

Nonequilibrium theory of Coulomb blockade in open quantum dots

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We develop a nonequilibrium theory to describe weak Coulomb blockade effects in open quantum dots. Working within the Bosonized description of electrons in the point contacts, we expose deficiencies in earlier applications of this method, and address them using a $1/N$ expansion in the inverse number of channels. At leading order this yields the self-consistent potential for the charging interaction. Coulomb blockade effects arise as quantum corrections to transport at the next order. Our approach unifies the phase functional and Bosonization approaches to the problem, as well as providing a simple picture for the conductance corrections in terms of renormalization of the dot's elastic-scattering matrix, which is obtained also by elementary perturbation theory. For the case of ideal contacts, a symmetry argument immediately allows us to conclude that interactions give no signature in the averaged conductance. Nonequilibrium applications to the pumped current in a quantum pump are worked out in detail.

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I. INTRODUCTION

Coulomb blockade is the phenomenon that transport through an almost isolated system is prohibited by the energy cost E_c to add or remove an electron. The Coulomb blockade can be lifted by fine tuning a gate voltage to a point of charge degeneracy, where the energy cost of adding one electron is zero, or by raising the bias voltage or the temperature above E_c . A fundamental question is to what extent the quantization of charge on the system, and hence the Coulomb blockade, is affected by the inclusion of that system in an electrical circuit. This question has been addressed both for metal particles coupled to electrodes via tunnel barriers, as a function of the dimensionless conductance of the tunnel barrier, and for semiconductor quantum dots coupled to electrodes via ballistic point contacts, as a function of the number of propagating channels in the point contacts. In each case, the Coulomb blockade is lifted when the conductance of the contact to the electrodes is larger than the conductance quantum e^2/h . Such quantum dots (or metal particles) are referred to as “open,” in contrast to “closed” quantum dots, which are connected to the electron reservoirs through tunneling contacts with a conductance smaller than the conductance quantum. The present paper deals with Coulomb blockade in open semiconductor quantum dots, which we shall refer to as “weak Coulomb blockade.”

In the literature, two approaches have been taken to this problem. One theory of weak Coulomb blockade was proposed by Flensberg,¹ Matveev,² and Furusaki and Matveev,^{3,4} who realized that a tractable description of the strong quantum charge fluctuations can be formulated using the one-dimensional nature of the point contacts. Whereas these theories accounted for the electron-electron interactions nonperturbatively, they neglected the effect of electrons co-

herently traversing the dot or being backscattered from inside. Extensions of the theories of Refs. 1–4 to the case of open quantum dots with coherent scattering of electrons were given by Yi and Kane⁵ for quantum dots in the quantum Hall regime and by Aleiner and Glazman⁶ and Brouwer and Aleiner⁷ for the general case (see Ref. 8 for a review).

The other theoretical approach starts from an effective theory in which the primary dynamic variables are the potential differences between the quantum dot and the electrodes.⁹ This approach has mainly been used to study metal grains coupled to the reservoirs via tunnel barriers with many channels,^{10–20} but recently it has been applied to semiconductor quantum dots, both without^{21–23} and with²⁴ coherent scattering of electrons from inside the quantum dot. The general conclusions of both approaches are the same: For the incoherent case and ideal point contacts (no backscattering in the contact), it was found that all charge quantization effects completely vanish.^{1–4,21} In all other cases—contacts with a small amplitude r_c for backscattering in the contact or coherent scattering from inside the dot—charge quantization gives a correction to the dot's capacitance or conductance. This correction is small, but it becomes more important as the temperature is lowered or the number of channels in the point contacts is reduced.

The previous works on weak Coulomb blockade all dealt with thermodynamic properties (capacitance) or time-independent transport. It is the goal of the present work to develop the general nonequilibrium theory for Coulomb blockade in open quantum dots, accounting for coherent scattering of electrons inside the dot. Our main result is an expansion in powers of $1/N$ for the current through the dot, where N is the total number of channels in the point contacts.

Two examples of nonequilibrium problems involving quantum interference in open and interacting mesoscopic

systems have been a particular motivation for this work. First, the possibility of applying time-dependent potentials via local gating of quantum dots led to the theoretical and experimental investigation of “quantum pumps,” in which a periodic perturbation of potentials inside the quantum dot, combined with quantum interference, leads to a dc current through the dot.²⁵ For quantum dots with time-dependent potentials, effects of coherent scattering inside the quantum dot, charge quantization, and electron-electron interactions were considered separately theoretically (see, e.g., Refs. 26–31), but not together. The second example that motivated this work is the experiment of Ref. 32, where the nonequilibrium electron distribution function of a current carrying diffusive wire was measured by tunneling spectroscopy. This allowed an experimental determination of the electron-electron collision integral, and led to the discovery of a different mechanism of energy relaxation.^{33,34}

Following previous studies of Coulomb blockade in open quantum dots, the only electron-electron interaction term we consider is the capacitive charging energy of the quantum dot. This requires relatively large quantum dots: For open quantum dots Coulomb blockade effects are small in the inverse of the number of channels N in the contacts connecting the dot to the electron reservoirs, whereas the residual parts of the Coulomb interaction are small in the inverse dimensionless conductance of the closed dot.^{6,8,35} The ratio of these two conductances equals the ratio of the dot’s dwell time and ergodic time, which is large for large quantum dots. This parameter justifies the use of the random matrix description of scattering by the dot, and means that the problem is effectively “zero dimensional.” The dimensionality of the system appears only in the corrections to this picture.

Unlike for the case of linear time-independent transport, where the capacitive interaction gives corrections to transport properties beyond the Hartree level only, interaction corrections to nonlinear or time-dependent transport exist already when interactions are described by a self-consistent (Hartree) potential V_d . As pointed out by Büttiker and co-workers, the reason is that, in nonequilibrium or time-dependent problems, a change in bias voltage or a change in a gate voltage potentials may change the number of electrons on the dot, which, in turn, changes V_d .^{26–28}

$$V_d(t) = -\frac{e}{C}[\langle \hat{N}_{\text{dot}}(t) \rangle - \mathcal{N}]. \quad (1.1)$$

Here C is the dot’s capacitance, $-e\langle \hat{N}_{\text{dot}} \rangle$ is the dot charge, and $-e\mathcal{N}$ the offset charge induced by the gates. The effect of a time-dependent or bias-dependent Hartree potential V_d on mesoscopic fluctuations of various transport properties—but neglecting the weak Coulomb blockade—was investigated in Refs. 30 and 36–39. To leading order in $1/N$, our theory recovers the self-consistent theory with the Hartree potential (1.1). In this sense, our result can be viewed as a formal confirmation of the self-consistent theory. Further, it gives a controlled method to find interaction corrections that cannot be described by means of a self-consistent potential.

A second motivation of this work is to compare and unify the results from two different approaches to these

corrections—the “one-dimensional” and “environmental” formalisms. Whereas both approaches have given equal results for the case of quantum dots without coherent scattering from inside the dot, results for the dc conductance of coherent dots, as reported by Brouwer and Aleiner⁷ and Golubev and Zaikin,²⁴ are in disagreement. Using a formalism closely related to that of Ref. 7, we are able to present detailed calculations in two limits: A formal expansion in the scattering matrix of the dot for an arbitrary number of channels N in the contacts, and as an expansion in powers of $1/N$. The former limit reproduces the result of Ref. 7, whereas the latter limit agrees with Ref. 24. The formal expansion in the scattering matrix S , however, is only allowed if the probability of coherent (back)scattering from either the quantum dot or the contact is small, and does not describe interaction corrections to the conductance of a fully coherent quantum dot (i.e., a dot with a unitary scattering matrix). With the same arguments, the result for the interaction correction to the dot’s capacitance, which was calculated in Ref. 8 using a formal expansion in S , is not applicable to a quantum dot with a unitary scattering matrix. The correct theory, in the limit of large N , will be given here. We will not compare nonperturbative results in the two formalisms that were obtained using re-Fermionization⁴ or instanton solutions.^{21,23}

With regard to the interaction corrections to conductance, our result is particularly simple: for a fully coherent dot there is no correction to the *ensemble averaged* conductance, to leading and subleading order in $1/N$. In particular, this applies to weak localization, the quantum correction to the ensemble averaged conductance that is suppressed by the application of a magnetic field. The reason for the absence of an interaction correction to the ensemble averaged conductance is the following.

In the Landauer description of transport, conductance is related to a sum of squares of moduli of elements $S_{ij}(\varepsilon)$ of the scattering matrix $S(\varepsilon)$, appropriately integrated over energy if the temperature is nonzero (see, e.g., Ref. 40). The ensemble average of the conductance depends on the ensemble averages $\langle |S(\varepsilon)_{ij}|^2 \rangle$ only. According to the random matrix theory of quantum transport, such an average is determined by symmetries only (presence or absence of time-reversal and spin-rotation symmetry). At leading and subleading order in $1/N$, we find that the interaction corrections can be expressed through a renormalized scattering matrix: $S(\varepsilon) \rightarrow S'(\varepsilon) \equiv S(\varepsilon) + \delta S(\varepsilon)$ [see Eq. (3.6) below]. Although this interaction correction changes the scattering matrix, and hence conductance, of a specific dot, it does not change the *symmetry* of the scattering matrix ensemble, leaving the ensemble averaged conductance (including the weak-localization correction) unchanged. We will revisit this argument in more detail in Sec. VII.

