Continuum model for the vibration of multilayered graphene sheets

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The vibration analysis of multilayered graphene sheets (MLGSs) using a continuum model is reported in this paper. An explicit formula is derived to predict the van der Waals (vdW) interaction between any two sheets of a MLGS. Based on the derived formula, a continuum-plate model is developed for the vibration of MLGSs. Our investigation indicates that the lowest natural frequency (classical natural frequency) of a MLGS for a given combination of m and n is independent of the vdW interaction, but that all of the other higher natural frequencies (resonant frequencies) are significantly dependent on this interaction. The mode shapes that are associated with the natural frequencies are investigated for double-layered and ten-layered graphene sheets. We find that the vibration modes that are associated with the classical natural frequency of all the sheets are in the same direction and have the same amplitude, whereas the vibration modes of the sheets that are associated with the resonant frequencies are different due to the influence of the vdW interaction. Thus various resonance modes can be obtained by varying the number of layers of a MLGS.

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I. INTRODUCTION

Recently, many continuum models have been proposed for the study of carbon nanotubes (CNTs). These include the atomistic-based continuum theory[1–3] for the mechanical properties of CNTs, the Euler-Bernoulli beam theory[4–6] for the bending and the critical buckling load, elastic cylindrical shell models[6–8] for the axial compression buckling and torsional buckling, space truss/frame models[9,10] for the Young's and shear moduli and the equivalent wall thickness, and the finite element technique,[11–13] which links the conventional finite element method with the atomistic-based potential for the bending and axial compression of CNTs. The results that have been obtained from these continuum models show a good agreement with experimental results or molecular dynamics simulations of single-walled carbon nanotubes (SWNTs), which indicates that with the suitable modification, conventional continuum mechanics can obtain results that are as accurate as molecular-dynamics simulation, but that are much more efficient, especially for large-scale simulations. It should be mentioned that most of these continuum models focus on SWNTs. For multiwalled carbon nanotubes (MWNTs), which are composed of several layers of carbon tubes, the establishment of a continuum model is more difficult, because van der Waals (vdW) forces occur between the tubes. Finding an appropriate description for the vdW force is the critical challenge to establishing a continuum model for the analysis of MWNTs. To investigate the mechanical behavior of double-walled CNTs, Ru[14] proposed a continuum model for the vdW interaction between two adjacent layers, and applied the developed vdW force model to buckling and vibration analyses[14–19] using the elastic shell and beam theory. He obtained an explicit formula for the critical axial strain of a double-walled CNT, and his vibration analysis shows that the vibration mode that is associated with the fundamental frequency is coaxial, whereas the vibration modes that are associated with the other higher natural frequencies are noncoaxial. Li and Chou[20–22] investigated the elastic and vibrational behavior of double-walled CNTs using the finite element method, in which the interlayer vdW forces are modeled by a nonlinear truss rod. Their simulations show that ultrahigh-frequency nanomechanical resonators can be achieved by using a MWNT. In addition, Pantano *et al.*[23] carried out a finite element simulation on the bending and buckling of a MWNT, in which the interlayer vdW force was treated as a function of the interlayer distance by using the Lennard-Jones potential. More recently, the authors[24] of this paper derived two explicit formulas for the interlayer vdW force between two adjacent layers and any two layers of a MWNT. These formulas indicate clearly the dependence of the interlayer vdW force (before and after buckling) on both the change in interlayer spacing and the tube radius. Based on these two formulas for the interlayer vdW force, a continuum algorithm was established for the buckling of a MWNT.

In contrast to CNTs, very limited work has been done on analyzing the mechanics of graphene sheets (GSs) using a continuum method, despite the fact that graphite possesses many superior properties,[25] such as good flexibility, low thermal expansion, and high electrical and thermal conductivity. Dubay and Kresse[26] calculated a set of force constants using *ab initio* density-functional theory, and then calculated the phonon dispersion relation of graphite using these force constants. Their numerical results are in reasonable agreement with the experimental results. Xu and Liao[27] studied the elastic response of a circular single-layered GS under a transverse central load using molecular dynamics, the closed-form elasticity solution, and the finite element method. Their simulation gave consistent predictions for the elastic deformation of a GS using molecular dynamics and conventional continuum mechanics. Graphite is composed of multiple layers of GSs that are attracted to each other through the vdW force. It has been reported[28] that single-layered GSs can be detected in carbon nanofilm, but so far single-layered GSs have not been separable from graphite. In this work, an explicit formula is derived for the vdW interaction between any two layers of a MLGS. Based on the derived formula for the vdW interaction, an efficient algorithm is established for the

