Zero-temperature optical conductivity of ultraclean Fermi liquids and superconductors

A. Rosch
Institut für Theoretische Physik, Universität zu Köln, D-50937 Köln, Germany

P. C. Howell
Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, D-76128 Karlsruhe, Germany
(Received 1 April 2005; published 13 September 2005)

We calculate the low-frequency optical conductivity $\sigma(\omega)$ of clean metals and superconductors at zero temperature neglecting the effects of impurities and phonons. In general, the frequency and temperature dependences of σ have very little in common. For small Fermi surfaces in three dimensions (but not in two dimensions) we find, for example, that Re $\sigma(\omega>0)\approx$const which corresponds to a scattering rate $\Gamma\propto\omega^2$ even in the *absence* of umklapp scattering when there is no T^2 contribution to Γ. In the main part of the paper we discuss in detail the optical conductivity of d-wave superconductors in two dimensions where Re $\sigma(\omega>0)\propto\omega^4$ for the smallest frequencies and the umklapp processes typically set in smoothly above a finite threshold ω_0 smaller than twice the maximal gap Δ. In cases where the nodes are located at $(\pm\pi/2,\pm\pi/2)$, such that direct umklapp scattering among them is possible, one obtains Re $\sigma(\omega)\propto\omega^2$.

DOI: 10.1103/PhysRevB.72.104510 PACS number(s): 74.25.Gz, 78.30.Er, 71.10.Ay

INTRODUCTION

Optical conductivity is a powerful tool[1] to study the properties of a strongly correlated metal. The frequency dependence in particular can give detailed information on the excitation spectrum of a system (gaps, phonons, magnons, interband transitions, etc.) which in general cannot be extracted from, for example, the temperature dependence of the conductivity.

In a superconductor the electronic contribution to the optical conductivity Re $\sigma(\omega)$ can be separated—at least in simple situations—into three different contributions. First, and most important, superconductivity implies the existence of a δ peak at $\omega=0$ whose (Drude) weight is given by the condensate fraction. Second, thermal excitations at small but finite temperatures, $T>0$, are expected to lead to a rather sharp peak centered again at $\omega=0$, whose width is identified with the scattering rate of the thermal excitations and is strongly temperature dependent (in cases where impurity scattering can be neglected). Finally, all other contributions at finite frequency are usually called "incoherent background." This background depends only weakly on temperature T. It is the goal of this paper to discuss the low-frequency properties of this incoherent background. More precisely, we consider the optical conductivity for frequencies $\omega>0$ at $T=0$ when thermal excitations are absent.

In most experimentally relevant situations the optical conductivity of (conventional) superconductors at low frequencies is dominated by elastic impurity scattering. The theory of optical conductivity in such systems was developed very early by Mattis and Bardeen.[2] Inelastic scattering is more important in strongly interacting superconductors and rather clean samples, and therefore the optical conductivity in d-wave superconductors has been studied quite extensively in the context of high-T_c superconductors (see, for example, Refs. 3–5 and references therein). Motivated by experiments, these investigations have mainly investigated the influence of scattering from collective modes and the interplay with impurity scattering. In this paper we systematically investigate the zero-temperature optical conductivity arising from the interplay of band-structure effects and electron-electron interactions taking into account all relevant vertex corrections. While the main focus of this paper is the investigation of d-wave superconductors, we also briefly discuss the optical conductivity of clean Fermi liquids and s-wave superconductors.

METHOD

According to the Kubo formula, the optical conductivity is given by

$$\text{Re }\sigma(\omega)=\frac{1}{\omega}\text{Im}\langle\langle J,J\rangle\rangle_\omega, \tag{1}$$

where $\langle\langle J,J\rangle\rangle_\omega$ is the the current-current correlator, $\langle\langle J,J\rangle\rangle_\omega=-i\int_0^\infty dte^{i(\omega+i0)t}\langle[J(t),J(0)]\rangle$. When calculating the optical conductivity perturbatively, it is important to take into account both vertex and self-energy corrections. For example, in a Galilean invariant system with a quadratic dispersion, $\varepsilon_k=k^2/(2m)$, vertex and self-energy corrections cancel exactly as the total electrical current is a conserved quantity. But even in clean non-Galilean invariant systems, i.e., for electrons moving in a periodic crystalline potential, massive cancellations between self-energy and vertex corrections occur, especially if there is little umklapp scattering close to the Fermi surface. To take into account vertex and self-energy corrections on the same footing, one in general has to solve an integral equation (a vertex equation or, equivalently, a linearized quantum Boltzmann equation) to obtain the correct conductivity even to lowest order in perturbation theory.

However, at zero temperature and in the absence of disorder one can avoid the substantial technical difficulties involved in solving multidimensional integral equations by the following argument: In general, one can express the conductivity in the form $\sigma(\omega)=\chi/[\Gamma(\omega)-i\omega]$ where χ is identified with the total optical weight and the (frequency-dependent) scattering rate $\mathrm{Re}\,\Gamma(\omega)$ can be calculated from the integral equations described above. However, for $|\Gamma(\omega)|\ll\omega$ this simplifies after multiplication with ω^2 to

$$\omega^2 \mathrm{Re}\,\sigma(\omega)=\omega^2 \mathrm{Re}\frac{\chi}{\Gamma(\omega)-i\omega}\approx\chi\,\mathrm{Re}\,\Gamma(\omega). \qquad (2)$$

Note that there is no contribution from the δ function at $\omega=0$ due to the ω^2 prefactor. For weak interactions Γ is small and therefore we can obtain $\sigma(\omega>0)$ in a straightforward perturbative expansion, i.e., without solving any integral equations, from the right-hand side of

$$\mathrm{Re}\,\sigma(\omega>0)=\frac{\mathrm{Im}\langle\langle\partial_t J,\partial_t J\rangle\rangle_\omega}{\omega^3}, \qquad (3)$$

provided that $|\Gamma(\omega)|\ll\omega$. As $\partial_t J$ is already linear in the interactions (see below), it is sufficient to leading order to evaluate the correlation function in Eq. (3) to zeroth order in the couplings. We will use this approximation only at $T=0$. At any *finite* temperature, the scattering rate $\Gamma(\omega\to 0)$ is constant and therefore the method described above will break down for $\omega\to 0$ but remains valid at higher frequencies where $|\Gamma(\omega)|\ll\omega$. Note that within the so-called memory-function approach[6] one uses essentially identical formulas to calculate $\Gamma(\omega)$.