The outline of our paper is as follows. In Sec. II we present a detailed discussion of the relevant existing works in the literature. This discussion will serve to introduce the necessary concepts and notations, and to explain the disagreement between the two existing theories of weak Coulomb blockade in coherent quantum dots. In Sec. III we proceed with a precise definition of the problem and an exposition of the formalism. In the diagrammatic language that we will

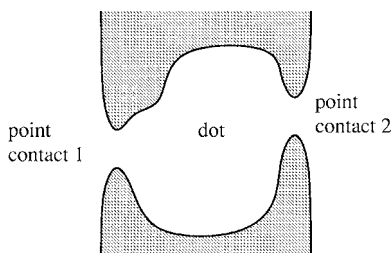


FIG. 1. Schematic drawing of the system under consideration: a quantum dot (center) coupled to source and drain reservoirs (left and right) via point contacts. The two point contacts have N_1 and N_2 propagating channels at the Fermi level. Direct reflection at the point contact is described by energy-independent reflection matrices r_{c1} and r_{c2} , so that their dimensionless conductances are $G_1 = (e^2/2\pi\hbar)(N_1 - \text{tr } r_{c1}r_{c1}^\dagger)$ and $G_2 = (e^2/h)(N_2 - \text{tr } r_{c2}r_{c2}^\dagger)$, respectively. Interactions in the dot are described by the charging energy E_c .

introduce later, the leading interaction correction to the current arises from a Fock-type diagram. The idea that interaction corrections to conductance in the presence of impurities should be thought of as due to scattering from an Hartree-Fock potential was introduced in the work of Matveev, Yue, and Glazman.^{41,42} In order to make contact to that work, we have added to Sec. III a calculation similar in spirit to that of Refs. 41 and 42 that already contains the basic structure of the interaction correction to the current. The full calculation of the current through the quantum dot, for the general non-equilibrium and time-dependent situation is then described in Sec. IV. Sections V and VI contain a detailed analysis of these results for nonlinear steady-state transport and adiabatic time-dependent transport, respectively. We conclude in Sec. VII. Finally, Appendixes A, B, and C contain materials on the correlation functions in the one-dimensional formalism, the ensemble average, and the case of a quantum dot with partially coherent scattering, respectively. A summary of our results, focusing on the relation between our work and previous approaches to weak Coulomb blockade appeared in Ref. 43.

II. PREVIOUS WORK

In this section, we present a more detailed overview of the results published in the literature. This will also serve to introduce some of the needed concepts and notations.

A schematic drawing of the system under consideration is shown in Fig. 1. It consists of a quantum dot coupled to two electron reservoirs by means of point contacts. The point contacts have N_1 and N_2 channels each (counting spin degeneracy). The two contacts are characterized by energy-independent reflection matrices r_{c1} and r_{c2} , of dimension N_1 and N_2 , respectively. In this notation, the conductance of each contact is

$$\bar{G}_i = \frac{e^2}{2\pi\hbar} g_i, \quad g_i = N_i - \text{tr } r_{ci}r_{ci}^\dagger, \quad i = 1, 2. \quad (2.1)$$

A contact with $N_i = g_i, i = 1, 2$ is called ballistic or ‘‘ideal’’; otherwise the contact is called ‘‘nonideal.’’ A dot with

$g_1 + g_2 \ll 1$ is referred to as closed; an open dot has $g_1 + g_2 \gg 1$. For future reference, we define the total number of channels in the point contacts $N = N_1 + N_2$, the total dimensionless conductance of the point contacts $g = g_1 + g_2$, and the $N \times N$ combined reflection matrix of both contacts,

$$r_c = \begin{pmatrix} r_{c1} & 0 \\ 0 & r_{c2} \end{pmatrix}. \quad (2.2)$$

The discussion of this section will be limited to the linear dc conductance G of the quantum dot. Our discussion of the literature will be limited to the cases of a ‘‘coherent’’ quantum dot, for which transport properties are expressed in terms of its N -dimensional unitary scattering matrix $S(\varepsilon)$, and an ‘‘incoherent’’ quantum dot, for which the dot itself is treated as a reservoir in equilibrium, with a self-consistent chemical potential.^{1,2} In the incoherent limit, coherent propagation across the dot, or backscattering from within, is not taken into account.

Neglecting interactions for electrons on the dot, for the incoherent case the conductance is nothing but the classical series conductance $G_{cl} = (e^2/h)g_1g_2/g$ of the two point contacts. For the coherent case, the conductance is given by the Landauer formula, which we write as

$$G_0 = \frac{e^2}{2\pi\hbar} \frac{N_1 N_2}{N} - \frac{e^2}{2\pi\hbar} \int d\varepsilon \left(-\frac{\partial f(\varepsilon)}{\partial \varepsilon} \right) \text{tr } \Lambda S(\varepsilon) \Lambda S^\dagger(\varepsilon). \quad (2.3)$$

Here $f(\varepsilon) = 1/[1 + \exp(\varepsilon/T)]$ is the Fermi distribution function and the matrix Λ is defined as

$$\Lambda_{ij} = \frac{\delta_{ij}}{N} \times \begin{cases} N_2 & \text{for } j = 1, \dots, N_1, \\ -N_1 & \text{for } j = N_1 + 1, \dots, N \end{cases}. \quad (2.4)$$

The introduction of the matrix Λ is a convenient way to separate the classical conductance for ballistic contacts [the first term in Eq. (2.3)] from the quantum corrections and the effect of nonideal contacts [second term in Eq. (2.3)].

The Landauer approach is an intrinsically single-particle picture, and as such cannot capture any effects of electron-electron interaction by itself. In a quantum dot, the most important part of the interaction^{6,8,35} is described by the Hamiltonian

$$\hat{H}_c = E_c (\hat{N}_{\text{dot}} - \mathcal{N})^2, \quad (2.5)$$

where the charging energy $E_c = e^2/2C$ is related to the geometric capacitance C of the quantum dot, \hat{N}_{dot} is the number of electrons on the quantum dot, and $\mathcal{N} = CV_g/e$ is a dimensionless gate voltage used to set the equilibrium value of N_{dot} .

The charging interaction (2.5) is responsible for the phenomenon of Coulomb blockade in closed dots at low temperatures: suppression of the conductance, except if the gate voltage \mathcal{N} is tuned to a point of charge degeneracy. As a dot is opened, a situation best realized in lateral semiconductor dots where point contacts may be controlled electrostatically, strong quantum charge fluctuations lead to the progressive diminishment of the Coulomb blockade. A theory of this weak Coulomb blockade was developed by Flensberg,¹

Matveev,² and Furusaki and Matveev,^{3,4} for the incoherent case. For a quantum dot with two nearly ideal single-channel point contacts ($|r_{c1}|, |r_{c2}| \ll 1$) and at temperatures $T \ll E_c$, the interactions were found to change the conductance of the contacts according to $g_j \rightarrow g_j + \delta g_j, j=1, 2$, where

$$\delta g_j = -2 \frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{\frac{e^C E_c}{\pi T}} \text{tr } r_{cj} r_{cj}^\dagger, \quad j=1, 2, \quad (2.6)$$

up to corrections of order $|r_{ci}|^4$. Here $C \approx 0.577$ is the Euler constant. As in the noninteracting case, the conductance G of the quantum dot is simply the series conductance of the two point contacts,

$$G = \frac{e^2}{2\pi\hbar} \left[1 - \frac{\Gamma(3/4)}{\Gamma(1/4)} \sqrt{\frac{e^C E_c}{\pi T}} \text{tr } r_c r_c^\dagger \right]. \quad (2.7)$$

The first term is the classical conductance with $N_1=N_2=2$, accounting for spin degeneracy. The second term shows the renormalization of the backscattering at the contacts; see Eq. (2.6). At low temperatures the system enters into a nonperturbative regime. We refer to Refs. 1, 3, 4, and 7 and for the expressions for the interaction corrections at different numbers of channels in the point contacts.

A different approach to the same problem was taken by Golubev and Zaikin,²³ and by Levy-Yeyati *et al.*²² who described the interaction Hamiltonian (2.5) through an effective theory for the electromagnetic environment of the junction. These authors also calculated the interaction correction to the conductance of the point contacts for an arbitrary transparency of the contacts using $1/N$ as a small parameter. They found

$$\delta g_j = -\frac{s_j g_j E_c}{3T}, \quad T \gg g E_c,$$

$$\delta g_j = -\frac{2s_j g_j}{g} \ln \frac{E_c g e^{1+C}}{2\pi^2 T}, \quad T \ll g E_c, \quad j=1, 2, \quad (2.8)$$

where s_1 and s_2 are the so-called Fano factors of the contacts,

$$s_j = \frac{1}{g_j} \text{tr } r_{cj} r_{cj}^\dagger (1 - r_{cj} r_{cj}^\dagger), \quad j=1, 2. \quad (2.9)$$

The appearance of the Fano factors is quite common to corrections to the conductance that are perturbative in the interaction; see Refs. 41 and 42. In fact, s_i is the only quadratic function of the product $r_{ci} r_{ci}^\dagger$ that vanishes both in the limits $r_{ci} \rightarrow 0$ (ideal contacts) and $r_{ci} \rightarrow 1$ (completely closed contacts). Equation (2.8) agrees with the large- N generalization of Eq. (2.7) for nearly ideal contacts.^{1,7}

The results of Refs. 1–4, being nonperturbative in E_c , are of conceptual importance, but describe the incoherent case only. They neglect entirely the possibility of coherent backscattering from within the dot, or coherent transmission from one contact to another. Thus Eq. (2.7) misses quantum interference effects, such as weak localization (the small negative magnetic-field dependent quantum interference correction to the ensemble averaged conductance) and conductance fluctuations, which one expects to be described by the Landauer formula (2.3) if the conductance of the contacts is large enough. Interaction corrections to quantum interference corrections cannot be captured by a simple renormalization of the point-contact conductances; one needs to consider the total conductance G of the quantum dot. Only at temperatures for which relaxation processes in the dot are dominant does one expect the separate renormalizations of the point-contact conductances of Eqs. (2.6) and (2.8) to be a sufficient description.

