1/2}}, \frac{L^{-1/2}}{d}\right). \tag{2.4}$$

In the regime of low Ω (and V), also the line density L is small and we cannot neglect the influence of the walls on evolution of L; therefore, we must incorporate in Eq. (2.4) terms depending on $L^{-1/2}/d$, which take into account these effects. Here, for the sake of simplicity, we will not consider corrections of the destruction contribution, like those proposed in Ref. 10. Indeed, in Ref. 10 it was shown that comparison with experiments indicates that wall effects are much more relevant in reducing the formation term. This is not evident *a priori* and, to our knowledge, it is still not completely understood in microscopic terms, but we take advantage of this fact to focus our attention on the most relevant features.

Choosing a quadratic dependence of ϕ_f on $L^{-1/2}/d$, one obtains the following equation for the evolution of vortex line density L:

$$\frac{dL}{dt} = -\eta\kappa L^2 + \left(\alpha_1 V + \beta_2\sqrt{\kappa\Omega} - \alpha_3\frac{\kappa}{d}\right)L^{3/2} + \left(\alpha_2\frac{V}{d} + \beta_3\frac{\sqrt{\kappa\Omega}}{d} - \beta_1\Omega - \beta_4\frac{V\sqrt{\Omega}}{\sqrt{\kappa}} - \alpha_4\frac{\kappa}{d^2}\right)L. \tag{2.5}$$

As in Ref. 17, we have chosen the linear term depending on $\sqrt{\Omega}$ as a production term, while we have chosen the negative sign in the terms independent of V and $\sqrt{\Omega}$, because the influence of walls hinders vortex formations, as was seen in Ref. 10. Thus, the first term in Eq. (2.5), namely, $-\eta\kappa L^2$, describes the collisions and recombinations of vortex lines among themselves, whereas the terms $-\alpha_3(\kappa/d)L^{3/2}$ and $-\alpha_4(\kappa/d^2)L$ describe the collisions of vortices with the walls of the container. The term $-\beta_1\Omega$ describes the ordering tendency of the rotation, which tends to straighten out the otherwise irregular vortex lines of the tangle, thus shortening them and reducing L. The positive terms in $(\kappa\Omega)^{1/2}$ are related to the Donnelly-Glaberson instability of Kelvin waves near straight vortex lines under counterflow velocities higher than a critical velocity proportional to $(\kappa\Omega)^{1/2}$. Thus, despite the lack for the moment of a rigorous derivation of the several terms in Eq. (2.5), their general trends may be given a physical interpretation. Finally, the term in $V\Omega^{1/2}$ reflects the nonadditive contributions of the rotation and the counterflow, discussed at length in Ref. 17.

Compared with Eq. (1.6), Eq. (2.5) contains the four additional coefficients α_2, α_3, α_4, and β_4, related to terms characterizing the influence of walls on the evolution of L. We will see that our analysis is able to reduce this number to only one independent coefficient, linked to the critical angular velocity Ω_c needed for the appearance of vortex lines in a rotating superfluid.

III. UNCOUPLED REGIME

We will study in detail the solutions of Eq. (2.5), focusing on the case in which wall effects are important in the presence of V and Ω. This is so when Ω and V are not too high, because in this case L is small, in such a way that interline separation, of the order of $L^{-1/2}$, becomes comparable to diameter d.

First of all, we study the behavior of Eq. (2.5) in the limiting uncoupled case with $\Omega=0$ and with $V=0$, respectively. This is convenient to identify the physical consequences of several terms and to be able to compare, below, with ensuing modifications when coupling is considered.

A. Counterflow only

In experiments on counterflow superfluid turbulence the vortex line density L is observed to develop from a low-density state TI, characterized by a low density of vortex lines, to a higher-density regime TII, that can be associated with the fully developed turbulent state. Furthermore, for small values of V, a laminar regime is also present in which vortices are absent. In this work, we have focused our interest on counterflow superfluid turbulence in the regime of low values of L, i.e., in the laminar and TI regimes, when V is small. For this reason, we neglect here the corrections considered in Refs. 10 and 11 to explain the transition from TI to TII turbulent regimes.

For $\Omega=0$ (no rotation), Eq. (2.5) reduces to the following extension of Vinen's equation:

$$\frac{dL}{dt} = -\eta\kappa L^2 + \left(\alpha_1 V - \alpha_5\frac{\kappa}{d}\right)L^{3/2} + \left(\alpha_2\frac{V}{d} - \alpha_4\frac{\kappa}{d^2}\right)L. \tag{3.1}$$

Detailed studies of the evolution equation for the line density L in counterflow only have been made in Refs. 10 and 11. Here, we observe only that this equation admits the solution $L=0$, corresponding to the absence of a vortex array, which is stable for V lower than a critical counterflow velocity, which, in this model, is defined by

$$V_c^H = \frac{\alpha_4\kappa}{\alpha_2 d}. \tag{3.2}$$

The values of this critical velocity depend strongly on temperature; accordingly to the data of Martin and Tough[18] they are $V_c^H=123\kappa/d$ (for $T=1.5$ K) and $V_c^H=93.2\kappa/d$ (for $T=1.7$ K). Such a strong variation is not surprising, because of the fast variation of the densities of the two components with temperature.