If one is only interested in the qualitative behavior of $\mathrm{Re}\,\sigma(\omega)$ at low ω, i.e., in the power law obtained for $\omega\to 0$, one can relax the condition $|\Gamma(\omega)|\ll\omega$ and replace it by $\mathrm{Re}\,\Gamma(\omega)\ll(1-c)\omega$ for $\omega<\omega^*$ where ω^* is some characteristic frequency and c (which can be of order 1) is obtained from $\mathrm{Im}\,\Gamma(\omega)\approx c\omega$ for $\omega\ll w^*$. As the latter condition is fulfilled in all cases discussed below, we expect that all our results are *qualitatively* correct at sufficiently low frequencies (a possible exception is discussed below) even in a strongly interacting systems. (Backflow and other Fermi-liquid renormalization effects[7] will only change prefactors, and multiparticle scattering processes are suppressed for $\omega\to 0$ due to the restricted phase space.)

In the following, we will first consider the optical conductivity of a clean Fermi liquid at $T=0$. This will serve as a reference for our results on d-wave superconductors presented in the second part.

METALS

In a one-band model, the electrical current is given by $\mathbf{J}=\Sigma_{\mathbf{k}\sigma}\mathbf{v}_\mathbf{k}c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}$ where $\mathbf{v}_\mathbf{k}=d\epsilon_\mathbf{k}/d\mathbf{k}$ is the velocity of electrons. In the presence of interactions and in the absence of Galilei invariance the current is not conserved with $\partial_t\mathbf{J}=i\Sigma_{\mathbf{k}\mathbf{k}'\mathbf{q}\sigma\sigma'}U_\mathbf{q}(\mathbf{v}_\mathbf{k}+\mathbf{v}_{\mathbf{k}'}-\mathbf{v}_{\mathbf{k}+\mathbf{q}}-\mathbf{v}_{\mathbf{k}'-\mathbf{q}})c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}+\mathbf{q},\sigma}c^\dagger_{\mathbf{k}'\sigma'}c_{\mathbf{k}'-\mathbf{q},\sigma'}$ for a density-density interaction $U_\mathbf{q}$. The change of current is proportional to the difference of incoming and outgoing velocities. In the following we will assume that the (screened) interaction $U_\mathbf{q}\approx U$ depends only weakly on the transferred momentum $\mathbf{q}$.

For a (normal) metal we therefore obtain at low frequencies and $T=0$ using Eq. (3),

$$\mathrm{Re}\,\sigma(\omega>0)\approx\frac{4\pi U^2}{\omega^3}\sum_{1234\mathbf{G}}f_1f_2(1-f_3)(1-f_4)\times(v_4^x+v_3^x-v_2^x-v_1^x)^2\delta_{1+2,3+4+\mathbf{G}}\times\{\delta[\omega-(\varepsilon_4+\varepsilon_3-\varepsilon_2-\varepsilon_1)]-(\omega\leftrightarrow-\omega)\}, \qquad (4)$$

where 1, ..., 4 denote the momenta $\mathbf{k}_1,\ldots,\mathbf{k}_4$ in the first Brillouin zone, $f_i=f(\epsilon_i)=f(\epsilon_{\mathbf{k}_i})$ are Fermi functions, and momentum is conserved modulo reciprocal-lattice vectors $\mathbf{G}$. To perform the momentum integrals it is useful to split $\mathbf{k}_i$ into a component perpendicular to the Fermi surface and an angular integration parallel to it. For small ω only a thin shell of width ω/v_F contributes for each of the three relevant momentum integrations, implying an ω^3 dependence which cancels the $1/\omega^3$ prefactor.

If the Fermi surface is sufficiently large such that umklapp scattering processes can take place, one therefore obtains the well-known result that

$$\mathrm{Re}\,\sigma(\omega>0)\approx\text{const},\quad\Gamma(\omega)\propto\omega^2 \qquad (5)$$

as the four velocities sum up to a finite value of the order of v_F in this case. The constant "incoherent background" corresponds according to Eq. (2) to a scattering rate $\Gamma(\omega)\propto\omega^2$ characteristic of a Fermi liquid with umklapp scattering in two or three dimensions.

Less well known is the corresponding result for a small Fermi surface ($k_F<G/4$) where umklapp scattering at the Fermi surface is not possible. Here the situations in two and three dimensions are quite different. For a generic (not too complex) Fermi surface in two dimensions, momentum conservation in the limit $\omega\to 0$ can only be fulfilled by choosing $\mathbf{k}_1=-\mathbf{k}_2$ and $\mathbf{k}_3=-\mathbf{k}_4$ (or $\mathbf{k}_1=\mathbf{k}_{3/4}$ and $\mathbf{k}_2=\mathbf{k}_{4/3}$). Therefore the sum of the velocities also vanishes linearly in ω for $\omega\to 0$ and one obtains from power counting

$$\mathrm{Re}\,\sigma(\omega>0)\propto\omega^2,\quad\Gamma(\omega)\propto\omega^4 \qquad (6)$$

for a small Fermi surface in $d=2$.