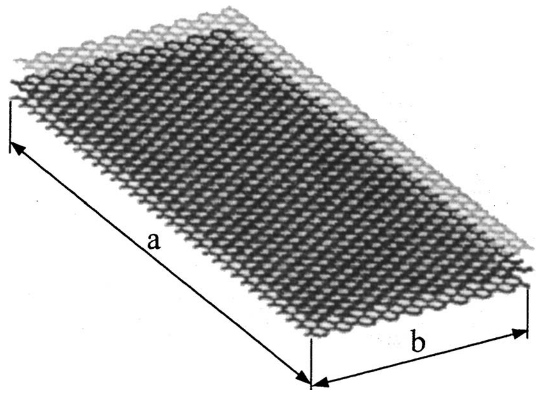

FIG. 1. A continuum plate model of a multilayered graphene sheet.

vibration analysis of MLGSs, in which an individual layer is modeled as a classical thin plate. Our numerical simulation for a double-layered GS shows that the two sheets resonate in opposite directions when the sheet is excited at the resonant frequency. This resonance characteristic is useful for the separation of a MLGS into individual single-layered sheets.

II. MODEL DEVELOPMENT

Consider a MLGS that consists of two or more single-layered GSs, as is shown in Fig. 1. The length of each sheet is a, the width is b, the thickness is h, the mass density is ρ, and the Young's modulus is E. We assume that the interlayer friction between any two adjacent layers is negligible. The governing equations for the vibration of a MLGS can therefore be derived as the N coupled equations, i.e.,

$$D\nabla^4 w_1 + \rho h \frac{\partial^2 w_1}{\partial t^2} = q_1$$

$$D\nabla^4 w_2 + \rho h \frac{\partial^2 w_2}{\partial t^2} = q_2$$

$$\vdots$$

$$D\nabla^4 w_N + \rho h \frac{\partial^2 w_N}{\partial t^2} = q_N, \tag{1}$$

where w_i $(i=1,2,\ldots,N)$ is the deflection of the ith sheet, q_i is the pressure that is exerted on sheet i due to the vdW interaction between layers, and D is the bending stiffness of the individual sheet. Note that the attractive vdW force that is obtained from the Lennard-Jones pair potential[29,30] is negative, the repulsive vdW force is positive, and the downward pressure is assumed to be positive in Eq. (1). As only infinitesimal vibration is considered, the net pressure due to the vdW interaction is assumed to be linearly proportional to the deflection between two layers, i.e.,

$$q_i = \sum_{j=1}^{N} c_{ij}(w_i - w_j) = w_i \sum_{j=1}^{N} c_{ij} - \sum_{j=1}^{N} c_{ij} w_j, \tag{2}$$

where N is the total number of layers of the MLGS.

The vdW interaction coefficients can be obtained through the Lennard-Jones pair potential,[24]

$$c_{ij} = -\left(\frac{4\sqrt{3}}{9a}\right)^2 \frac{24\varepsilon}{\sigma^2}\left(\frac{\sigma}{a}\right)^8 \left[\frac{3003\pi}{256}\sum_{k=0}^{5}\frac{(-1)^k}{2k+1}\binom{5}{k} \times \left(\frac{\sigma}{a}\right)^6 \frac{1}{(\bar{z}_i - \bar{z}_j)^{12}} - \frac{35\pi}{8}\frac{35\pi}{8}\sum_{k=0}^{2}\frac{(-1)^k}{2k+1}\frac{1}{(\bar{z}_i - \bar{z}_j)^6}\right], \quad i \neq j. \tag{3}$$

It is obvious from Eq. (3) that $c_{ij}=c_{ji}$, and that the pressure is caused by the vdW interaction between all of the layers, rather than between two adjacent layers only.

Suppose that all of the edges are simply supported, then the deflection of all of the layers can be approximated by a periodic solution of the form

$$w_k(x,y,t) = A_k \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}e^{i\omega t}, \tag{4}$$

where $i=\sqrt{-1}$, ω is the frequency of natural vibration, $A_k(k=1,2,\ldots,N)$ are N unknown coefficients, a is the length and b the width of the GS, and m and n are the half wave numbers in the direction of x and y, respectively.