The first results for the coherent case, concerning interaction corrections to transport in open dots described by a unitary scattering matrix, were obtained by Brouwer and Aleiner⁷ (see also the review Ref. 8). They found $G = G_0 + G_{BA}$ with

$$G_{BA} = -\frac{e^2}{2\pi\hbar} \left(\frac{1}{\pi} \sin \frac{\pi}{N} \right) \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \text{tr } S^\dagger(-\tau_1) \Lambda S(\tau_2) \Lambda \times \int_{t_0}^\infty d\sigma \frac{(2\sigma + \tau_2 + \tau_1) \pi^2 T^2}{\sinh[(\sigma + \tau_1) \pi T] \sinh[(\sigma + \tau_2) \pi T]} \left(\frac{\sinh[(\tau_1 + \tau_2 + \sigma + t_0) \pi T] \sinh[(\sigma - t_0) \pi T]}{\sinh[(\tau_1 + t_0) \pi T] \sinh[(\tau_2 + t_0) \pi T]} \right)^{1/N}. \quad (2.10)$$

Here $t_0 = \pi/E_c N e^C$ is a charge relaxation time and $S(\tau)$ is the Fourier transform of the scattering matrix $S(\varepsilon)$, which is defined as

$$S(\tau) = \int \frac{d\varepsilon}{2\pi} S(\varepsilon) e^{-i\varepsilon\tau}, \quad (2.11)$$

$$S^\dagger(\tau) = \int \frac{d\varepsilon}{2\pi} S^\dagger(\varepsilon) e^{-i\varepsilon\tau}. \quad (2.12)$$

For a dot with two single-channel spin-polarized point contacts, $N_1=N_2=1$, there is an additional interaction correction to the conductance that depends explicitly on the gate voltage \mathcal{N} . Equation (2.10) is obtained as a formal expansion in the scattering matrix S . Such an expansion is controlled if S

is subunitary, as is the case, e.g., in an effective description of inelastic processes in the dot.⁴⁴ Noting that only times up to the “thermal time horizon” \hbar/T contribute to the interaction correction of Eq. (2.10), Brouwer and Aleiner argued that their formal expansion is also justified for unitary S if the temperature is sufficiently high that the contribution from scattering times $\lesssim \hbar/T$ is small. For a chaotic dot with mean dwell time τ_d and ideal point contacts, this translates to the condition $T \gg \hbar/\tau_d$.

The result of Ref. 7 captures the Landauer part of the conductance exactly, whereas the interaction correction (2.10) reproduces the singular behavior of Eq. (2.7) for the scattering matrix $S(\tau) = r_c \delta(\tau) + \dots$, provided that the scattering from inside the dot, represented by “ \dots ,” is from far beyond the “thermal time horizon.” Using known statistical properties of the scattering matrix for chaotic quantum dots, Brouwer and Aleiner calculated the ensemble averages of the conductance and the conductance fluctuations for $T \gg \hbar/\tau_d$, and found that both the weak localization correction to the conductance and the conductance fluctuations are slightly enhanced by interactions.

In this paper we question the arguments used to justify the formal expansion in the scattering matrix used to obtain Eq. (2.10). As we will argue below, the interaction correction of Eq. (2.10) is not the correct interaction correction to the conductance of a coherent quantum dot at any temperature, despite the fact that it is small if $T \gg \hbar/\tau_d$. Our main arguments will be given in the next sections, where we present two calculations of the interaction correction to the conductance: The first is a first-order-in- E_c calculation of the conductance, which differs from Eq. (2.10) if that result is expanded in E_c . The second is a calculation to all orders in E_c , but to leading order in $1/N$. This calculation confirms that Eq. (2.10) is found if the amplitude for coherent scattering from the dot is small—i.e., transport through the dot is mainly incoherent (but not fully incoherent)—but we find a different interaction correction if transport is fully coherent.

The main shortcoming of Eq. (2.10) can already be seen noting that, although for $T \gg \hbar/\tau_d$ most scattering processes have delay time $\gg \hbar/T$ and thus do not contribute to the interaction correction, the remaining interaction correction is dominated by scattering processes with delay time $\lesssim \hbar/T$. Hence it is necessary that the theory describes this time range accurately; its contribution being small is not sufficient to justify Eq. (2.10). That Eq. (2.10) does *not* describe delay times $\lesssim \hbar/T$ accurately can be seen by considering the example of a “dot” that consists of a single ballistic channel in which there is an interaction of the form (2.5). The channel is of length L , so the 2×2 scattering matrix describing ballistic propagation from one end to the other is

$$S(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \delta(\tau - \tau_d), \quad (2.13)$$

where $\tau_d = L/v_F$, v_F being the Fermi velocity inside the channel, is the time to propagate through the channel. We assume $\tau_d \gg t_0$. Then, substituting Eq. (2.13) into Eq. (2.10) gives a conductance that is *larger* than the conductance quantum $e^2/2\pi\hbar$, $G = e^2/2\pi\hbar + G_{\text{BA}}$, with

$$G_{\text{BA}} = \frac{e^2}{2\pi\hbar} \frac{\pi T \tau_d}{\hbar} e^{-2\pi T \tau_d/\hbar}, \quad T \gg \hbar/\tau_d,$$

$$G_{\text{BA}} = \frac{e^2}{2\pi\hbar} \frac{\pi\hbar}{8T\tau_d}, \quad T \ll \hbar/\tau_d. \quad (2.14)$$

On the other hand, it is well known that in the absence of backscattering the conductance of such a system cannot deviate from the quantized value e^2/h , see, e.g., Refs. 45–47, so that the interaction correction to the conductance must be zero for all temperatures. Hence although the error made by using Eq. (2.10) is exponentially small for $\tau_d \gg \hbar/T$, it is substantial for $\tau_d \lesssim \hbar/T$.

Very recently, Golubev and Zaikin extended their environmental formalism to the case of a fully coherent dot.²⁴ Again using $1/N$ as an expansion parameter, they found $G = G_0 + G_{\text{GZ}}$, with

$$G_{\text{GZ}} = \frac{e^2}{2\pi^2\hbar} \text{Im} \int d\varepsilon d\omega \left(-\frac{\partial f(\varepsilon)}{\partial \varepsilon} \right) \kappa(\omega) \text{tr}[\Lambda S(\varepsilon) \Lambda S^\dagger(\varepsilon + \omega) - \Lambda S(\varepsilon) S^\dagger(\varepsilon + \omega) S(\varepsilon) \Lambda S^\dagger(\varepsilon)] [1 - 2f(\varepsilon + \omega)], \quad (2.15)$$

where $\kappa(\omega)$ is an effective interaction kernel that describes the fluctuations of the dot potential. Since these fluctuations are small as $1/N$ if N is large, the correction (2.15) is a small correction to the Landauer result G_0 . Golubev and Zaikin performed the ensemble average over S and found that the leading interaction correction for the coherent case is the same as in the incoherent limit [i.e., the classical combination of the corrections Eq. (2.8)] as long as $T \gg \hbar/\tau_d$, and saturates when $T \lesssim \hbar/\tau_d$. The agreement with the incoherent limit is no surprise, since the leading-in- N contributions to the conductance are generally insensitive to the presence of quantum coherence.

The authors of Ref. 24 did not analyze the quantum interference corrections to the conductance and thus attempted no comparison with the results of Refs. 7 and 8. Our full calculation, which is described in the next sections and which includes quantum interference corrections, agrees with Eq. (2.15) and extends it to the case of general time-dependent transport—the focus of this paper.

Technically, our formalism is very close to that of Refs. 7 and 8. Within the same formalism, we can obtain Eq. (2.10) using a formal expansion in the scattering matrix S and Eq. (2.15) as a formal expansion in $1/N$. As we discussed above, expansion in powers of S requires a relaxation mechanism, and Eq. (2.10) can be justified only if relaxation is strong, so that, effectively, $|S(\varepsilon)| \ll 1$.