The situation is quite different for a system with a small Fermi surface in three dimensions, where momentum conservation on the Fermi surface does *not* require that the relevant moments are located opposite to each other. Therefore the sum of the four velocities in Eq. (4) will generically *not* vanish and one finds

$$\mathrm{Re}\,\sigma(\omega>0)\approx\text{const},\quad\Gamma(\omega)\propto\omega^2 \qquad (7)$$

for a small Fermi surface in $d=3$: Even *without* umklapp processes the scattering rate varies as $\Gamma(\omega)\propto\omega^2$! Note that the frequency and temperature dependence of the conductivity are drastically different in this case. $\Gamma(T)$ does *not* vary as T^2 but the two-particle scattering rate is exponentially suppressed; multiparticle processes lead to a power law $\Gamma\propto T^{2n-2}$ where the integer n depends on the size of the Fermi surface,[8] $n\sim G/(2k_F)$. The disparate behavior of $\Gamma(\omega,T=0)\sim U^2\omega^2$ and $\Gamma(\omega=0,T)\sim U^2e^{-\Delta/T}+U^nT^{2n-2}$ can easily be understood once one realizes that, in the absence of umklapp scattering, on the one hand the current is not conserved while on the other hand the momentum is conserved. As explained in detail, e.g., in Refs. 8 and 9, the component of the current "perpendicular" to the momentum does decay rapidly giving rise to the frequency independent incoherent background of Eq. (7). The dc conductivity is, however, determined by the long-time decay of the component of the current "parallel" to the momentum and therefore by the decay rate of the momentum, i.e., by umklapp processes which are very rare for small Fermi surfaces. It is likely that the rather general results, Eqs. (6) and (7), have been discussed before in the literature but we are not aware of a directly relevant reference.

It should be clear from the discussion of Eq. (7) given above that also in the *presence* of umklapp scattering, when $\Gamma(\omega,T)\approx a(k_BT)^2+b(\hbar\omega)^2$, there is in general no simple relation between the constants a and b. We emphasize this fact as in the experimental literature such a relation has sometimes been claimed to exist[1,10,11] but is actually not observed.[11,12] Note that recent progress in the experimental methods allows precise measurements of the optical conductivity at low frequencies and temperatures.[12]

SUPERCONDUCTORS

We now turn to the calculation of the $T=0$ optical conductivity in superconductors neglecting again phonons and impurities. Our main interest is the case of a d-wave superconductor in $d=2$ on a square lattice with unit lattice spacing. To describe the superconducting state we use weakly interacting Bogoliubov quasiparticles (QPs), $d^\dagger_{\mathbf{k}\sigma}=u_\mathbf{k}c^\dagger_{\mathbf{k}\sigma}-\sigma v_\mathbf{k}c_{-\mathbf{k}\bar\sigma}$, which diagonalize the BCS Hamiltonian $H_{\mathrm{BCS}}=\Sigma_{\mathbf{k}\sigma}\varepsilon_\mathbf{k}c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+\Sigma_\mathbf{k}\Delta^*_\mathbf{k}c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}+\mathrm{H.c.}=\Sigma_{\mathbf{k}\sigma}E_\mathbf{k}d^\dagger_{\mathbf{k}\sigma}d_{\mathbf{k}\sigma}$ where $c^\dagger_{\mathbf{k}\sigma}$ is the electron creation operator and $E_\mathbf{k}=\sqrt{\varepsilon^2_\mathbf{k}+\Delta^2_\mathbf{k}}$ the BCS energy. The electric current is given by

$$\mathbf{J}=\sum_{\mathbf{k}\sigma}\mathbf{v}_\mathbf{k}c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}=\sum_{\mathbf{k}\sigma}\mathbf{v}_\mathbf{k}d^\dagger_{\mathbf{k}\sigma}d_{\mathbf{k}\sigma}, \tag{8}$$

where it is important to realize that it is the *bare* velocities $\mathbf{v}_\mathbf{k}=d\varepsilon_\mathbf{k}/d\mathbf{k}$ rather than $dE_\mathbf{k}/d\mathbf{k}$ that enter if the current is expressed in terms of the BCS quasiparticles.

Within the BCS approximation the current is conserved, $[J,H_{BCS}]=0$, and there is no optical weight at finite frequencies. To calculate the optical conductivity it is therefore essential to include the interaction of the quasiparticles. The Hamiltonian for the QPs is given by $\mathcal{H}=H_{\mathrm{BCS}}+H_{\mathrm{int}}$, where the (properly normal ordered) local density-density interaction $H_{\mathrm{int}}=2U\Sigma_i n_{i\downarrow}n_{i\uparrow}$ can be rewritten as

$$\begin{aligned}H_{\mathrm{int}}=U\sum r_{13}r_{24}d^\dagger_{4\uparrow}d^\dagger_{3\downarrow}d^\dagger_{2\downarrow}d^\dagger_{1\uparrow}+\mathrm{H.c.}+2\tilde r_{12}r_{34}d^\dagger_{4\uparrow}d^\dagger_{3\downarrow}d^\dagger_{2\sigma}d_{1\sigma}\\+\mathrm{H.c.}+r_{12}r_{34}d^\dagger_{4\sigma}d^\dagger_{3\bar\sigma}d_{2\bar\sigma}d_{1\sigma}+\tilde r_{14}\tilde r_{23}d^\dagger_{4\sigma}d^\dagger_{3\sigma'}d_{2\sigma'}d_{1\sigma},\end{aligned} \tag{9}$$

where $r_{ij}=r_{ji}=u_1v_2+v_1u_2$, $\tilde r_{ij}=\tilde r_{ji}=u_1u_2-v_1v_2$, and $i\equiv\mathbf{k}_i$. The momentum sums conserve crystal momentum and the spin sums are *only* over repeated indices. This expression can be derived by keeping the fluctuations around mean-field theory in the BCS approach. The various terms describe not only the scattering of quasiparticles (and holes) but also the breaking up and recombination of Cooper pairs.