The substitution of Eqs. (2) and (4) into Eq. (1) gives

$$\left\{D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - \sum_{j=1}^{N} c_{kj} - \rho h\omega^2\right\}A_k + \sum_{j=1}^{N} c_{kj}A_j = 0 \quad (k=1,2,\ldots,N). \tag{5}$$

The natural frequency can then be obtained by solving the eigenvalue equation

$$[\rho h\omega^2 \mathbf{I}_{N\times N} - \mathbf{H}_{N\times N}]\begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{Bmatrix} = 0, \tag{6}$$

where $\mathbf{I}$ is an identity matrix and the elements in the matrix $\mathbf{H}$ are

$$h_{ij} = c_{ij}, \quad i \neq j, \tag{7}$$

and

$$h_{ii} = D\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - \sum_{\substack{j=1 \\ j\neq i}}^{N} c_{ij}. \tag{8}$$

Obviously, the natural frequencies that are obtained from Eq. (6) are relative to the wave numbers of m and n, and especially in the case of a double-layered GS, the existence condition for a nonzero solution of A_1 and A_2 leads to two sets of explicit formulas for the natural frequencies

TABLE I. Classical natural frequencies and resonant frequencies (THz) of a square five-layered GS with width $b=10$ nm.

m	n	ω_1	ω_2	ω_3	ω_4	ω_5
1	1	0.069	1.135	2.193	3.052	3.604
1	2	0.173	1.146	2.198	3.056	3.608
1	3	0.346	1.184	2.219	3.071	3.620
2	1	0.173	1.146	2.198	3.056	3.608
2	2	0.276	1.166	2.209	3.064	3.614
2	3	0.449	1.219	2.237	3.084	3.632
3	1	0.346	1.184	2.219	3.071	3.620
3	2	0.449	1.219	2.237	3.084	3.632
3	3	0.622	1.292	2.278	3.114	3.657

$$\omega_1 = \sqrt{\frac{D}{\rho h}}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right] \tag{9}$$

and

$$\omega_2 = \sqrt{\frac{D}{\rho h}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^2 - 2\frac{c_{12}}{\rho h}}. \tag{10}$$

It is observed from Eq. (9) that ω_1 is the same as the natural frequency of a simply supported single-layered GS that is not relative to the vdW interaction. As $c_{12}=-108$ (GPa/nm), which is calculated from Eq. (3), a comparison of Eqs. (9) and (10) gives us $\omega_1<\omega_2$, which means that the vdW interaction between layers raises the natural frequency. This phenomenon is expected, because when a layer deflects away from the other layers, the vdW force that is created is attractive, whereas when the layer deflects toward the other layers, the vdW force is repulsive. Hence the vdW interaction always has an effect against deflection, and thus exerts a restriction on the MLGS that leads to the raising of its natural frequencies. As can be seen from Eq. (6), the number of solutions to the natural frequencies is the same as the number of layers. To distinguish the lowest natural frequency that is independent of the vdW interaction from the other higher natural frequencies that are significantly dependent on the vdW interaction for a given combination of m and n, we refer to the lowest solution ω_1 as the classical natural frequency, and to the other higher natural frequencies ω_k as the resonant frequencies. Here, $\omega_1<\omega_k$ with $k=2,3,\ldots N$.

III. DISCUSSION

Suppose that each layer has the same length a and width b. The initial interlayer separation between the two adjacent layers is assumed to be 0.34 nm. The vdW interaction coefficients c_{ij} are calculated using the derived Eq. (3), and the parameters are taken to be $\varepsilon=2.968$ meV and $\sigma=3.407$ Å.[31] It should be noted that the vdW interaction coefficient between two adjacent layers is -108 (GPa/nm), which is almost the same as that of an MWNT (Ref. 24) when the radius is large enough. To calculate the natural frequencies, each layer is modeled as an individual classical thin plate with the same length, width, and thickness. For all of the considered examples, the thickness of each GS is taken to be $h=0.34$ nm. The Young's modulus of a carbon GS $E=1.02$ TPa, the Poisson ratio $\nu=0.16$,[32] and the mass density $\rho=2250$ kg/m^3.