B. Rotation only

Ion experiments[19] in rotating helium II which is slowly accelerated are sufficiently sensitive for evidencing the entry of a single vortex line. For a cylindrical container of diameter about $d=0.1$ cm, the appearance of the first line is at about $\Omega_0=1.6$ rad/s. Since the minimal vorticity required to create one vortex is the quantum of vorticity κ, it is natural to expect that no vortex line will appear for situation where $2\pi rv<\kappa$; taking $v=\Omega r$ it follows that the minimum value of Ω for the appearance of one vortex would be, in this naive approximation, $\Omega\simeq(2/\pi)(\kappa/d^2)$. A more refined analysis by Hall and Vinen[20] predicts for $\Omega_{c_{\text{theor}}}$ the value

$$\Omega_{c_{\text{theor}}} = \frac{2\kappa}{\pi d^2}\ln\frac{d}{2r_0}, \tag{3.3}$$

where r_0($r_0\simeq1$ Å) is the core radius of a single vortex line: the theoretical value $\Omega_{c_{\text{theor}}}$, for the container used in Ref. 19 (where $d=d_0=0.1$ cm) was about $\Omega^{(0)}_{c_{\text{theor}}}=1$ rad/s. As one sees the Ω critical value (3.3) depends only on κ and r_0, besides d itself.

We will show that Eq. (2.5) predicts a result that agrees with the behavior predicted by Hall and Vinen. Putting then $V=0$ in Eq. (2.5) one obtains

$$\frac{dL}{dt} = -\eta\kappa L^2 + \left(\beta_2\sqrt{\kappa\Omega} - \alpha_3\frac{\kappa}{d}\right)L^{3/2} - \left(\beta_1\Omega - \beta_3\frac{\sqrt{\kappa\Omega}}{d} + \alpha_4\frac{\kappa}{d^2}\right)L. \tag{3.4}$$

This equation admits the solution $L=0$, corresponding to the absence of vortex lines. We obtain, first of all, the critical value Ω_c which indicates the formation of the (ordered) vortex array. Thus, we study the stability of the solution $L=0$. This is done by perturbing this state with a small perturbation δL and studying its evolution. We have, from Eq. (3.4),

$$\frac{d\delta L}{dt} = \left[-2\eta\kappa L + \frac{3}{2}\left(\beta_2\sqrt{\kappa\Omega} - \alpha_3\frac{\kappa}{d}\right)L^{1/2} - \beta_1\Omega + \beta_3\frac{\sqrt{\kappa\Omega}}{d} - \alpha_4\frac{\kappa}{d^2}\right]\delta L. \tag{3.5}$$

Substituting $L=0$ in this equation, we obtain

$$\frac{d\delta L}{dt} = \left(-\beta_1\Omega + \beta_3\frac{\sqrt{\kappa\Omega}}{d} - \alpha_4\frac{\kappa}{d^2}\right)\delta L. \tag{3.6}$$

Thus, at Ω very small, the $L=0$ solution is stable, because the expression before δL in the right-hand side of Eq. (3.6) is negative. However, it would become stable again for high values of Ω, after a region of values of Ω for which this solution would be unstable. Such a range of values, writing $\Omega \equiv \varpi^2\kappa/d^2$, is

$$\frac{\beta_3 - \sqrt{\beta_3^2 - 4\alpha_4\beta_1}}{2\beta_1} \leqslant \varpi \leqslant \frac{\beta_3 + \sqrt{\beta_3^2 - 4\alpha_4\beta_1}}{2\beta_1}. \tag{3.7}$$

Thus, we must study also the nonzero steady-state solutions of Eq. (3.4), which are

$$L_{\pm}^{1/2} = \frac{\beta_2\varpi - \alpha_3}{2\eta d} \pm \frac{1}{2\eta d} \times \sqrt{(\beta_2\varpi - \alpha_3)^2 + 4\eta(\beta_3\varpi - \beta_1\varpi^2 - \alpha_4)}. \tag{3.8}$$

First note that, in order that $L=0$ is stable until it is substituted by the solution (3.8), the expression $\beta_3\varpi - \beta_1\varpi^2 - \alpha_4$ [appearing in Eq. (3.8) as well as in Eq. (3.6)] must have a double zero, i.e., the coefficients β_3, β_1, and α_4 must satisfy the relation

$$\beta_3^2 = 4\beta_1\alpha_4. \tag{3.9}$$

In this case, we will find the solution $L=0$ (laminar regime) stable for

$$\Omega < \Omega_c = \left(\frac{\beta_3}{2\beta_1}\right)^2\frac{\kappa}{d^2} \tag{3.10}$$