While $[J,H_{\mathrm{BCS}}]=0$, the current J decays in the presence of the interactions between QPs, $\partial_t J=-i[J,H_{\mathrm{int}}]$. It is now straightforward (albeit somewhat tedious) to evaluate the contributions to Eq. (3) to lowest order in the interactions, and we obtain

$$\mathrm{Re}\,\sigma(\omega)=\frac{\pi U^2}{\omega^3}[\phi''_{\mathrm{pp}}(\omega)+\phi''_{\mathrm{pq}}(\omega)+\phi''_{\mathrm{qq}}(\omega)-(\omega\leftrightarrow-\omega)],$$

$$\begin{aligned}\phi''_{\mathrm{pp}}(\omega)=\sum_{1234\mathbf{G}}&(r_{12}r_{34}-r_{13}r_{24})^2\delta_{1+2+3+4-\mathbf{G}}(v^x_1+v^x_2+v^x_3+v^x_4)^2\\&\times[(1-f_1)(1-f_2)(1-f_3)(1-f_4)-f_1f_2f_3f_4]\\&\times\delta[\omega-(E_1+E_2+E_3+E_4)],\end{aligned}$$

$$\begin{aligned}\phi''_{\mathrm{pq}}(\omega)=4\sum_{1234\mathbf{G}}&(\tilde r_{12}r_{34}-\tilde r_{14}r_{23})^2(v^x_1-v^x_2-v^x_3-v^x_4)^2\\&\times\delta_{1-2-3-4+\mathbf{G}}[f_1(1-f_2)(1-f_3)(1-f_4)\\&-(1-f_1)f_2f_3f_4]\delta[\omega-(-E_1+E_2+E_3+E_4)],\end{aligned}$$

$$\begin{aligned}\phi''_{\mathrm{qq}}(\omega)=\sum_{1234\mathbf{G}}&(1-f_1)(1-f_2)f_3f_4(v^x_4+v^x_3-v^x_2\\&-v^x_1)^2\delta_{1+2-3-4+\mathbf{G}}\times[4(r_{12}r_{34}+\tilde r_{14}\tilde r_{23})^2+2(\tilde r_{14}\tilde r_{23}\\&-\tilde r_{13}\tilde r_{24})^2]\delta[\omega-(E_1+E_2-E_3-E_4)].\end{aligned} \tag{10}$$

The first (second, third) contribution comes from the first (second, last two) scattering terms in the Hamiltonian (9). At zero temperature, obviously only the first term $\phi_{\mathrm{pp}}(\omega)$ survives as all QPs have positive energies and $f_i=f(E_{\mathbf{k}_i})=0$ at $T=0$.

In the case of an ultraclean s-wave superconductor, a direct consequence of Eq. (10) is that the gap in the optical conductivity (ignoring phonons) is of size 4Δ while it is 2Δ for dirty superconductors[2] as has previously been noted by Orenstein *et al.*[3] Obviously one has to ask whether this result will also hold to higher order in perturbation theory. To answer this question, one has to investigate whether symmetries and corresponding selection rules allow for an optical transition from the ground state of the superconductor to a two-quasiparticle excited state by an operator of the form $\Sigma_{\mathbf{k}\mathbf{k}'}\alpha^{\sigma\sigma'}_{\mathbf{k}\mathbf{k}'}\tilde d^\dagger_{\mathbf{k}\sigma}\tilde d^\dagger_{-\mathbf{k}'\sigma'}$ where $\tilde d^\dagger$ are the creation operators of the fully renormalized "true" quasiparticles of the system (which can only be identified with the BCS quasiparticles for weak interactions). Symmetries strongly restrict the form of $\alpha^{\sigma\sigma'}_{\mathbf{k}\mathbf{k}'}$. Translational invariance on the lattice, for example, implies that $\alpha^{\sigma\sigma'}_{\mathbf{k}\mathbf{k}'}=\alpha^{\sigma\sigma'}_\mathbf{k}\delta(\mathbf{k}-\mathbf{k}')$ in the absence of impurities with $\alpha^{\sigma\sigma'}_\mathbf{k}=-\alpha^{\sigma'\sigma}_{-\mathbf{k}}$ as the quasiparticles are fermions. If the superconductor does not break time-reversal invariance one has $(\alpha^{\uparrow\uparrow}_\mathbf{k})^*=(\alpha^{\downarrow\downarrow}_\mathbf{k})$ and $(\alpha^{\uparrow\downarrow}_\mathbf{k})^*=-(\alpha^{\uparrow\downarrow}_\mathbf{k})$ and in a crystal with inversion symmetry one has $\alpha^{\sigma\sigma'}_\mathbf{k}=\alpha^{\sigma'\sigma}_\mathbf{k}$. In the absence of spin-orbit coupling, i.e., if spins are rotationally invariant one finds that $\alpha^{\uparrow\downarrow}_\mathbf{k}=-\alpha^{\downarrow\uparrow}_\mathbf{k}$ and $\alpha^{\sigma\sigma}_\mathbf{k}=0$. From this we can conclude that, in the absence of disorder and in the presence of inversion symmetry, $\alpha_{\mathbf{k}}^{\sigma\sigma'}$ vanishes and the optical gap is therefore 4Δ for an s-wave superconductor in the *absence* of spin-orbit coupling. In the presence of impurities, however, the gap[2] is only 2Δ. Interestingly, the symmetry analysis suggests that even in a generic inversion-symmetric clean crystal, high-order spin-orbit processes could possibly induce relevant low-energy processes not included in Eq. (10) which lead to a gap of size 2Δ. All the low-order results presented below may therefore not be valid in the presence of sizable spin-orbit coupling. Note also that phonons and other low-energy collective modes with energies smaller than 2Δ can induce optical weight in the frequency window $2\Delta<\omega<4\Delta$. The precise functional form of the optical conductivity of an s-wave superconductor for $\omega \gtrsim 4\Delta$ will not be discussed in detail here. It depends on the dimension and on the angular dependence of Δ. Generically the onset will be smooth and of the form $(\omega-4\Delta)^2$. Therefore a precise experimental determination of Δ using a feature close to 4Δ will be rather difficult. For all conventional s-wave superconductors we anyhow expect that impurity scattering will dominate even for the cleanest available samples leading to the well-known 2Δ gap which is much easier to detect.