Let us first consider a square five-layered GS with width $b=10$ nm. It can be seen from Eq. (6) that there are five equations that give five different natural frequencies $\omega_1<\omega_2<\omega_3<\omega_4<\omega_5$ for every combination of m and n. Here, the lowest natural frequency ω_1 is the natural frequency of a single-layered GS that is independent of the vdW interaction, whereas the other four higher natural frequencies from ω_2 to ω_5 are the resonant frequencies that are caused by the vdW interaction. If the vdW interaction is ignored, then each sheet of the five-layered GS behaves like an individual sheet, and the five natural frequencies are reduced to the same value as the natural frequency of a single-layered GS. Five sets of natural frequencies from ω_1 to ω_5 are obtained from Eq. (6) for m, $n=1$, 2, 3, as is shown in Table I. It can be seen that the natural frequency $\omega_1(m,n)$ varies significantly as the mode order increases. For example, the fundamental frequency $\omega_1(1,1)=0.069$ THz and the sixth-order classical natural frequency $\omega_1(3,3)=0.622$ THz, which is nine times $\omega_1(1,1)$. For the resonant frequency $\omega_2(m,n)$, the sixth-order resonant frequency $\omega_2(3,3)=1.292$ is 1.14 times the first-order resonant frequency $\omega_2(1,1)=1.135$, which demonstrates that the effect of the mode order on the resonant frequency ω_2 cannot be neglected. However, all of the other higher resonant frequencies from ω_3 to ω_5 are insensitive to the mode order, for example, $\omega_3(3,3)/\omega_3(1,1)=1.04$, $\omega_4(3,3)/\omega_4(1,1)=1.02$, and $\omega_5(3,3)/\omega_5(1,1)=1.01$, which indicates that the vdW interaction plays a dominant role in the higher-order resonant frequencies but that the effect of the mode order can be neglected.

Table II shows the natural frequencies $\omega_1\sim\omega_{10}$ of a square ten-layered GS with width $b=10$ nm. Compared to the results for a five-layered GS, as is shown in Table I, the classical natural frequencies $\omega_1(m,n)$ are the same, and are equal to the natural frequency of a single-layered GS for any combination of (m,n), whereas the resonant frequencies $\omega_{10}(m,n)$ are very close to $\omega_5(m,n)$ of a five-layered GS. Thus the interval between any two adjacent resonant fre-

TABLE II. Classical natural frequencies and resonant frequencies (THz) of a square ten-layered GS with width $b=10$ nm.

m	n	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9	ω_{10}
1	1	0.069	0.572	1.132	1.674	2.184	2.645	3.042	3.364	3.600	3.743
1	2	0.173	0.593	1.143	1.682	2.190	2.649	3.046	3.368	3.603	3.746
1	3	0.346	0.665	1.181	1.708	2.210	2.666	3.061	3.381	3.616	3.758
2	1	0.173	0.593	1.143	1.682	2.190	2.649	3.046	3.368	3.603	3.746
2	2	0.276	0.631	1.163	1.696	2.200	2.658	3.054	3.375	3.610	3.753
2	3	0.449	0.724	1.216	1.732	2.228	2.682	3.074	3.393	3.627	3.769
3	1	0.346	0.665	1.181	1.708	2.210	2.666	3.061	3.381	3.616	3.758
3	2	0.449	0.724	1.216	1.732	2.228	2.682	3.074	3.393	3.627	3.769
3	3	0.622	0.842	1.290	1.785	2.270	2.716	3.104	3.420	3.652	3.794

quencies is only half that of a five-layered GS. It can be seen from Table II that at first the difference between the first few resonant frequencies is large; for example, the relative difference between the first two resonant frequencies $\omega_2(1,1)$ and $\omega_3(1,1)$ is 98%. As the resonant frequency increases, the difference between a pair of adjacent resonant frequencies decreases greatly; for example, the relative difference between $\omega_9(1,1)$ and $\omega_{10}(1,1)$ is reduced to 4%. As is expected, the resonant frequencies will approach a constant as the number of layers increases.