(and also for $\Omega \geqslant \Omega_c$). Further, for

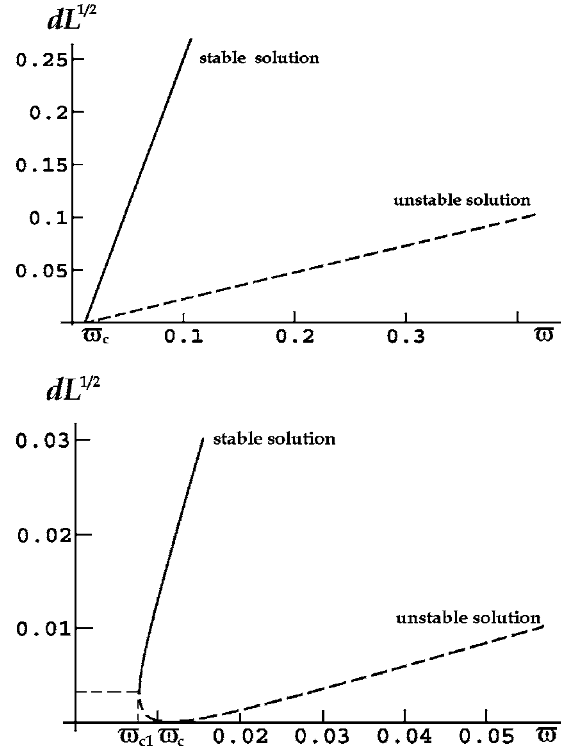

FIG. 1. Stability diagram of stationary solutions (3.12) of Eq. (3.4) (a) choosing for the coefficients the values obtained below in Sec. IV (case $\Omega_c = \Omega^* = \Omega_{c1}$; (b) choosing $\Omega^* = 0.95\Omega_c$.

$$\Omega \geqslant \Omega_{c1} = \left(\frac{\alpha_3\sqrt{\beta_1} + \beta_3\sqrt{\eta}}{\beta_2\sqrt{\beta_1} + 2\beta_1\sqrt{\eta}}\right)^2\frac{\kappa}{d^2}, \tag{3.11}$$

we have also the two nonzero solutions (3.8), which, under the hypothesis (3.9), can be written

$$L_{\pm}^{1/2} = \frac{\beta_2}{2\eta}\frac{\sqrt{\Omega} - \sqrt{\Omega_c}}{\sqrt{\kappa}}\left[1 \pm \sqrt{1 - \frac{4\eta\beta_1}{\beta_2^2}\left(\frac{\sqrt{\Omega} - \sqrt{\Omega_c}}{\sqrt{\Omega} - \sqrt{\Omega^*}}\right)^2}\right], \tag{3.12}$$

where we have put $\Omega^* = (\alpha_3^2/\beta_2^2)(\kappa/d^2)$. As is seen, the two critical angular velocities Ω_c and Ω_{c1} (and also Ω^*) are proportional to $1/d^2$. This result is in agreement with the theoretical result (3.3) of Hall and Vinen.

For $\Omega \geqslant \Omega_{c1}$ both solutions (3.12) are non-negative and correspond to values of $L^{1/2}$ acceptable from a mathematical point of view. We study therefore their stability. Substituting Eq. (3.12) in Eq. (3.5), we deduce that the solution L_+ is stable, where it exists, while L_- is unstable.

For high values of Ω, the stable solution of Eq. (3.5) must reduce to Eq. (1.1). This happens if $\beta_2 + (\beta_2^2 - 4\beta_1\eta)^{1/2} = 2\sqrt{2}\eta$. As a consequence, η, β_1, and β_2 must satisfy the relation $\sqrt{2}\beta_2 = 2\eta + \beta_1$. The result (3.12) generalizes the known result (1.1) to very small values of Ω (see Fig. 1).

If, in addition to Eq. (3.9), also

$$\beta_2\beta_3 - 2\alpha_3\beta_1 = 0, \tag{3.13}$$

it follows that $\Omega_c = \Omega^* = \Omega_{c1}$.