III. FORMALISM

A. Definition of the problem

The system under consideration—a quantum dot coupled to source and drain reservoirs via point contacts—has been introduced at the beginning of Sec. II. There are two point contacts, with N_1 and N_2 channels each (including spin degeneracy). Electrons on the dot interact via the simple interaction Hamiltonian (2.5). We refer to Refs. 6, 8, and 35 for a

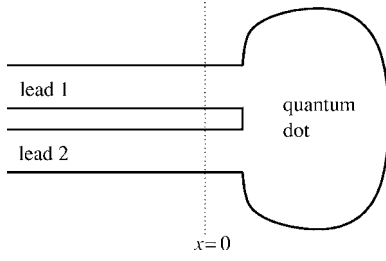


FIG. 2. Representation of the source and drain reservoirs as one-dimensional ideal leads. The lead-dot interface is at $x=0$. The one-dimensional description (3.1) is valid from $x=-\infty$ up to slightly beyond $x=0$.

microscopic justification of Eq. (2.5) for quantum dots with a large (internal) dimensionless conductance. In using Eq. (2.5) we ignore those parts of the interaction describing superconducting pairing and exchange.⁸

Following Flensberg¹ and Matveev² we describe the electron dynamics in the point contacts by a one-dimensional Hamiltonian. We locate the lead-dot interface at $x=0$, taking dot and lead at $x > 0$ and $x < 0$, respectively; see Fig. 2. Since there is no backscattering of electrons that have left the dot through the point contacts and into the reservoirs back into the dot, we may represent the reservoirs by extending the one-dimensional description to all $x < 0$,

$$\hat{H} = iv_F \sum_{j=1}^N \int_{-\infty}^0 dx \left(\hat{\psi}_{jL}^\dagger(x) \frac{\partial}{\partial x} \hat{\psi}_{jL}(x) - \hat{\psi}_{jR}^\dagger(x) \frac{\partial}{\partial x} \hat{\psi}_{jR}(x) \right) + E_c (\hat{N}_{\text{dot}} - \mathcal{N})^2. \quad (3.1)$$

Here the index j labels the channels in the two point contacts, including spin, N is the total number of channels in the two point contacts, $\hat{\psi}_{jL}$ and $\hat{\psi}_{jR}$ are annihilation operators for left-moving and right-moving electrons in channel j , respectively. Because of scattering by the dot, these are not independent fields. In the one-dimensional Hamiltonian (3.1), the kinetic energy is linearized around the Fermi energy, which is appropriate if the Fermi energy is not too close to a threshold at which a new channel in the point contact is opened. The coordinate x in the one-dimensional Hamiltonian (3.1) arises as the Fourier transform of the one-dimensional Hamiltonian in momentum representation; see Ref. 2. Except for the immediate vicinity of the point contact, it cannot be identified with a distance to the lead-dot interface.

It is convenient to follow Aleiner and Glazman,⁶ and write the number of electrons on the dot as

$$\hat{N}_{\text{dot}} = N_{\text{ref}} - \sum_j \int_{-\infty}^0 dx [\hat{\psi}_{jL}^\dagger(x) \hat{\psi}_{jL}(x) + \hat{\psi}_{jR}^\dagger(x) \hat{\psi}_{jR}(x)]. \quad (3.2)$$

Here N_{ref} is the total number of electrons in the system (reservoirs and dot), which is time independent. Now the interaction does not contain the dot degrees of freedom. Thus the relation between the operators for right-moving electrons entering the quantum dot and left-moving electrons exiting the

quantum dot is given by the noninteracting formula. In the general time-dependent case one then has⁴⁸

$$\hat{\psi}_{jL}(0, t) = \sum_{k=1}^N \int d\tau S_{jk}(t, t - \tau) \hat{\psi}_{kR}(0, t - \tau),$$

$$\hat{\psi}_{jL}^\dagger(0, t) = \sum_{k=1}^N \int d\tau \hat{\psi}_{kR}^\dagger(0, t - \tau) (S^\dagger)_{kj}(t - \tau, t), \quad (3.3)$$

where $S(t, t')$ is the scattering matrix of the dot. Notice that the scattering matrix has two time arguments. For time-independent transport, $S(t, t')$ is a function of the time difference $t - t'$ only.

The relation (3.3), together with the one-dimensional Hamiltonian (3.1), Eq. (3.2) for \hat{N}_{dot} , and the boundary conditions set by the reservoirs completely define the problem. Notice that we never explicitly introduce the dot degrees of freedom into the problem. If the scattering from the dot is not fully coherent, Eq. (3.3) has to be replaced by a different boundary condition. We return to the case of incoherent scattering in Sec. III D. A model for partially coherent scattering is discussed in Appendix C.

B. First order in E_c

It was shown by Matveev, Yue, and Glazman^{41,42} that, at first order in the interaction, interaction corrections to conductance in the presence of impurities may be understood in terms of the elastic scattering of electrons from the Hartree-Fock potential created by the impurities. This gave a simple and intuitive picture of the renormalization of backscattering in the interacting one-dimensional electron gas, without recourse to sophisticated formalism. We now outline a similar calculation for transport through a quantum dot.

For linear transport, it is sufficient to consider the scattering off the Hartree-Fock potential in equilibrium. With the replacement (3.2) the Hartree-Fock potential acts in the leads, not in the dot. In equilibrium there is no Hartree potential as $\langle \hat{N}_{\text{dot}} \rangle = \mathcal{N}$. For the Fock potential we find

$$\hat{H}_F = \sum_{i,j=1}^N \int_{-\infty}^0 dx dy [\hat{\psi}_{iL}^\dagger(x) V_{ij}(x, y) \hat{\psi}_{jR}(y) + \text{H.c.}], \quad (3.4)$$

where ‘‘H.c.’’ denotes the Hermitian conjugate, plus forward-scattering terms that do not affect the linear conductance at lowest order in E_c . The Fock potentials are written in terms of the density matrices, which may be evaluated using the boundary conditions (3.3),

$$V_{ij}(x, y) = -2E_c \langle \hat{\psi}_{jR}^\dagger(y) \hat{\psi}_{iL}(x) \rangle$$

$$= -2E_c \int_0^\infty d\tau S_{ij}(\tau)$$

$$\times \frac{iT}{2 \sinh\{\pi T(\tau - v_F[x + y] + i\lambda)\}}, \quad (3.5)$$

where λ is a positive infinitesimal. Thus the effect of the Coulomb interaction is to establish a nonlocal Fock potential

in the leads within a region of order the thermal length v_F/T from the contacts. It is now straightforward to find the change in the scattering matrix due to this potential,

$$\delta S(\epsilon) = i \int \frac{d\omega}{2\pi} \kappa(\omega) \{ [2f(\epsilon - \omega) - 1] S(\epsilon - \omega) + [2f(\epsilon + \omega) - 1] S(\epsilon) S^\dagger(\epsilon + \omega) S(\epsilon) \}, \quad (3.6)$$

where

$$\kappa(\omega) = - \frac{E_c}{(\omega - i0^+)^2}, \quad (3.7)$$

0^+ being a positive infinitesimal. The first term in Eq. (3.6) represents the effect of direct $R \rightarrow L$ (in to out) scattering by the Fock potential (3.4). The second corresponds to the dot scattering $R \rightarrow L$, followed by the Fock potential scattering $L \rightarrow R$, and ending with the dot scattering $R \rightarrow L$ once more. Substitution of Eq. (3.6) into the Landauer formula (2.3) immediately yields the result (2.15) for the interaction correction to conductance, with $\kappa(\omega)$ given by Eq. (3.7) above.

We may repeat the derivation for the incoherent case that is considered in Refs. 1–4, 22, and 23. In this case, the inclusion of the Fock potential leads to a change of the reflection matrix r_c for direct reflection at the contacts. Since electrons leaving the dot are incoherent with those entering it, they do not contribute to the Fock potential (3.5). The renormalized reflection matrix is thus

$$\delta r_c(\epsilon) = i \int \frac{d\omega}{2\pi} \kappa(\omega) \{ [2f(\epsilon - \omega) - 1] r_c + [2f(\epsilon + \omega) - 1] r_c r_c^\dagger r_c \}. \quad (3.8)$$

Taking into account the energy dependence of the renormalized reflection matrix r_c , we find that the interaction correction to the dimensionless conductance of contact $j, j=1,2$, becomes

$$\delta g_j = 2s_j g_j \text{Im} \int d\epsilon \int \frac{d\omega}{2\pi} \times \left(- \frac{\partial f(\epsilon)}{\partial \epsilon} \right) [1 - 2f(\epsilon + \omega)] \kappa(\omega), \quad (3.9)$$

where s_j was defined in Eq. (2.9). Although this looks very similar to Eq. (2.15) with the scattering matrix chosen as $S(\epsilon) = r_c$, the effect of using a nonunitary scattering matrix is significant. Notice that for a unitary scattering matrix, the trace in Eq. (2.15) vanishes as ω^2 , so that the regulator in Eq. (3.7) plays no role. In Eq. (3.9) on the other hand, the imaginary part of the integrand comes from the region near $\omega=0$ and yields a finite, regulator-independent contribution to the conductance,

$$\delta g_j = -s_j g_j \frac{E_c}{3T}, \quad j=1,2, \quad (3.10)$$

which coincides with the high-temperature limit of Eq. (2.8).²³ Recovering this same result, together with the low-temperature limit of Eq. (2.8), as the zero level spacing limit of scattering from a coherent dot is subtle; see Sec. V and

Ref. 24. With this physical picture of the interaction corrections, we now proceed with the formal development of the theory.

C. Effective action

We are interested in the current through the point contacts, at the dot-lead interface $x=0$. The current in channel j is

$$\hat{I}_j = -e v_F [\hat{\psi}_{jR}^\dagger(0) \hat{\psi}_{jR}(0) - \hat{\psi}_{jL}^\dagger(0) \hat{\psi}_{jL}(0)]. \quad (3.11)$$

Below, the expectation value of the current is calculated using standard methods of nonequilibrium many-body theory.⁴⁹ The current $\hat{I}(t)$ is related to the state of the system at a reference time t_{ref} at which the Hamiltonian is not time dependent. The expectation value $I(t)$ is then found from a thermodynamical average at the time t_{ref} . An important issue here is that the total number of electrons N_{ref} be kept constant—especially when observables at two different values of the gate voltage \mathcal{N} are compared.⁸ In Ref. 8 this was implemented by taking this thermodynamical average in a canonical ensemble. In our case, however, we can take the average in a grand canonical ensemble, since, in the nonequilibrium formalism, it is sufficient that the dynamics conserves the number of electrons.