To examine the effect of the vdW interaction, the classical natural frequency ω_1 and the resonant frequency ω_2 of a double-layered GS are calculated using Eqs. (9) and (10), and are plotted in Fig. 2 for $m=1$ and various n. The values of ω_1 are the same as those of a five- or ten-layered GS for the same combination of m and n, which confirms again that the classical natural frequencies ω_1 are independent of both the vdW interaction and the number of layers of a MLGS, and can thus be calculated from Eq. (9). It is observed that the influence of the vdW interaction is most significant on the lowest order resonant frequency $\omega_2(1,1)$. As the mode order increases, the influence of the vdW interaction on the resonant frequency decreases gradually until it can be ignored altogether when the mode order is large enough.

The classical natural frequency ω_1 and the resonant frequencies from ω_2 to ω_5 of a five-layered GS are obtained from Eq. (6) for $n=1$ and various m, and are presented in Fig. 3. Similar conclusions can be obtained for the effect of the vdW interaction on the resonant frequencies. It can be seen from Figs. 2 and 3 that the difference between ω_1 and ω_2 of a double-layered GS is almost the same as the difference between ω_1 and ω_5 of a five-layered GS. The intermediate resonant frequencies, ω_2, ω_3, and ω_4, of the five-layered GS are inserted between ω_1 and ω_5 at almost the same interval, which again indicates that the more layers there are, the closer the resonant frequencies.

Figure 4 shows the first $(m=1,n=1)$ and fourth $(m=2,n=2)$ order mode shapes that are associated with the classical natural frequencies ω_1 for a square double-layered GS with width $b=10$ nm. The amplitude ratio $A_1/A_2=1$ is obtained by substituting Eq. (9) into Eq. (5). This implies that the resonances of the two sheets are always in the same direction and have the same amplitude when a double-layered GS is excited at a classical natural frequency. This is because the classical natural frequency ω_1 is independent of the vdW interaction, as can be seen from Eq. (9), and thus the two

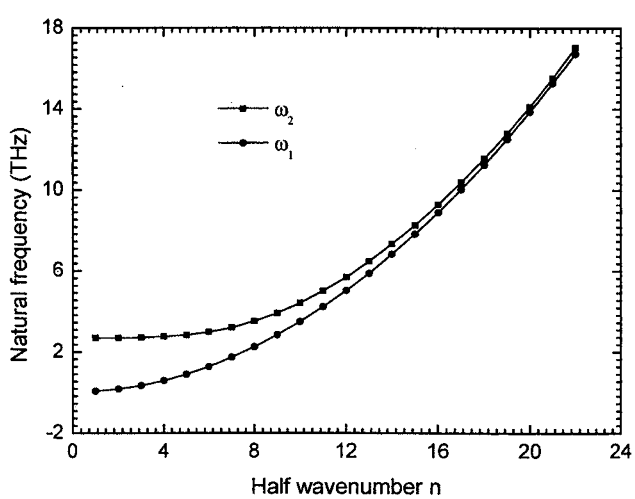

FIG. 2. Classical natural frequencies and resonant frequencies of a square double-layered GS with a half wave number $m=1$ vs a number of half wave numbers n.

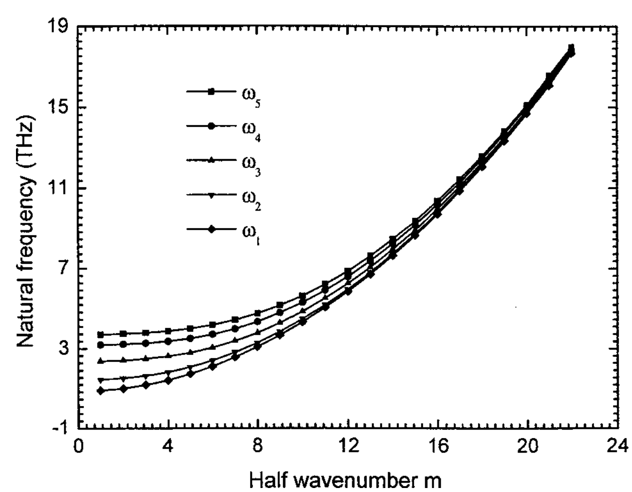

FIG. 3. Classical natural frequencies and resonant frequencies of a square five-layered GS with a half wave number $n=1$ vs a number of half wave numbers m.