The metastability region of the vortex-free regime is not evidenced in experiments. For this reason, in the following, we will suppose $\Omega_c=\Omega^*=\Omega_{c1}$. Under this hypothesis Eq. (3.4) becomes

$$\frac{dL}{dt}=-\eta\kappa L^2+\beta_2\sqrt{\kappa}(\sqrt{\Omega}-\sqrt{\Omega_c})L^{3/2}-\beta_1(\sqrt{\Omega}-\sqrt{\Omega_c})^2 L \tag{3.14}$$

whose stable nonzero stationary solution is simply

$$L_R^{1/2}=\frac{1}{2\eta}\left(1+\sqrt{1-\frac{4\eta\beta_1}{\beta_2^2}}\right)\frac{(\sqrt{\Omega}-\sqrt{\Omega_c})}{\sqrt{\kappa}}. \tag{3.15}$$

We will determine, now, for the container used in Ref. 16 the theoretical value $\Omega^{(1)}_{c_{\text{theor}}}$. For the container used in Ref. 19 (where $d=d_0=0.1$ cm), this value was about $\Omega^{(0)}_{c_{\text{teor}}}=1$ rad/s. In the experiment described in Ref. 16, the channel has a square cross section of side 1 cm. By substituting the square with a circle having diameter $d=d_1=1$ cm, we find that the critical angular velocity $\Omega^{(1)}_{c_{\text{theor}}}$ of Ref. 16 is approximately

$$\Omega^{(1)}_{c_{\text{theor}}}=\Omega_c^{(0)}\frac{d_0^2}{d_1^2}\frac{\ln(d_1/2r_0)}{\ln(d_0/2r_0)}; \tag{3.16}$$

since $d_1=10d_0$, one obtains

$$\Omega^{(1)}_{c_{\text{theor}}}=1.15\times 10^{-2}\ \text{rad/s}. \tag{3.17}$$

Since the theoretical critical angular velocity $\Omega_c^{(1)}$ is somewhat lower than the lowest angular velocity used in experiments described in Ref. 16, we conclude that in the situations examined in Ref. 16 the vortex-free regime is not evidenced.

Finally, observe that, in our model the logarithmic dependence of Ω_c on d does not appear. This is right if the considered values of d are of the same order of magnitude. For example, from the result (3.17), using Eq. (3.10) we can easily determine for the coefficient $\mathcal{L}=(\beta_3/2\beta_1)^2$, in the range of values of d of the order of 1 cm, the value

$$\mathcal{L}_{\text{cylinder}}=\left(\frac{\beta_3}{2\beta_1}\right)^2=11.3. \tag{3.18}$$

The term $\ln(d/r_0)$ in Eq. (3.3) is not forbidden in our analysis, but it cannot be derived from it because it has been partially based on the dimensional analysis, which is not able to derive this dimensionless term. One could incorporate it by comparing Eq. (3.10) with Eq. (3.3), in which case one has

$$\mathcal{L}_{\text{cylinder}}=\left(\frac{\beta_3}{2\beta_1}\right)^2=\frac{2}{\pi}\ln\frac{d}{2r_0}. \tag{3.19}$$

This logarithmic dependence should then be taken into account not only here but also in relations (3.9) and (3.13)—and also (4.1)—which yield α_2, α_3, and α_4 in terms of β_3/β_1.

C. Rotation between two concentric cylinders

Equation (3.14) should be applicable to other situations with rotation, as for instance to describe some features of superfluid turbulence in a Couette flow between two concentric cylinders.[6,21–23] To do so, we must take into account that the value (3.10) does refer in general to the vorticity of the fluid, rather than to the critical angular velocity of the cylinder—as is known, in a rotating cylinder $\omega=2\Omega$, but when we consider two concentric cylinders, the vorticity ω depends in a more complicated way on the radii and the velocities of both cylinders, as we will clarify below. Further, we must take into account that the minimal vorticity required to create one single row of vortices inside the annulus depends on the geometry of the system.

Thus, we consider now the laminar flow between infinitely long concentric cylinders rotating at angular velocities Ω_1 and Ω_2. In this case (as in classical fluids) the flow is given by a combination of solid body rotation and potential flow, i.e.,[6]

$$v=Ar+\frac{B}{r}, \tag{3.20}$$

where

$$A=\frac{R_2^2\Omega_2-R_1^2\Omega_1}{R_2^2-R_1^2},\quad B=-\frac{R_1^2R_2^2(\Omega_2-\Omega_1)}{R_2^2-R_1^2}. \tag{3.21}$$

The vorticity ω for this flow is given by

$$\omega=2|A|=\frac{2(R_2^2\Omega_2-R_1^2\Omega_1)}{R_2^2-R_1^2}. \tag{3.22}$$

We consider first the critical angular velocity for the appearance of vortices in an annular region of fluid between cylinders rotating at the same angular velocity. In this case, the velocity distribution is given by Eq. (3.20) with $B=0$ and $A=\Omega_2=\Omega_1=\Omega$.

As observed by Donnelly and Fetter,[24,6] in the annulus there is relative motion between normal fluid and superfluid. In fact, the normal fluid velocity is distributed according to Eq. (3.20), while in the absence of vortices, the superfluid motion is potential. Therefore, on the inner part of the annulus we have $v_s>v_n$, while on the outer part $v_s<v_n$. Donnelly and Fetter speculate that, in general, vortices will appear when their presence can minimize the relative velocity V between the two fluids. Since $v_n\leqslant\Omega R_2$ and $v_s\geqslant\Omega R_1$, one deduces that a single row of vortices can appear in the annulus if $\Omega R_2-\Omega R_1\geqslant\kappa/\pi d$, or

$$\Omega>\frac{\kappa}{\pi d^2} \tag{3.23}$$

with d taken here as $d=R_2-R_1$.