The one-dimensional dynamics of the Hamiltonian (3.1) applies to the region $x < 0$ only; the dynamics in the quantum dot is described by the boundary condition (3.3). This boundary condition is difficult to implement in the calculations, and it is advantageous to reformulate the problem in terms of a one-dimensional Hamiltonian of the form (3.1), where the x integration extends over the entire real axis. In this formulation of the problem, the boundary condition (3.3), or its equivalent for a dot in which scattering is not fully coherent, is replaced by an effective action \mathcal{S} that involves the fermion operators at $x = \eta$, where η is a positive infinitesimal. Such a formulation of the problem has been given by Aleiner and Glazman⁶; see also Ref. 8. Here we will use an equivalent but simpler effective action formulation.

In order to derive the effective action, a fictitious half-infinite N -channel lead is side-coupled to the point contacts, at position $x = \eta$, where η is a positive infinitesimal. The fictitious lead couples only to electrons moving out of the dot, i.e., to left movers. The total Hamiltonian \hat{H} is written as

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \quad (3.12)$$

where \hat{H}_0 is the Hamiltonian of the full interacting system (including the quantum dot, the reservoirs, and the fictitious lead), with ideal coupling between the fictitious lead and the point contact; see Fig. 3 (solid lines). Hence in the system described by Hamiltonian \hat{H}_0 , all electrons entering the point contact from the dot will exit through the fictitious lead, whereas all electrons entering from the fictitious lead will exit towards the source or drain reservoirs. The Hamiltonian \hat{H}_1 represents an impurity in the contact to the fictitious lead, which ensures that electrons coming from the dot in channel j have amplitude $-ir_j$ to be transmitted coherently towards

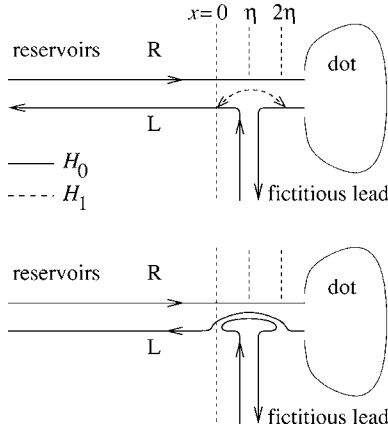


FIG. 3. Inclusion of a fictitious lead at the point contact. Top panel: With the Hamiltonian H_0 , the fictitious lead couples ideally to outgoing electrons (solid lines), so that all electrons that exit the dot enter into the fictitious lead. The Hamiltonian H_1 describes scattering of electrons exiting the dot into the physical lead and back-scattering of electrons coming from the fictitious lead (dotted line). The amplitude for this scattering process in channel j is $ir_{j,j} = 1, \dots, N$. Bottom panel: If $r_j=1$, the fictitious lead is decoupled again and the original problem is restored.

the source or drain reservoirs (cf. dashed lines in Fig. 3) and that electrons coming in from the fictitious lead have amplitude $-ir_j$ to be reflected into the fictitious lead,

$$\hat{H}_1 = \sum_{j=1}^N 2v_F v_j [\hat{\psi}_{jL}^\dagger(0) \hat{\psi}_{jL}(2\eta) + \hat{\psi}_{jL}^\dagger(2\eta) \hat{\psi}_{jL}(0)], \quad (3.13)$$

where

$$r_j = \frac{2v_j}{1+v_j^2}, \quad j = 1, \dots, N. \quad (3.14)$$

The parameters r_j (or v_j) describe how well the fictitious lead is coupled to the point contact. In the limit $r_j \rightarrow 1$, $j = 1, \dots, N$, the fictitious lead is fully decoupled. We will take the limit $r_j \rightarrow 1$ at the end of the calculation. In writing Eq. (3.13), the same regularization of fermion operators has been employed as in Ref. 8.

The operator identity (3.3) applies to left-moving fermions at the moment when they exit the dot. With the inclusion of the fictitious lead, this means that one should take ψ_L at $x=2\eta$, not at $x=0$. Hence Eq. (3.3) gives a relation between $\hat{\psi}_L(2\eta)$ and $\hat{\psi}_R(0)$,

$$\hat{\psi}_{jL}(2\eta, t) = \sum_{k=1}^N \int d\tau S_{jk}(t, t-\tau) \hat{\psi}_{kR}(0, t-\tau). \quad (3.15)$$

The advantage of separating the Hamiltonian into the contributions \hat{H}_0 and \hat{H}_1 is that the problem described by the Hamiltonian H_0 alone is exactly solvable; see Sec. III E. The effect of \hat{H}_1 is then treated in perturbation theory. Using Eq. (3.15), the expectation value of the current at time t can be expressed in terms of an effective action \mathcal{S} ,

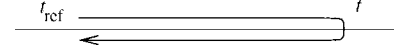


FIG. 4. Integration contour for Eq. 3.16.

$$I_j(t) = \langle T_c e^{-i\hat{\mathcal{S}}_j(t)} \rangle_0, \quad (3.16)$$

where the current operator \hat{I}_j is given by Eq. (3.11) above, “ c ” denotes the Keldysh contour, see Fig. 4, T_c denotes contour ordering along the Keldysh contour, the time dependence of the operators is that of the interaction picture with respect to \hat{H}_0 , and the average $\langle \dots \rangle_0$ denotes an average with respect to H_0 at reference time t_{ref} . The effective action \mathcal{S} is

$$\begin{aligned} \mathcal{S} = & \int_c dt_1 \hat{H}_1(t_1) = 2v_F \int_c dt_1 \int_c d\tau \sum_{j=1}^N \sum_{k=1}^N v_j \\ & \times [\psi_{jL}^\dagger(0, t_1) S_{jk}(t_1, t_1 - \tau) \psi_{kR}(0, t_1 - \tau) \\ & + \psi_{kR}^\dagger(0, t_1 - \tau) (S^\dagger)_{kj}(t_1 - \tau, t_1) \psi_{jL}(0, t_1)], \end{aligned} \quad (3.17)$$

where we dropped reference to the infinitesimal η . It is important to note that in this effective action, the contour ordering occurs according to the “contour time” t_1 , not according to the “scattering delay time” τ .

Both the current operator (3.11) and the effective action (3.17) do not contain any reference to the electrons after they have passed through the point contact, so that one may send the upper integration boundary in Eq. (3.1) to infinity and set

$$\begin{aligned} \hat{H}_0 = & iv_F \sum_{j=1}^N \int_{-\infty}^{\infty} dx \left(\hat{\psi}_{jL}^\dagger(x) \frac{\partial}{\partial x} \hat{\psi}_{jL}(x) - \hat{\psi}_{jR}^\dagger(x) \frac{\partial}{\partial x} \hat{\psi}_{jR}(x) \right) \\ & + E_c (\hat{N}_{\text{dot}} - \mathcal{N})^2, \end{aligned} \quad (3.18)$$

where, as before, \hat{N}_{dot} is expressed in terms of the fermion fields for $x < 0$ only; see Eq. (3.2). In the system described by the Hamiltonian \hat{H}_0 , the fermion fields ψ_R have chemical potentials μ_R corresponding to the chemical potentials in the reservoirs, whereas the fermion fields ψ_L have the chemical potential μ_L of the fictitious lead. For the physically relevant case $r_j \rightarrow 1, j=1, \dots, N$, the fictitious lead is fully decoupled from the point contact. In that limit, all physical observables become independent of the choice of μ_L .

The effective action (3.17) is different from the effective action used in Refs. 6–8: the kernel in the latter action is a matrix L , which is a function of the scattering matrix S of the quantum dot.

D. Relaxation inside quantum dot

If the fictitious lead used for the above derivation of the effective action is fully coupled to the point contact (i.e., if $r_j=0, j=1, \dots, N$), every electron exiting the quantum dot will escape towards the fictitious lead, not towards the (physical) reservoirs. Setting the chemical potential μ_L of the fictitious lead such that no net current is drawn, one finds that each electron that escapes into the fictitious lead is replaced by another one—without phase relationship.^{50,51} Hence coupling to the fictitious lead allows for complete

phase and energy relaxation of electrons exiting the dot, just before they pass through the point contact to the reservoirs.

The simplest model for incomplete relaxation in the quantum dot is to use the coupling to the fictitious lead with $0 < r_j < 1$. In this case, the electrons are allowed to escape into the fictitious lead and relax, but with finite probability only. Hence imperfect coupling to the fictitious lead describes a quantum dot with a finite probability that the electron exits the dot without relaxation. Of course, this model is oversimplified, because all relaxation is localized at the point contact. A more realistic but still phenomenological model for relaxation in the quantum dot is discussed in Appendix C.

A connection can be made between the effective action (3.17) for general r_j and the “fully incoherent” limit of the problem, with nonideal contacts.^{1-4,22,23} Hereto, the fictitious lead is interpreted as the “quantum dot,” the scattering matrix $S_{ij}(t, t')$ is replaced by the identity matrix, $S_{ij}(t, t') = \delta_{ij} \delta(t - t')$, and one identifies the reflection matrix r with the contact’s reflection matrix r_c . In this case, energy and phase relaxation inside the dot is still complete, but a non-zero value of the reflection probabilities r_j^2 allows one to describe a nonideal coupling between dot and reservoirs.

E. Correlation functions

In order to formulate a perturbation theory in the effective action (3.17), we need the contour-ordered fermion correlation functions of the Hamiltonian \hat{H}_0 . The equilibrium correlators were calculated using the Bosonization method in Ref. 8. The new elements in the nonequilibrium case are few. We discuss them now in physical terms, with the details being relegated to Appendix A.