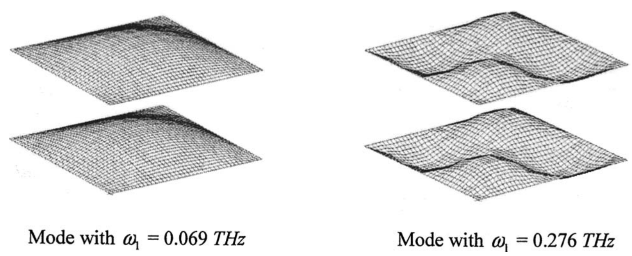

FIG. 4. Two identical resonance modes that are associated with the classical natural frequencies ω_1=0.069 THz $(m=1,n=1)$ and ω_1=0.276 THz $(m=2,n=2)$ for a simply supported square double-layered GS.

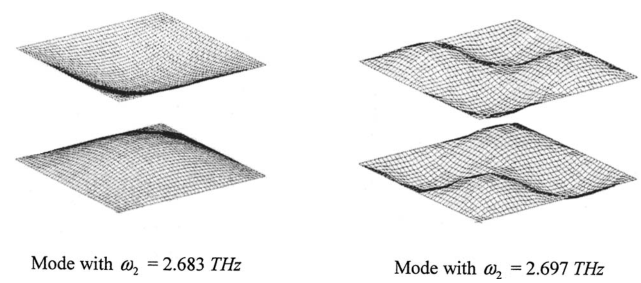

FIG. 5. Two nonidentical resonance modes that are associated with the resonant frequencies ω_2=2.683 THz $(m=1,n=1)$ and ω_2=2.697 THz $(m=2,n=2)$ for a simply supported square double-layered GS.

sheets vibrate independently at the same classical natural frequency ω_1. Regardless of the number of layers a MLGS has, all of the sheets vibrate in the same direction and with the same amplitude when the MLGS is excited at the classical natural frequency ω_1. We refer to this kind of vibration as identical vibration. However, for vibration that is associated with the resonant frequency, each sheet of a MLGS vibrates either in a different direction or at a different amplitude. We refer to this kind of vibration as nonidentical vibration.

Figure 5 shows the first- and fourth-order mode shapes that are associated with the resonant frequencies ω_2 for a square double-layered GS with width b=10 nm. It can be seen from Eq. (10) that the vibration that is associated with the resonant frequency ω_2 is dependent on the vdW interaction. The substitution of Eq. (10) into Eq. (5) gives $A_1/A_2=-1$, which indicates that the vibrations of the two sheets are always of the same amplitude, but are in opposite directions due to the vdW interaction. Note that graphite is usually formed by stacked GSs, and current research efforts involve the attempt to peel GSs from graphite. Thus the characteristic of the opposite vibration of two sheets is useful for separating a double-layered GS into two individual single-layered sheets. The opposite resonant vibrations of the two sheets can be produced by exciting a double-layered GS at the resonant frequency, which can help overcome the vdW attraction between the two sheets to allow a single-layered sheet to be peeled from a double-layered GS.

To illustrate that various resonance patterns for MLGSs can be obtained, the amplitude ratios of a square ten-layered MLGS are calculated and presented in Table III for the classical and resonant frequencies with $m=1$ and $n=1$. For the vibration mode that is associated with the classical natural frequency ω_1, all of the amplitude ratios are equal to unity, which indicates again that all of the sheets resonate identically, regardless of the number of layers, when a MLGS is excited at a classical natural frequency. This phenomenon is expected, because the classical natural frequencies for any given combination of m and n are the same for MLGSs with any number of layers, and are independent of the vdW interaction, as discussed above for Table I for the five-layered GS (Table I), for the ten-layered GS (Table II), and for the double-layered GS (Fig. 2). The amplitude ratios for the vibration modes that are associated with the resonant frequencies are not the same for all of the sheets, which indicates that the resonances of the sheets that are associated with the resonant frequencies from ω_2 to ω_{10} are nonidentical. The difference in the amplitude ratios is caused by the vdW interaction. Because the vdW interaction is dependent on the distance between layers, the amplitude ratios vary with the number of layers of a MLGS, and thus various resonance patterns can be obtained as the number of layers changes. As can be seen from Table III, there are two kinds of vibration modes that are associated with the resonant frequencies:

TABLE III. Amplitude ratios of a square ten-layered GS for the vibration modes that are associated with various resonant frequencies $(m=1,n=1)$.