A careful calculation was addressed in Ref. 24 by Donnelly and Fetter, which, using free-energy minimization, found the following expression for the critical velocity at which the first array of quantized vortices appears in the annulus:

$$\Omega_0=\frac{\kappa}{\pi d^2}\ln\left(\frac{2d}{\pi r_0}\right). \tag{3.24}$$

We can also write

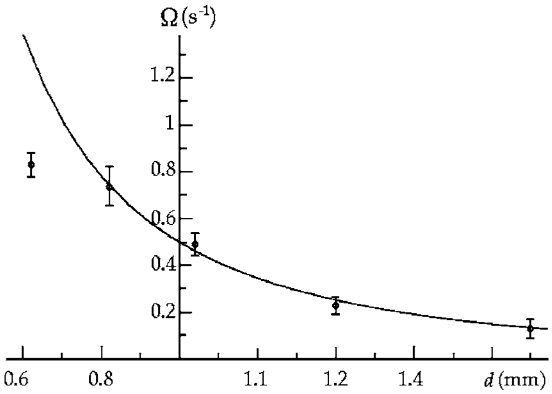

FIG. 2. Values of Ω_c as a function of d from this work. Points are experimental data from Ref. 25.

$$\Omega_0 = \frac{\kappa}{d^2}\mathcal{L}_{\text{annulus}} \quad \text{with } \mathcal{L}_{\text{annulus}} = \frac{1}{\pi}\ln\left(\frac{2d}{\pi r_0}\right). \quad (3.25)$$

As one sees, we obtain complete agreement with Eq. (3.10), choosing $(\beta_2/2\beta_1)^2=\mathcal{L}_{\text{annulus}}$ and neglecting the logarithmic factor. Taking into account that $\Omega=\omega/2$ [see Eq. (3.22)], we note that in this geometry the minimal vorticity required to create one single row of vortices inside the annulus is

$$\omega_{\text{crit}}^{\text{annulus}} = \frac{2\kappa\mathcal{L}_{\text{annulus}}}{d^2}. \quad (3.26)$$

For example, if we choose $d\approx 10^{-3}$ m and $r_0\approx 10^{-10}$ m the logarithmic factor appearing in Eq. (3.3) is $\ln(d/2r_0)=15.4$ whereas the logarithmic factor in Eq. (3.25) is $\ln(2d/\pi r_0)=15.7$, while if we choose $d\approx 10^{-2}$ m these two logarithmic factors assume the values 17.7 and 18.0, respectively. We deduce from these results that

$$\omega_{\text{crit}}^{\text{annulus}} \simeq \frac{1}{2}\omega_{\text{crit}}^{\text{cylinder}}. \quad (3.27)$$

This result can be explained by observing that we are dealing with two very different geometries: the vorticity $\omega_{\text{crit}}^{\text{cylinder}}$ is the critical ω needed for the appearance of a *single vortex* in a rotating cylinder, while the vorticity $\omega_{\text{crit}}^{\text{annulus}}$ is the critical ω needed for the appearance of a *row of vortices* in the rotating annulus.

Experiments to detect the appearance of the first row of vortices in an annulus with second sound were carried out by Bendt and Donnelly.[25] Here we have fitted the experimental data with the function (3.10) obtaining $\mathcal{L}_{\text{annulus}}=[(\beta_3/2\beta_1)^2]_{\text{annulus}}=4.97$. In Fig. 2 are reported the experimental data of Ref. 25 and our results. As one can see, except for the narrowest gap, the experimental results are in good agreement with our model.

In Couette flow, it is usual to keep still the outer cylinder (radius R_2) and to make the internal cylinder (radius R_1) rotate at an angular speed Ω_1. This induces in the region within the annulus a vorticity ω of the order of

$$\omega = \frac{2R_1^2\Omega_1}{R_2^2 - R_1^2}. \quad (3.28)$$

If, as usual in the study of Couette flow, $R_2-R_1 \ll (R_1+R_2)/2 \simeq R_1$, the vorticity ω is

$$\omega = \frac{R_1\Omega_1}{R_2 - R_1} = \frac{R_1\Omega_1}{d}. \quad (3.29)$$

According to Eq. (3.29), the value of the critical Ω_1 of the internal cylinder needed to produce vortex lines in the annular region is

$$\Omega_{1,\text{crit}} = \frac{R_2 - R_1}{R_1}\omega_{\text{crit}}; \quad (3.30)$$

since in this case the geometry of the sample is the same as in the rotating annulus, we use for ω_{crit} the value (3.27). One obtains

$$\Omega_{1,\text{crit}} \simeq \frac{R_2 - R_1}{R_1}\frac{1}{2}\mathcal{L}_{\text{cylinder}}\frac{\kappa}{d^2} = \frac{2\kappa}{\pi R_1 d}\ln\frac{d}{2r_0}. \quad (3.31)$$