We specify the nonequilibrium conditions applied to the system through the chemical potentials $\mu_{jL}(t+x/v_F)$ and $\mu_{jR}(t-x/v_F)$ of the left- and right-moving electrons. Although the chemical potentials of the fictitious leads should not appear in physical quantities in the $r_j \rightarrow 1$ limit, we retain them in order to illustrate how this happens.

In the absence of scattering, i.e., setting $\hat{H}_1 = 0$, the Hamiltonian is quadratic and a mean-field treatment is thus exact. The average density of left-moving electrons at the contact is simply related to their chemical potential

$$\langle \hat{\rho}_{jL}(0, t) \rangle = \frac{1}{2\pi v_F} \mu_{jL}(t). \quad (3.19)$$

For right movers, however, the presence of the charging interaction in the lead means that the *electrochemical potential* μ_c is the relevant quantity

$$\langle \hat{\rho}_{jR}(0, t) \rangle = \frac{1}{2\pi v_F} \mu_{c_j}(t), \quad (3.20)$$

$$\mu_{c_j}(t) = \mu_{jR}(t) + eV_d(t), \quad (3.21)$$

where $V_d(t)$ was defined in Eq. (1.1). The average current at the contact is

$$\begin{aligned} I_j(t) &= -e v_F [\langle \hat{\rho}_{jR}(0, t) \rangle - \langle \hat{\rho}_{jL}(0, t) \rangle] \\ &= -\frac{e}{2\pi} \{ \mu_{jR} - \mu_{jL} - 2E_c [\langle \hat{N}_{\text{dot}}(t) \rangle - \mathcal{N}] \}. \end{aligned} \quad (3.22)$$

Thus with an excess charge in the dot $\langle \hat{N}_{\text{dot}} \rangle \neq \mathcal{N}$, there is a contribution from the interaction. We solve Eq. (3.20) by writing the excess charge as an integral of the current,

$$\langle \hat{\rho}_{jR}(t) \rangle = \frac{1}{2\pi v_F} \mu_{jR}(t) - \frac{E_c}{\pi} \sum_{k=1}^N \int_0^t dt' \langle \hat{\rho}_{kR}(t') \rangle - \langle \hat{\rho}_{kL}(t') \rangle, \quad (3.23)$$

where we have introduced the convention that fields without a position label are taken to be evaluated at the origin. The solution of Eq. (3.23) is

$$\begin{aligned} \langle \hat{\rho}_{jR}(t) \rangle &= \frac{1}{2\pi v_F} \left(\mu_{jL}(t) + \sum_{k=1}^N \int_0^\infty d\tau \pi_{jk}(\tau) \right. \\ &\quad \left. \times [\mu_{kR}(t - \tau) - \mu_{kL}(t - \tau)] \right), \end{aligned} \quad (3.24)$$

where we defined the kernel

$$i_{mk}(\tau) = \delta_{mk} \delta(\tau - 0^+) - \frac{E_c}{\pi} e^{-E_c N \tau / \pi} \theta(\tau), \quad (3.25)$$

0^+ being a positive infinitesimal. The kernel $i_{jk}(\tau)$, which appears frequently in the following development, relates the current at the contacts to the chemical potentials in the leads, accounting properly for the effect of the charging interaction. Note the appearance of the *RC* time $\pi/E_c N$ characteristic of an ideal *N*-channel contact.

Combining the above results, we find that the current in channel *j* is given by

$$I_j(t) = -\frac{e}{2\pi} \sum_{k=1}^N \int_0^\infty d\tau \pi_{jk}(\tau) [\mu_{kR}(t - \tau) - \mu_{kL}(t - \tau)]. \quad (3.26)$$

Note that in the dc bias situation Eq. (3.26) simplifies to

$$I_j = -\frac{e}{2\pi} \sum_{k=1}^N \left(\delta_{jk} - \frac{1}{N} \right) (\mu_{kR} - \mu_{kL}), \quad (3.27)$$

so that for the two-terminal setup

$$I_j = -\frac{e}{2\pi} \Lambda_{jj} (\Delta\mu_1 - \Delta\mu_2), \quad (3.28)$$

where $\Delta\mu_{1,2} = \mu_{R1,2} - \mu_{L1,2}$ are the chemical potential differences in the two contacts, and Λ_{ij} is the matrix defined in Eq. (2.4).

Correlators of many fermion operators will be expressed in terms of the single-fermion Green functions,

$$-i \langle T_c \hat{\psi}_{mL}(s) \hat{\psi}_{nL}^\dagger(t) \rangle = G_{mnL}(s, t), \quad (3.29)$$

$$-i\langle T_c \hat{\psi}_{mR}(t) \hat{\psi}_{nR}^\dagger(s) \rangle = G_{mnR}(t, s). \quad (3.30)$$

These read

$$G_{mnL}(t, s) = -\frac{\delta_{mn} T e^{-i[\phi_{mL}(t) - \phi_{mL}(s)]}}{2v_F \sinh\{\pi T [t - s - i\lambda \operatorname{sgn}_c(t - s)]\}},$$

$$G_{mnR}(t, s) = -\frac{\delta_{mn} T e^{-i[\phi_{mR}(t) - \phi_{mR}(s)]}}{2v_F \sinh\{\pi T [t - s - i\lambda \operatorname{sgn}_c(t - s)]\}}, \quad (3.31)$$

where λ is a positive infinitesimal and

$$\phi_{mL}(t) = \int_0^\infty d\tau \mu_{mL}(t - \tau),$$

$$\phi_{mR}(t) = \int_0^\infty d\tau \left(\mu_{mL}(t - \tau) + \sum_{k=1}^N \int_0^\infty d\tau' i_{mk}(\tau') \right. \\ \left. \times [\mu_{kR}(t - \tau - \tau') - \mu_{kL}(t - \tau - \tau')] \right) \quad (3.32)$$

are the integrals of the electrochemical potentials of right and left movers. It is straightforward to see that Eq. (3.31) is

consistent with Eqs. (3.19) and (3.24). The single-fermion Green function can be written as a sum of a Keldysh part, which does not depend on the contour positions, and a delta function,

$$G_{mnL}(t, s) = \frac{1}{2} G_{mnL}^K(t, s) - \frac{i}{2v_F} \delta_{mn} \delta_c(t, s),$$

$$G_{mnR}(t, s) = \frac{1}{2} G_{mnR}^K(t, s) - \frac{i}{2v_F} \delta_{mn} \delta_c(t, s). \quad (3.33)$$

Here we abbreviated

$$\delta_c(s, t) = \operatorname{sgn}_c(s - t) \delta(s - t). \quad (3.34)$$

Now the contour-ordered average of a product of fermion operators is calculated as

$$(-i)^{n+m} \langle T_c \hat{\psi}_{l_1L}(s_1) \hat{\psi}_{k_1L}^\dagger(t_1) \cdots \hat{\psi}_{l_nL}(s_n) \hat{\psi}_{k_nL}^\dagger(t_n) \hat{\psi}_{k'_1R}(t'_1) \hat{\psi}_{l'_1R}^\dagger(s'_1) \cdots \hat{\psi}_{k'_mR}(t'_m) \hat{\psi}_{l'_mR}^\dagger(s'_m) \rangle \\ = \sum_{P, Q} (-1)^{P+Q} \prod_{i=1}^n G_{l_i k_{P(i)}L}(s_i, t_{P(i)}) \prod_{j=1}^m G_{k'_j l'_{Q(j)}R}(t'_j, s'_{Q(j)}) \prod_{i, j} \left(\frac{f(t_i - t'_j) f(s_i - s'_j)}{f(t_i - s'_j) f(s_i - t'_j)} \right)^{1/N}, \quad (3.35)$$

where P and Q are permutations of the numbers $i=1, \dots, n$ and $j=1, \dots, m$, respectively, and the function $f(t-t')$ is

$$\ln f(t-t') = \ln \frac{\lambda E_C N e^C}{\pi} - \frac{E_C N}{\pi} \int_0^\infty d\zeta e^{-E_C N \zeta / \pi} \\ \times \ln \frac{\sinh\{\pi T [t - t' + \zeta - i0^+ \operatorname{sgn}_c(t - t')]\}}{\sinh(\pi T \zeta)}. \quad (3.36)$$

In the last equation, 0^+ is a positive infinitesimal and $C \approx 0.577$ is the Euler constant. We separate $\ln f(t-t')$ into a contour-independent (Keldysh) part and a contour-dependent part,

$$\ln f(t-t') = \ln |f(t-t')| + iN \kappa_0(t-t') \operatorname{sgn}_c(t-t'), \quad (3.37)$$

where

$$\kappa_0(t-t') = \frac{\pi}{N} (1 - e^{-E_C N(t-t')/\pi}) \theta(t'-t). \quad (3.38)$$

Here $\theta(x)=1$ if $x>0$ and 0 otherwise.