A d-wave superconductor in two dimensions (as realized in high-temperature superconductors) with point nodes along the diagonals of the quadratic Brillouin zone has a vanishing gap in nodal direction. For frequencies small compared to the maximal gap $\Delta, \omega \ll \Delta$, all QPs are created in the vicinity of the nodes, so we expand the dispersion around them. Writing $\mathbf{k}=\mathbf{k}_{\text{node}}+\boldsymbol{\kappa}$ the most generic band structure consistent with the square symmetry of the lattice is

$$\varepsilon_{\boldsymbol{\kappa}}=\frac{v_F}{\sqrt{2}}(\kappa_x+\kappa_y)+\frac{1}{2m^*}(\kappa_x^2+\kappa_y^2)+D\kappa_x\kappa_y+L(\kappa_x^3+\kappa_y^3) + F\kappa_x\kappa_y(\kappa_x+\kappa_y)+\mathcal{O}(\kappa^4), \tag{11}$$

where the constants D,L,F determine the deviation of the dispersion from that of a free-electron gas and m^* is an effective mass.

There are four qualitatively different terms that appear in the sum for ϕ''_{pp} in Eq. (10), which are sketched in Fig. 1: (i) $\mathbf{G}=(2\pi,2\pi)$ and hence all four QPs in one node; (ii) $\mathbf{G}=(2\pi,0)$ and two QPs in each of two "perpendicular" nodes; (iii) $\mathbf{G}=\mathbf{0}$ and one QP in each node; (iv) $\mathbf{G}=\mathbf{0}$ and two QPs in each of two opposite nodes. These give rise to very different dependences on ω and doping, as we now discuss.

The role of umklapp scattering is determined by the distance of the nodes from $(\pi/2,\pi/2)$ which we denote by

$$\delta k_{\text{node}}=\left|\mathbf{k}_{\text{node}}-\left(\frac{\pi}{2},\frac{\pi}{2}\right)\right|=\left|k_F-\frac{\pi}{\sqrt{2}}\right|. \tag{12}$$

In processes of type (i) the four QPs have very similar velocities (recall that it is the *normal-state* velocity that contributes) and so the contribution to $\phi''_{\text{pp}}(\omega)$ is large. However, it is only possible to create four QPs of arbitrarily low energy if the nodes are situated exactly at $(\pi/2,\pi/2)$; otherwise there is an excess momentum $4\delta k_{\text{node}}$ which must be carried by the QPs, so that at least one of them is situated a finite distance from the node. Accordingly, absorption can only occur for frequencies above the threshold

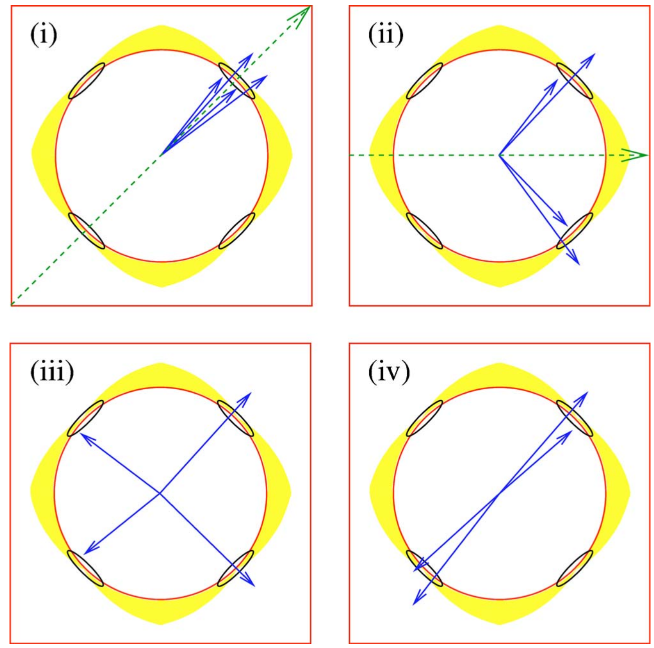

FIG. 1. (Color online) The four possible types of scattering process at $T=0$, in which four QPs are created. The circle represents the Fermi surface, the solid arrows the QP momenta, and the dashed arrows the reciprocal-lattice vector $\mathbf{G}$. The shaded region indicates the size of the superconducting gap and the ellipses a constant energy contour in the vicinity of each node.