According to the analyses in Refs. 24–27, the value of the critical Ω_1 of the internal cylinder needed to produce vortex lines in the annular region is

$$\Omega_{1,c} = \frac{R_2^2 - R_1^2}{R_1^2}\frac{\kappa}{\pi d^2}\ln\frac{2d}{\pi r_0} \approx \frac{2\kappa}{\pi R_1 d}\ln\frac{2d}{\pi r_0}. \quad (3.32)$$

As we have already observed, the two logarithmic factors appearing in Eqs. (3.3) and (3.25) are very similar. Thus, our prediction (3.31) for the critical value of the rotating speed of the inner cylinder is reasonably close to the free-energy prediction (3.32), having in mind that the macroscopic theory does not yield the factors inside the logarithm, which require a more microscopic theory.

The second-sound attenuation data of Swanson and Donnelly (see Fig. 1 of Ref. 26) show that the line density L is proportional to the angular velocity of the inner cylinder. The stability of this flow has been studied from a theoretical point of view by Barenghi[27] and confirmed by experiments of Swanson and Donnelly.[26] They found a temperature-dependent critical Reynolds number (linked to the angular velocity of the inner cylinder) beyond which the Couette flow is no longer stable. Therefore, the solution (3.15) of Eq. (3.14) describes the Couette flow between two rotating concentric cylinders in this regime.

IV. COUPLED REGIME: LOW VALUES OF Ω

Swanson and coauthors[16] showed that in combined rotation and counterflow the ordered vortex array produced by rotation becomes unstable when counterflow propagates along the rotation axis. In the simultaneous presence of counterflow and rotation, they showed that also a very low rotation eliminates the counterflow critical velocity V_c^H present in the absence of rotation.

As was mentioned in Sec. II, the coupled regimes with low Ω and V, and consequently with small L, should be especially sensitive to wall effects because in them the interline separation $L^{-1/2}$ may become comparable to d. Experiments[16] in this regime show that even the few lines introduced by very slow rotation ($f=0.0073$ Hz, i.e., $\Omega=0.0458$ rad/s) eliminate the critical velocity V_c^H present in the absence of rotation. Here, we will study the region of values of V and Ω for which the laminar regime ($L=0$) is stable. As recalled in Sec. III B, in the container vortex lines are present only if the angular velocity Ω is higher than the critical angular velocity Ω_c defined in Eq. (3.10). This implies that the vortex production mechanism is active only if $\Omega>\Omega_c$. As a consequence, we can make the hypothesis that Eq. (2.5) can be obtained from Eq. (1.6) by simply substituting $\sqrt{\Omega}$ with $\sqrt{\Omega}-\sqrt{\Omega_c}$. This happens if Eqs. (3.9) and (3.13) are satisfied and if in addition.

$$\beta_3\beta_4=2\alpha_2\beta_1. \tag{4.1}$$

Under these hypotheses Eq. (2.5) becomes

$$\frac{dL}{dt}=-\eta\kappa L^2+[\alpha_1 V+\beta_2\sqrt{\kappa}(\sqrt{\Omega}-\sqrt{\Omega_c})]L^{3/2} - \left(\beta_4\frac{\sqrt{\Omega}-\sqrt{\Omega_c}}{\sqrt{\kappa}}V+\beta_1(\sqrt{\Omega}-\sqrt{\Omega_c})^2\right)L \tag{4.2}$$

and the result is

$$V_c^H=\frac{\alpha_4}{\alpha_2}\frac{\kappa}{d}=\frac{\beta_1}{\beta_4}\sqrt{\kappa\Omega_c}. \tag{4.3}$$

Observe that both the coefficients α_2 and β_4 present a step: the coefficient α_2 in correspondence to the TI-TII transition (see Ref. 11), β_4 in correspondence to the formation of helical waves (see Ref. 17).

A. The vortex-free regime

In combined counterflow and rotation, in order to establish whether the vortex-free regime is also present, we must study the stability of the solution $L=0$. Reasoning as in the previous cases, i.e., writing the equation for the evolution of the perturbation δL around $L=0$, one obtains from Eq. (4.2)

$$\frac{d\delta L}{dt}=(\sqrt{\Omega}-\sqrt{\Omega_c})\left(\frac{\beta_4}{\sqrt{\kappa}}V+\beta_1(\sqrt{\Omega}-\sqrt{\Omega_c})\right)\delta L. \tag{4.4}$$

Supposing $\Omega<\Omega_c$, the stability condition is then

$$\beta_4 V+\beta_1\sqrt{\kappa}(\sqrt{\Omega}-\sqrt{\Omega_c})<0. \tag{4.5}$$

This inequality singles out a region of the plane $(V,\sqrt{\Omega})$ (placed in the first quadrant), delimited by a portion of straight line, which intercepts the axis in correspondence with the two critical values Ω_c and V_c^H, defined in Eqs. (3.2) and (3.10), respectively. Consequently V_c^H and Ω_c^R are the highest values of V and Ω, respectively, for which the laminar regime is present. This agrees with the experimental observation that even a very small angular velocity (but higher than the critical one) eliminates the critical counterflow velocity V_c^H.