Note that the result (3.35) is compatible with the symmetry of the Hamiltonian H_0 . At first glance this is $U(N) \times U(N)$, the two factors corresponding to the right and left

movers. Certainly this is the case for the noninteracting model with $E_c \rightarrow 0$, where Eq. (3.35) reduces to the usual Wick's theorem result. The interaction, however, leads to the chiral anomaly, as in the Schwinger model, that reduces the symmetry to $SU(N) \times SU(N) \times U(1)$. Physically, this is because the interaction can scatter right movers to left movers and vice versa. This allows a second type of correlation function to exist, characterized by the $SU(N)$ -invariant antisymmetric tensor $\varepsilon_{i_1 i_2 \dots i_N}$. We will refer to such correlators as anomalous. They describe the scattering of right movers to left movers (and vice versa) by the interaction.^{2,8} Below we list the first of these nontrivial correlators for $N=1$ and for $N=2$. For $N=1$ one has

$$-i\langle T_c \hat{\psi}_R(t') \hat{\psi}_L^\dagger(s) \rangle = \frac{E_c e^C f(s-t')}{2\pi^2 v_F f(0)} e^{-i[2\pi N + \phi_R(t') - \phi_L(s)]},$$

$$-i\langle T_c \hat{\psi}_L(t) \hat{\psi}_R^\dagger(s') \rangle = -\frac{E_c e^C f(t-s')}{2\pi^2 v_F f(0)} e^{-i[\phi_L(t) - \phi_R(s') - 2\pi N]}. \quad (3.39)$$

For $N=2$ the relevant anomalous correlators are

$$\begin{aligned}
& (-i^2) \langle T_c \hat{\psi}_{1R}(0, t'_1) \hat{\psi}_{1L}^\dagger(0, t'_2) \hat{\psi}_{2R}(0, t'_3) \hat{\psi}_{2L}^\dagger(0, t'_4) \rangle_0 \\
&= \left(\frac{E_c e^C}{\pi^2 v_F f(0)} \right)^2 e^{-i[2\pi N + \phi_{R1}(t'_1) - \phi_{L1}(t'_2) + \phi_{R2}(t'_3) - \phi_{L2}(t'_4)]} \\
&\quad \times [f(t_2, t'_1) f(t_4, t'_1) f(t_2, t'_3) f(t_4, t'_3)]^{1/2}, \\
& (-i^2) \langle T_c \hat{\psi}_{1L}(0, s_1) \hat{\psi}_{1R}^\dagger(0, s'_2) \hat{\psi}_{2L}(0, s_3) \hat{\psi}_{2R}^\dagger(0, s'_4) \rangle_0 \\
&= \left(\frac{E_c e^C}{\pi^2 v_F f(0)} \right)^2 e^{i[\phi_{L1}(s_1) - \phi_{R1}(s'_2) + \phi_{L2}(s_3) - \phi_{R2}(s'_4) - 2\pi N]} \\
&\quad \times [f(s_1, s'_2) f(s_1, s'_4) f(s_3, s'_2) f(s_3, s'_4)]^{1/2}. \quad (3.40)
\end{aligned}$$

Notice that the “vacuum angle” describing the U(1) symmetry breaking is just $2\pi N$.

For the perturbation theory calculation of the current, we need contour-ordered correlators of the form

$$\langle T_c \hat{J}_j(t) \hat{A}(t'_1) \cdots \hat{A}(t'_n) \rangle,$$

where the symbols $\hat{A}(t')$ represent creation or annihilation operators for left- or right-moving fermions. There is a simple relation between such correlators and the corresponding correlator without current operator,

$$\langle T_c \hat{J}_j(t) \hat{A}_1(t'_1) \cdots \hat{A}_n(t'_n) \rangle = \langle T_c \hat{A}_1(t'_1) \cdots \hat{A}_n(t'_n) \rangle \sum_{m=1}^n F_m(t, t'_m). \quad (3.41)$$

Here the F_m depend on whether the corresponding operator A_m is a creation or annihilation operator for left- or right-moving fermions. For an annihilation operator for left-moving and right-moving fermions in channel k , one has

$$\begin{aligned}
F_{j,kL}(t, t_1) &= -\frac{eT}{2i} \int_0^\infty d\tau \pi_{jk}(t - t_1 - \tau) \coth(\pi T \tau) \\
&\quad - \frac{e}{2} \operatorname{sgn}_c(t - t_1) i_{jk}(t - t_1), \quad (3.42)
\end{aligned}$$

respectively. Factors F_m for the creation operators for left-moving and right-moving fermions in channel k are $-F_{j,kL}$ and $-F_{j,kR}$, respectively.

IV. CALCULATION OF THE CURRENT

With the help of the correlators listed in the previous section, the perturbation expansion (3.16) can be evaluated. In this section, we first describe a full evaluation of $I(t)$ up to second order in the effective action (first order for $N=1$). We

then reorganize the perturbation theory, and calculate $I(t)$ to all orders in S , but in a perturbation expansion in $1/N$, which turns out to correspond to a loop expansion.⁵²

A. Expansion in effective action

To *zeroth order* in the effective action, one finds that the current in channel j is given by

$$\begin{aligned}
I_{0,j}(t) &= \frac{1}{2} i e v_F [G_{jjR}^K(t, t) - G_{jjL}^K(t, t)] \\
&= \frac{e}{2\pi\hbar} \int dt_1 \sum_{k=1}^N i_{jk}(t - t_1) [\mu_{kL}(t_1) - \mu_{kR}(t_1)], \quad (4.1)
\end{aligned}$$

in agreement with Eq. (3.26) and with $i_{jk}(t - t_1)$ defined in Eq. (3.25) above.

To *first order* in the effective action one has

$$I_{j,1}(t) = -i \langle \hat{S} \hat{J}_j(t) \rangle_0. \quad (4.2)$$

Substituting Eq. (3.17) for the effective action and using the fermion correlation functions of the previous section, one finds $I_{j,1}(t) = 0$, except for $N=1$. In that case, one has

$$\begin{aligned}
I_1(t) &= 4v_F v \operatorname{Im} \int_c dt_1 \int d\tau e^{-2\pi i N S(t_1, t_1 - \tau)} [-F_L(t, t_1) \\
&\quad + F_R(t, t_1 - \tau)] \left(\frac{-iE_c e^C f(\tau)}{2\pi^2 v_F f(0)} \right) e^{-i[\phi_R(t_1 - \tau) - \phi_L(t_1)]}. \quad (4.3)
\end{aligned}$$

In this equation, it is important to note that all contour dependence should depend on t_1 only; the time difference τ should not enter into the contour signs. We also note that the argument of $f(\tau)$ is always positive, so that $f(\tau)$ does not depend on the contour position, only the contour-dependent part of the function F that contributes to the integration, and that the contribution from right-moving fermions is zero by causality, so that

$$\begin{aligned}
I_1(t) &= -\frac{2e v E_c e^C}{\pi^2 f(0)} \int dt_1 i(t, t_1) \int d\tau f \\
&\quad \times (\tau) \operatorname{Re} S(t_1, t_1 - \tau) e^{-i[\phi_R(t_1 - \tau) - \phi_L(t_1)] - 2\pi i N}. \quad (4.4)
\end{aligned}$$

One verifies that for a time-independent situation— $S(t, t - \tau)$ independent of t and chemical potentials μ_L and μ_R independent of time—the current is zero. The *second-order* calculation proceeds similarly. We first restrict our attention to the case $N > 2$. Using matrix notation, we then find

$$\begin{aligned}
I_{j,2}(t) &= -\frac{e}{2} (2v_F)^2 \int_c dt_1 ds_1 \int d\tau_1 d\sigma_1 \left(\frac{f(\tau_1) f(\sigma_1)}{f(t_1 - s_1 + \sigma_1) f(s_1 - t_1 + \tau_1)} \right)^{1/N} \\
&\quad \times \operatorname{tr} v^2 [i_j(t, t_1) \operatorname{sgn}_c(t - t_1) - i_j(t, s_1) \operatorname{sgn}_c(t - s_1)] \times G_L(s_1, t_1) S(t_1, t_1 - \tau_1) G_R(t_1 - \tau_1, s_1 - \sigma_1) S^\dagger(s_1 - \sigma_1, s_1). \quad (4.5)
\end{aligned}$$

Here i_j is an $N \times N$ diagonal matrix, closely related to the kernel defined in Eq. (3.25),

$$(i_j)_{mn}(\tau) = \delta_{mn} i_{jm}(\tau). \quad (4.6)$$

At this point, we write the Green functions G_R and G_L as the sum of the Keldysh component and a delta function; see

Eq. (3.33). Doing this, there will be four terms: one term with delta functions for both G_L and G_R , two terms with one delta function and one Keldysh Green function, and one term with two Keldysh Green functions. The first term is easily shown to vanish. In the second and third term, the interaction function f cancels, and one finds

$$I_{j,2,0}(t) = 2iv_F e \int dt_1 \text{tr} v^2 i_{jL}(t, t_1) \left(G_L^K(t_1, t_1) - \int d\tau_1 d\sigma_1 S(t_1, t_1 - \tau_1) \right. \\ \left. \times G_R^K(t_1 - \tau_1, t_1 - \sigma_1) S^\dagger(t_1 - \sigma_1, t_1) \right). \quad (4.7)$$

Here the last index “0” indicates that, apart from the difference between $i_{jk}(\tau)$ and $\delta_{jk}\delta(\tau)$, Eq. (4.7)—together with the zeroth-order result of Eq. (4.1)—reproduces precisely what one would obtain from the scattering approach for noninteracting fermions. Finally, in the remaining term, the interaction contribution is retained. In fact, it is just because of the contour dependence of the function f that one finds a nonzero result. We then find

$$I_{j,2,ee}(t) = -2eiv_F^2 \int dt_1 \int^{t_1} ds_1 \int d\tau_1 d\sigma_1 \left| \frac{f(\tau_1)f(\sigma_1)}{f(t_1 - s_1 + \sigma_1)f(s_1 - t_1 + \tau_1)} \right|^{1/N} \sin[\kappa_0(s_1 - t_1 + \tau_1)] \\ \times \text{tr} v^2 i_j(t, t_1) [G_L^K(s_1, t_1) S(t_1, t_1 - \tau_1) G_R^K(t_1 - \tau_1, s_1 - \sigma_1) (S^\dagger)(s_1 - \sigma_1, s_1) \\ - G_L^K(t_1, s_1) S(s_1, s_1 - \sigma_1) G_R^K(s_1 - \sigma_1, t_1 - \tau_1) (S^\dagger)(t_1 - \tau_1, t_1)]. \quad (4.8)$$