	The amplitude ratios of various sheets to the top sheet								
Associated natural frequencies (THz)	A_2/A_1	A_3/A_1	A_4/A_1	A_5/A_1	A_6/A_1	A_7/A_1	A_8/A_1	A_9/A_1	A_{10}/A_1
$\omega_1=0.069$	1	1	1	1	1	1	1	1	1
$\omega_2=0.572$	0.9043	0.7184	0.4614	0.1590	–0.1590	–0.4614	–0.7184	–0.9043	−1
$\omega_3=1.132$	0.6249	0.0036	–0.6209	–1.0076	–1.0076	–0.6209	0.0036	0.6249	1
$\omega_4=1.674$	0.1874	–0.7944	–1.1199	–0.5161	0.5161	1.1199	0.7944	–0.1874	−1
$\omega_5=2.184$	–0.3652	–1.2448	–0.3947	1.0047	1.0047	–0.3947	–1.2448	–0.3652	1
$\omega_6=2.645$	–0.9778	–1.0189	0.9893	1.0042	–1.0042	–0.9893	1.0189	0.9778	−1
$\omega_7=3.042$	–1.5888	–0.0360	1.6098	–0.9850	–0.9850	1.6098	–0.0360	–1.5888	1
$\omega_8=3.364$	–2.1374	1.4901	0.3653	–1.9248	1.9248	–0.3653	–1.4901	2.1374	−1
$\omega_9=3.600$	–2.5696	3.1206	–2.5043	0.9532	0.9532	–2.5043	3.1206	–2.5696	1
$\omega_{10}=3.743$	–2.8445	4.3538	–5.4438	6.0152	–6.0152	5.4438	–4.3538	2.8445	−1

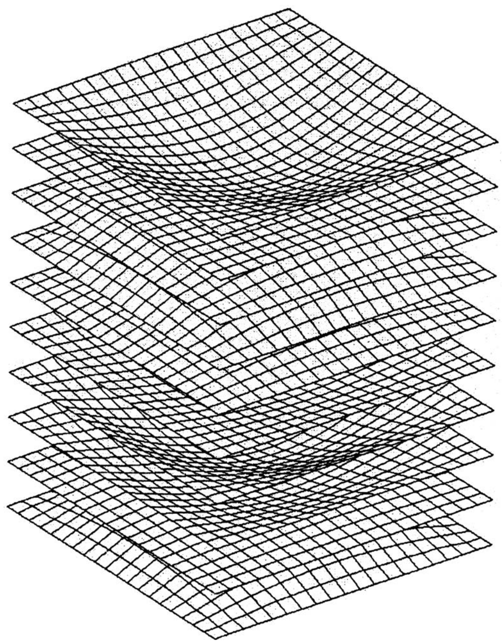

FIG. 6. Nonidentical resonance mode ($m=1$ and $n=1$) of a square ten-layered GS for the associated resonant frequency ω_4.

modes that are symmetrical about the middle plane of the MLGS, such as the modes that are associated with ω_2, ω_4, ω_6, ω_8, and ω_{10}; and modes that are antisymmetrical, such as the modes that are associated with ω_1, ω_3, ω_5, ω_7, and ω_9. To illustrate clearly the resonance modes that are shown in Table III, the resonance mode that is associated with $\omega_4 = 1.674$ THz is plotted in Fig. 6. It is observed that the mode is symmetrical about the middle plane of the ten-layered MLGS.

IV. SUMMARY

In conclusion, an explicit formula is derived to predict the vdW interaction between any two sheets of a MLGS. The derived formula indicates that the vdW interaction is significantly dependent on the interlayer spacing. A continuum-plate model is proposed for the vibration of MLGSs based on the derived formula for the vdW interaction. The number of natural frequencies of a MLGS is the same as the number of layers for any given combination of wavenumbers m and n, and the lowest natural frequency (classical natural frequency) ω_1 is independent of the vdW interaction and the number of layers of a MLGS, and thus the classical natural frequencies can be calculated from the simple formula in Eq. (9). However, all of the other resonant frequencies ω_k ($k \neq 1$) are significantly dependent on the vdW interaction. The associated mode shapes of all the sheets for the classical natural frequencies of a MLGS are in the same direction and of the same amplitude. However, the vibration modes that are associated with the resonant frequencies are different because of the influence of the vdW interaction, and thus various resonance modes can be achieved by varying the number of layers of a MLGS.

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