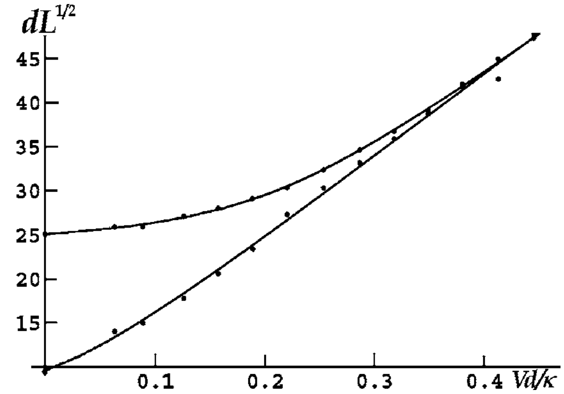

FIG. 3. Values of $dL^{1/2}$ as function of Vd/κ, for f = (a) 0.0073 and (b) 0.05 Hz from this work. Experimental data are from Ref. 16.

B. The turbulent regime

We study now, using Eq. (4.2), the counterflow-rotation superfluid turbulence, for low values of Ω and V, but outside of the region of the plane $(V,\sqrt{\Omega})$ which characterizes the laminar regime.

In rotation only, for values of Ω beyond Ω_c, the stable stationary solution of Eq. (3.14) is the solution (3.15). In counterflow-rotation regimes, the nonzero stationary solutions of Eq. (4.2) are the solutions of the equation

$$-L+\left(\frac{\alpha_1}{\eta\kappa}V+\frac{\beta_2}{\eta}\frac{\sqrt{\Omega}-\sqrt{\Omega_c}}{\sqrt{\kappa}}\right)L^{1/2}-\left(\frac{\beta_4}{\eta}\frac{\sqrt{\Omega}-\sqrt{\Omega_c}}{\sqrt{\kappa}}V + \frac{\beta_1}{\eta}\frac{(\sqrt{\Omega}-\sqrt{\Omega_c})^2}{\kappa}\right)=0. \tag{4.6}$$

A fitting with experimental data reported in Fig. 1 of Ref. 16, reported also in our Fig. 3, allows us to obtain the values for the coefficients appearing in Eq. (4.2); these values are reported in Table I. It is seen that, in this regime of slow rotation, the coefficients α_1, β_4, β_2, and β_1 depend on angular velocity (or, alternatively, on anisotropy, to which we will not refer, because we do not know about it in detail).

In the execution of the fitting the quantity β_4/α_1 has been chosen in such a way that the ratio β_1/β_4 obtained for the two values of angular velocities considered is equal: in fact, from Eq. (4.3) one has $(\beta_1/\beta_4)=V_c^H/\sqrt{\kappa\Omega_c}$. The values $\beta_4/\alpha_1=1.645$ and $\beta_1/\beta_4=5.886$ were obtained.

V. CONCLUSIONS

In summary, we have proposed in this paper an evolution equation (2.5) for the vortex line density L, which includes

TABLE I. Values of the coefficients appearing in Eq. (4.2) obtained from the data of Ref. 16.

f(Hz)	α_1/η	β_4/η	β_2/η	β_1/η
0.0073	0.0936	0.154	3.15	0.916
0.05	0.0843	0.139	2.22	0.816

the influence of counterflow velocity V, angular velocity Ω, and channel diameter d. When the effects of Ω and d are neglected, Eq. (2.5) reduces to the usual Vinen equation (1.3). When the effect of d is neglected we recover the results in Ref. 17.

We have studied the solutions of (2.5) and their stability in several regimes, with the main aim of identifying the physical outcomes of this equation and of evaluating the parameters by comparison with experimental data. In this work we analyze situations where V, Ω, and d are different from zero, but small, in which case $L^{-1/2}$ may become comparable to the diameter of the channel. In particular, we obtained the region of the plane $(V, \sqrt{\Omega})$ for which the laminar regime is stable $(L=0)$ and the solution for $L \neq 0$ just beyond this stability region.