$$\omega_0 \approx 4v_{\text{F}}\delta k_{\text{node}}. \tag{13}$$

Processes of type (ii) resemble those of type (i), since again the velocities add. However, the fact that the nodes are at right angles to one another reduces the threshold frequency as the excess momentum $(\Sigma\mathbf{k}_{\text{node}})-\mathbf{G}=(2\sqrt{2}\delta k_{\text{node}},0)=\sqrt{2}\delta k_{\text{node}}[(1,1)+(1,-1)]$ can be split into two components parallel to the Fermi surface at the nodes where the velocity of the QPs $v_\Delta=dE_k/dk_\parallel=d\Delta_k/dk_\parallel$ is much smaller. This leads to a considerably smaller threshold frequency

$$\omega_0' \approx 4v_\Delta\delta k_{\text{node}} \sim \frac{\Delta}{\varepsilon_{\text{F}}}\omega_0, \tag{14}$$

where this simplified formula is only valid if $\delta k_{\text{node}}<\Delta/v_{\text{F}}$ when corrections to the Dirac spectrum close to the nodes can be neglected. Note that the construction described above reduces the available phase space for scattering and so the contribution close to ω_0' is smaller than that of type (i) processes by a factor of v_Δ/v_{F}.

In most realistic situations (including most of the cuprates) the point node will *not* be located close to $(\pi/2,\pi/2)$ and δk_{node} will be larger than Δ/v_F. In this case the gap for umklapp processes will depend on details of the band structure. For sufficiently large Fermi surfaces (e.g., optimally doped Bi-2212 according to Ref. 13, see also Fig. 2), the gap

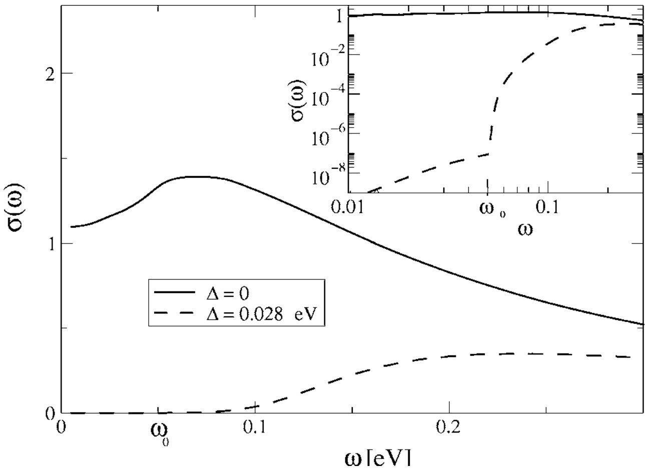

FIG. 2. Optical conductivity, Re $\sigma(\omega>0)$, for an ultraclean d-wave superconductor (dashed line) using the band structure of optimally doped Bi-2212 taken from Ref. 13 and a d-wave gap of size $\Delta=0.028$ eV (arbitrary units on the y axis). For reference, the solid line shows the $T=0$ optical conductivity in the normal state ($\Delta=0$) which is constant for low frequencies due to umklapp processes; see Eq. (5). The δ peak at $\omega=0$ is not shown. For lowest frequencies, $\omega<\omega_0$, one finds $\sigma\propto\omega^4$ in the superconducting phase as can be seen on the logarithmic scale of the inset. This regime is, however, practically not observable due to the small prefactor. Instead one finds a very smooth onset [see Eq. (21)] for $\omega>\omega_0\approx 0.05$ eV $<2\Delta$ Eq. (15). Note that the nodes are *not* close to $(\pi/2,\pi/2)$ for the band structure considered in this figure.

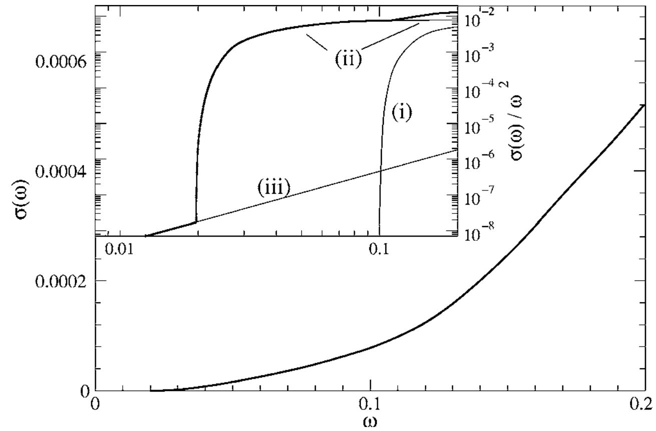

FIG. 3. Zero-temperature optical conductivity, Re $\sigma(\omega>0)$, for an ultraclean d-wave superconductor from a numerical evaluation of Eq. (10) for a model with nodes close to $(\pi/2,\pi/2)$ [$\Delta=0.31$, $v_F=1$, $v_F/v_\Delta=5$, $\omega_0=0.1$, and $\omega_0'=0.02$]. Inset: logarithmic plot of Re $\sigma(\omega>0)/\omega^2$. The thin lines show the contributions from the various processes shown in Fig. 1. Processes of type (i) and (ii) are gapped by ω_0 and ω_0', respectively. Due to the smooth onset of type-(i) umklapp processes, Eq. (20), there is almost no feature at ω_0. The numerical results are fully consistent with the power laws of Eqs. (17)–(21).

for umklapp processes in a d-wave superconductor will be smaller than 2Δ,

$$\omega_0 < 2\Delta, \qquad (15)$$

as typically an umklapp process will exist where two QPs are located at the nodes and the two other somewhere else on the Fermi surface.