Note that in Eq. (2.5) there are nine parameters, which correspond to the different terms obtained from dimensional considerations. We have shown that consistency arguments with qualitative stability trends indicate that only six of such coefficients are truly independent. For instance, we may take η and α_1 [the coefficients already appearing in Vinen's equation (1.3)], and the four coefficients β_1, β_2, β_3, and β_4. The other three coefficients are determined by the relations (3.9), (3.13), and (4.1). Note that the three coefficients β_1, β_2, and β_4 have been introduced in Ref. 17 and concern terms not related to d, i.e., to the presence of walls. The additional contributions are the terms related to the walls, i.e., those where d appears, which are related to coefficients β_3, α_2, α_3, and α_4; it is seen that the relations (3.9), (3.13), and (4.1) ensure that only one of these four coefficients (β_3 in our choice) is actually independent and can be determined from the value of Ω_c [Eq. (3.10)]. The explicit expressions for the coefficients α_2, α_3, and α_4 are then

$$\alpha_2 = \frac{\beta_3\beta_4}{4\beta_1}, \quad \alpha_3 = \frac{\beta_2\beta_3}{2\beta_1}, \quad \alpha_4 = \frac{\beta_3^2}{4\beta_1}, \tag{6.1}$$

and their respective values are $\alpha_2=0.285$, $\alpha_3=5.829$, and $\alpha_4=11.492$.

We have seen that Eq. (3.14), with the value of Ω_c corresponding to the different geometry, is able to describe Couette flow. We think that the more complete equation (4.2) may be able to describe also the more general phenomenon of the simultaneous presence of rotation and heat flux in the flow of helium II between concentric cylinders.

A microscopic understanding of the several terms appearing in Eq. (2.5) is an urgent task. In particular, this task would be especially challenging and rewarding in two aspects: a best knowledge of the effects of the interaction of vortices with the walls, and an understanding of the physical mechanism for the coupling of counterflow and rotation.

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[1] W. F. Vinen and J. J. Niemela, J. Low Temp. Phys. **128**, 167 (2002).
[2] *Quantized Vortex Dynamics and Superfluid Turbulence*, edited by C. F. Barenghi, R. J. Donnelly, and W. F. Vinen (Springer, Berlin, 2001).
[3] R. J. Donnelly, J. Phys.: Condens. Matter **11**, 7783 (1999).
[4] W. F. Vinen, Phys. Rev. B **61**, 1410 (2000).
[5] C. F. Barenghi, J. Phys.: Condens. Matter **11**, 7751 (1999).
[6] R. J. Donnelly, *Quantized Vortices in Helium II* (Cambridge University Press, Cambridge, U.K., 1991).
[7] W. F. Vinen, Proc. R. Soc. London, Ser. A **240**, 493 (1957).
[8] K. W. Schwarz, Phys. Rev. B **31**, 5782 (1988).
[9] T. Lipniacki, Phys. Rev. B **64**, 214516 (2001).
[10] M. S. Mongiovì and D. Jou, Phys. Rev. B **71**, 094507 (2005).
[11] M. S. Mongiovì and D. Jou, J. Phys.: Condens. Matter **17**, 4423 (2005)
[12] A. P. Finne *et al.*, Nature (London) **424**, 1022 (2003).
[13] M. Tsubota, T. Araki, and C. F. Barenghi, Phys. Rev. Lett. **90**, 205301 (2003).
[14] M. Tsubota, T. Araki, and C. F. Barenghi, J. Low Temp. Phys. **134**, 471 (2004).
[15] M. Tsubota, C. F. Barenghi, T. Araki, and A. Mitani, Phys. Rev. B **69**, 134515 (2004); J. Low Temp. Phys. **134**, 471 (2004).
[16] C. E. Swanson, C. F. Barenghi, and R. J. Donnelly, Phys. Rev. Lett. **50**, 190 (1983).
[17] D. Jou and M. S. Mongiovì, Phys. Rev. B **69**, 094513 (2004).
[18] K. P. Martin and J. T. Tough, Phys. Rev. B **27**, 2788 (1983).
[19] R. E. Packard and T. M. Sanders, Phys. Rev. A **6**, 799 (1972).
[20] H. E. Hall and W. F. Vinen, Proc. R. Soc. London, Ser. A **238**, 204 (1956).
[21] C. J. Swanson, Int. J. Eng. Sci. **36**, 1481 (1998).
[22] K. L. Hendersen and C. F. Barenghi, Physica B **284-288**, 67 (2000).
[23] K. L. Hendersen, in *Quantized Vortex Dynamics and Superfluid Turbulence* (Ref. 2), p. 138.
[24] R. J. Donnelly and A. L. Fetter, Phys. Rev. Lett. **17**, 747 (1966).
[25] P. J. Bendt and R. J. Donnelly, Phys. Rev. Lett. **19**, 214 (1967).
[26] C. J. Swanson and R. J. Donnelly, Phys. Rev. Lett. **67**, 1578 (1991).
[27] C. F. Barenghi, Phys. Rev. B **45**, 2290 (1992).