To obtain the frequency dependence, we ignore in a first step the coherence factors and velocity prefactors and evaluate the integral

$$\sum_{1,2,3} \delta[\omega-(E_1+E_2+E_3+E_{-(1+2+3)})] \approx \frac{c_i\omega^5}{(v_F v_\Delta)^3} \qquad (16)$$

for $\delta k_{\text{node}}=0$ and small ω in each of the four cases ($i=1$–4) shown in Fig. 1. This can be done by scaling the momenta perpendicular and parallel to the Fermi surface at the node by ω/v_F and ω/v_Δ, respectively. In cases (i) and (iv) shown in Fig. 1, when all nodes are parallel to each other, c_1 and c_4 are constants of order 1. The situation is slightly more complicated in the cases (ii) and (iii) where by choosing a proper rescaling procedure we find $c_2\sim c_3\sim v_\Delta/v_F$. Equation (16) does not include the effect of the velocity prefactor $(\Sigma v_i^x)^2$ and of the combination $(r_{12}r_{34}-r_{13}r_{24})^2$ of coherence factors in Eq. (10). At the nodes, the coherence factors

$$(1/\sqrt{2})\sqrt{1\pm\frac{\varepsilon_{\mathbf{k}}}{E_{\mathbf{k}}}}$$

are rapidly varying functions of order 1. For $\delta k_{\text{node}}=0$ they change the result only quantitatively but not qualitatively (as we have checked numerically) but can become important for $\delta k_{\text{node}}\neq 0$ as discussed below. For the umklapp processes (i) and (ii) the velocities just add up to give a finite prefactor of order v_F^2. If the nodes are located at $(\pi/2,\pi/2)$, we therefore obtain

$$\sigma(\omega) \propto \frac{U^2}{v_F v_\Delta^3}\omega^2, \quad \text{type (i) for } \delta k_{\text{node}}=0 \qquad (17)$$

and similarly

$$\sigma(\omega) \propto \frac{U^2}{v_F^2 v_\Delta^2}\omega^2, \quad \text{type (ii) for } \delta k_{\text{node}}=0. \qquad (18)$$

These power laws can also be observed for $\omega\gg\omega_0,\omega_0'$ if δk_{node} is finite but small. This can be seen in the inset of Fig. 3 which discusses the various regimes based on a numerical evaluation of Eq. (10).

Due to momentum conservation, the leading contribution to Σv_i^x vanishes for the nonumklapp processes (iii) and (iv). But band-structure effects break Galilean invariance and one obtains a low-energy contribution even in the absence of umklapp. The leading term is given by $D\kappa_x\kappa_y$ in Eq. (11) which leads to $v_x=D\kappa_y$. Although the sum $\Sigma_i v_i^x$ still vanishes at this order for processes of type (iv), it remains finite if the geometry is determined by (iii) and we obtain from a scaling analysis

$$\sigma(\omega) \propto \frac{U^2D^2}{v_F^6 v_\Delta^2}\omega^4, \quad \text{type (iii) for } \omega\to 0. \qquad (19)$$

While this term is suppressed by the tiny factor $\omega^2 v_\Delta/\varepsilon_F^2 v_F$ compared to Eq. (17), it is nevertheless the leading $\omega\to 0$ correction when the nodes are located *away* from $(\pi/2,\pi/2)$. Equation (19) therefore describes the typical low-frequency optical conductivity of a two-dimensional d-wave superconductor in the absence of impurities (cf. insets of Figs. 2 and 3). Processes from the scattering geometry (iv) are always subleading and only give rise to contributions $\propto\omega^6$. It is worth noting that the prefactor of Eq. (19)—not shown in the equation—turns out to be numerically very small, approximately a factor of 20 smaller than the prefactor of Eq. (17) and more than a factor of 100 smaller than the corresponding numerical prefactor of Eq. (18) if we assume a local interaction U. In general completely different matrix elements enter the various scattering processes (i)–(iv) and therefore their relative magnitude depends on details of the relevant interactions. But the smallness of the contribution may imply that in actual measurements the low-frequency ω^4 regime is never observable; see Figs. 2 and 3.

As the nonumklapp contribution (19) to the optical conductivity is very small and difficult to detect experimentally, it is worthwhile to investigate the precise form of the onset of umklapp terms at $\omega > \omega_0, \omega_0'$ in the generic case when the nodes are not located at $(\pi/2, \pi/2)$. Consider, for example, the scattering geometry (i) in Fig. 1. At $\omega=\omega_0$ the components $\kappa_\parallel$ of all four momenta *parallel* to the Fermi surface will be zero, so $\varepsilon_{\mathbf{k}}=E_k$ and therefore the coherence factors $(r_{12}r_{34}-r_{13}r_{24})^2$ of Eq. (10) will vanish. As a consequence the onset of umklapp processes will be very smooth and of the form

$$\sigma(\omega) \propto \frac{U^2}{v_F v_\Delta^3}(\omega-\omega_0)^2, \quad \text{type (i) for } \omega \gtrsim \omega_0 \qquad (20)$$

and

$$\sigma(\omega) \propto \frac{U^2}{v_F^2 v_\Delta^2}(\omega-\omega_0')^2, \quad \text{type (ii) for } \omega \gtrsim \omega_0' \qquad (21)$$

as we have checked numerically; see Figs. 2 and 3. Formulas for ω_0 and ω_0' are given in Eqs. (13)–(15) above. The prefactors in Eqs. (20) and (21) are only valid for very small $\omega_0, \omega_0' \ll k_{\mathrm{F}} v_\Delta^2 / v_{\mathrm{F}}$ when one can use a Dirac spectrum for the nodal quasiparticles; however, the frequency dependence close to the onset frequency is also quadratic for larger values of ω_0 and ω_0' as we have again checked numerically for example in Fig. 2 which shows the optical conductivity in a model which uses the band structure[13] of Bi-2212.

All results shown above rely on the fact that at lowest energies the nodal dispersion takes the form of a Dirac cone, $E_{\mathbf{k}}=\sqrt{(v_F k_\perp)^2+(v_\Delta k_\parallel)^2}$. But already at a very low energy scale, $E_c=m^* v_\Delta^2/2 \sim \Delta^2/\varepsilon_{\mathrm{F}} \ll \Delta$, one has to take into account the curvature of the Fermi surface which bends contours of equal energy into a banana shape. It is therefore important to check which of the results calculated above remain unmodified at this crossover scale—the existence of such a small energy scale will otherwise make the experimental determination of power laws extremely difficult. Fortunately, it turns out that our results in the scattering geometry (ii) and (iii), i.e., Eqs. (18) and (19), are not affected by E_c and remain valid up to energies of the order of the maximal gap Δ. This can most easily be seen by rewriting momentum conservation in polar coordinates while scaling $k-k_F$ with ω/v_F and the polar angle ϕ with $\omega/(k_F v_\Delta)$. Using the same analysis for geometry (i), one finds a crossover at the energy E_c and Eq. (17) has to be multiplied by a factor E_c/ω for $E_c \ll \omega \ll \Delta$.

CONCLUSIONS

The frequency dependence of the optical conductivity at zero temperature and finite frequencies describes how the electrical current can decay. The example of a Fermi liquid with a small Fermi surface shows that the temperature and frequency dependencies of $\sigma(\omega, T)$ have very little in common and may result from completely different processes. The zero-temperature optical conductivity of d-wave superconductors turns out to be rather complex even for frequencies much smaller than the maximal gap. While we hope that our calculation can serve as a reference for the interpretation of the incoherent background, a direct observation of the predicted power laws will be difficult as the calculated contributions turn out to be both small in size and very smooth in their frequency dependence (see Figs. 2 and 3). Therefore it will be very difficult even in very clean crystals to separate the predicted effects from the effects of elastic impurity scattering.

An interesting open question is whether and how spin-orbit interactions modify the results presented in this paper. Based on a symmetry analysis, we argued that spin-orbit interactions can open new channels for current relaxation in a superconductor—even in the presence of inversion symmetry. Neglecting such relativistic effects, we believe that our results are valid even in strongly interacting superconductors at sufficiently low frequencies when multiparticle scattering is suppressed due to phase-space restrictions. This will also be the case if the interactions are mediated, e.g., by (short-ranged) spin-fluctuations,[4,5] provided the system is not located directly at a quantum-critical point.

At small but finite temperatures thermal excitations induce a characteristic sharp peak in the low-frequency optical conductivity. The calculation of this prominent feature in a d-wave superconductor taking into account the relevant vertex corrections and approximate conservation laws[8,9] is left as a challenge for the future—while the $T=0$ results of this paper can provide a reference for this calculation, the simple methods used here will not be sufficient to describe the finite-temperature regime.

ACKNOWLEDGMENTS

We thank M. Grüninger, P. Hirschfeld, D. van der Marel, J. Orenstein, M. Scheffler, and P. Wölfle for useful discussions and the SFB 608 and the Emmy Nöther program of the DFG for financial support.

[1] M. Dressel and G. Grüner, *Electrodynamics of Solids* (Cambridge University Press, Cambridge, England, 2002).
[2] D. C. Mattis and J. Bardeen, Phys. Rev. **111**, 412 (1958).
[3] J. Orenstein, S. Schmitt-Rink, and A. E. Ruckenstein, in *Electronic Properties of High-T_c Superconductors*, edited by H. Kuzmany (Springer, Berlin, 1990), p. 254.
[4] J. P. Carbotte and E. Schachinger, Phys. Rev. B **69**, 224501 (2004); E. Schachinger and J. P. Carbotte, *ibid.* **65**, 064514 (2002).
[5] S. M. Quinlan, P. J. Hirschfeld, and D. J. Scalapino, Phys. Rev. B **53**, 8575 (1996); P. J. Hirschfeld, W. O. Putikka, and D. J. Scalapino, *ibid.* **50**, 10250 (1994).
[6] W. Götze and P. Wölfle, Phys. Rev. B **6**, 1226 (1972); R. Zwanziger, in *Lectures in Theoretical Physics* (Interscience, New York, 1961), Vol. 3; H. Mori, Prog. Theor. Phys. **34**, 423 (1965).
[7] D. Pines and P. Nozières, *The Theory of Quantum Liquids* (Benjamin, New York, 1966), Vol. 1.
[8] A. Rosch and N. Andrei, J. Low Temp. Phys. **126**, 1195 (2002).
[9] A. Rosch and N. Andrei, Phys. Rev. Lett. **85**, 1092 (2000); E. Shimshoni, N. Andrei, and A. Rosch, Phys. Rev. B **68**, 104401 (2003).
[10] L. Degiorgi, Rev. Mod. Phys. **71**, 687 (1999).
[11] P. E. Sulewski, A. J. Sievers, M. B. Maple, M. S. Torikachvili, J. L. Smith, and Z. Fisk, Phys. Rev. B **38**, 5338 (1988); C. Julien, J. Ruvalds, A. Virosztek, and O. Gorochov, Solid State Commun. **79**, 875 (1991); P. Tran, S. Donovan, and G. Grüner, Phys. Rev. B **65**, 205102 (2002).
[12] M. Scheffler, Ph.D. thesis, University of Stuttgart, Stuttgart, 2004.
[13] M. R. Norman, M. Randeria, H. Ding, and J. C. Campuzano, Phys. Rev. B **52**, 615 (1995).