For $N=2$ there is an additional oscillating contribution to the current,

$$I_{j,\text{osc}}(t) = 8ev_1v_2 \left(\frac{2E_c e^C}{\pi^2 f(0)} \right)^2 \text{Im} e^{-2\pi i N} \int dt_1 \int^{t_1} dt_2 dt'_1 dt'_2 \sin(\phi(t_2, t'_1)) \\ \times \det[i_j(t, t_1) e^{i\phi_L(t_1)} S(t_1, t'_1) e^{-i\phi_R(t'_1)}, e^{i\phi_L(t_2)} S(t_2, t'_2) e^{-i\phi_R(t'_2)}] \\ \times |f(t_1, t'_1) f(t_1, t'_2) f(t_2, t'_1) f(t_2, t'_2)|^{1/2}, \quad (4.9)$$

where we defined

$$\det(B(1), \dots, B(N)) \\ = \frac{1}{N!} \sum_P \sum_{k_1, \dots, k_N} (-1)^P B(1)_{k_1, P(k_1)} \dots B(N)_{k_N, P(k_N)}, \quad (4.10)$$

where $B(j)$ is an $N \times N$ matrix, $j=1, \dots, N$. Equations (4.4), (4.8), and (4.9) represent the interaction corrections to the current.

This result is found to agree precisely with the linear dc conductance (2.10) calculated in Refs. 7 and 8 in the dc bias limit, to first order in the bias. To see this, we set $\mu_{R1} - \mu_{R2} = -eV$, expand to linear order in the bias voltage, take $S(t, t-\tau)$ to depend on τ only, set $\mu_{jL}=0$ for all channels $j=1, \dots, N$ in the fictitious lead, use the approximations $|f(\tau)| \approx \lambda \pi T / v_F \sinh[\pi T |\tau + t_0|]$, $\kappa_0(\tau) = \pi \theta(-\tau - t_0) / N$ valid for $\tau \ll t_0$ and $\tau \gg t_0$, and set $v_j=1$, so that $r_j=1$ [see Eq. (3.14)], which corresponds to detaching the fictitious lead and thus restoring the original system. The oscillating current corrections (4.4) and (4.9) reproduce the periodic-in- \mathcal{N} inter-

action corrections to the dot’s “electrochemical capacitance” $C_c = dQ/dV$ found in Ref. 6 if we set $\mu_{jL}=0$, choose a time-dependent chemical potential $\mu_{jR}(t) = eVt$ for all channels, and expand in V . [In order to recover the nonperiodic interaction correction to the capacitance reported in Ref. 8 from Eqs. (4.7) and (4.8) one has to include time dependencies in both μ_L and μ_R in order to ensure that no charge escapes through the fictitious lead.]

However, truncating at second order in the effective action \mathcal{S} is not a satisfactory description for a fully coherent quantum dot as there is no *a priori* reason why the effective action is small in this case. This will be made explicit in the next subsection, where we report a calculation to all orders in \mathcal{S} (but for N large). However, that Eqs. (4.4), (4.8), and (4.9) cannot be the correct interaction correction for a fully coherent dot becomes clear when one realizes that the current in fact depends on the chemical potentials μ_{jL} , $j=1, \dots, N$, of the fictitious reservoir, despite the fact that one has taken the limit $r_j \rightarrow 1$, $j=1, \dots, N$. (Note that this dependence on the fictitious chemical potentials is not manifest in the linear response calculation of Refs. 6–8 because $\mu_{jL}=0$, $j=1, \dots, N$, in equilibrium.)

Of course, truncating the perturbation expansion at second order in \mathcal{S} is justified if there is a reason why \mathcal{S} is small. This is the case, e.g., if the r_j are small, $j=1, \dots, N$. Then the fictitious lead is strongly coupled to the point contact, and serves as a source of relaxation. In that case the chemical potentials μ_{jL} must be chosen such that no current flows through the fictitious lead at any point in time. A more realistic (but still phenomenological) model of relaxation is to couple to quantum dot itself (rather than the point contact) to a fictitious lead with many weakly coupled channels.^{44,50,51,53–55} As shown in Appendix C, this model results in an effective action \mathcal{S} similar to Eq. (3.17), but with a subunitary scattering matrix S . If relaxation is strong, S is small, and expanding in the action is justified. For this situation, we recover the interaction corrections found in Refs. 6–8.

B. Large number of channels

We have not been able to do a full calculation to all orders in \mathcal{S} . However, by organizing each order in the effective

action to the (formal) power of N it carries, we have been able to calculate the current to all orders in \mathcal{S} while expanding in $1/N$. The reason that an expansion in $1/N$ is possible is that the correlator (3.35) of interacting fermions admits a systematic expansion around the noninteracting correlator, by expanding

$$f^{1/N} = 1 + \frac{1}{N} \ln f + \dots \quad (4.11)$$

Counting the power of $1/N$ is done keeping the RC time $\pi/E_c N$ and the dwell time τ_d constant, so that the only factors $1/N$ arise from the expansion (4.11) and from explicit summations over the channel indices. Oscillating contributions with an explicit dependence on the dimensionless gate voltage \mathcal{N} , involve N th order in perturbation theory and cannot be considered in this method.

Nonoscillating contributions to the current occur to even order in the effective action only. The nonoscillating current contribution of order $2n$ in the action reads

$$\begin{aligned} I_{j,n}(t) &= \frac{1}{(2n)!} (-i)^{2n} \langle \mathcal{S}^n I_j \rangle_0 \\ &= \frac{1}{(n!)^2} (-2iv_F)^{2n} \sum_{k_1, l_1, k'_1, l'_1} \dots \sum_{k_n, l_n, k'_n, l'_n} \int_c dt_1 \dots dt_n ds_1 \dots ds_n \int d\tau_1 \dots d\tau_n d\sigma_1 \dots d\sigma_n v_{k_1} \dots v_{k_n} \\ &\quad \times v_{l_1} \dots v_{l_n} S_{k_1, k'_1}(t_1, t_1 - \tau_1) \dots S_{k_n, k'_n}(t_n, t_n - \tau_n) (S^\dagger)_{l'_1, l_1}(s_1 - \sigma_1, s_1) \dots (S^\dagger)_{l'_n, l_n}(s_n - \sigma_n, s_n) \\ &\quad \times \langle T_c \hat{\psi}_{k_1 L}^\dagger(t_1) \hat{\psi}_{k'_1 R}(t_1 - \tau_1) \hat{\psi}_{l'_1 R}^\dagger(s_1 - \sigma_1) \hat{\psi}_{l_1 L}(s_1) \dots \hat{\psi}_{k_n L}^\dagger(t_n) \hat{\psi}_{k'_n L}(t_n - \tau_n) \hat{\psi}_{l'_n R}^\dagger(s_n - \sigma_n) \hat{\psi}_{l_n L}(s_n) \hat{I}_j(t) \rangle_0. \end{aligned} \quad (4.12)$$

As before, the time differences denoted by Greek symbols do not affect the contour ordering for the fermion operators.

The average of the product of $4n$ fermion operators and one current operator in Eq. (4.12) admits a standard diagrammatic representation if the factors $1/N \ln f$ in the expansion (4.11) of the general average (3.35) are considered “interaction lines.” Without interaction lines, the average is simply given by Wick’s theorem: Denoting each single-fermion Green function by a solid line, Wick’s theorem expresses the average as a sum over partitions in “bubbles” of alternating single-fermion Green functions for right-moving and left-

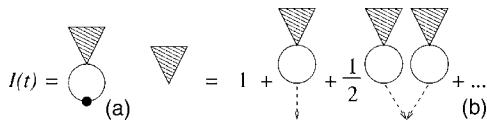


FIG. 5. (a) Diagrammatic representation of leading-in- $1/N$ contribution to the current. The current vertex i is denoted with a filled circle, the bubble of single-fermion Green function with a solid loop, and the tree structure with a hatched triangle. The defining equation for the tree structure is shown in (b), where the dotted arrow represents the interaction line.

moving fermions. One bubble contains the current vertex i . In our formal expansion in $1/N$, each bubble contributes a factor N , as it contains a trace over the channel index. Each interaction line, on the other hand, contributes a factor $1/N$. Only diagrams in which all bubbles are connected by interaction lines contribute to the current. This means that, to leading order in $1/N$, the relevant diagrams have a “tree” structure; see Fig. 5: Every bubble has precisely one connection to the current vertex via intermediate bubbles and interaction lines. Subleading contributions to the current contain “loops.”

To leading order in $1/N$, we only have to examine the tree diagrams, plus the current contribution that is zeroth order in the action; see Eq. (4.1). Combining these, we find

$$\begin{aligned} I_j(t) &= \frac{e}{2\pi\hbar} \int dt_1 \sum_{k=1}^N i_{jk}(t-t_1) [\mu_{Lk}(t_1) - \mu_{Rk}(t_1)] \\ &\quad + e \int dt_1 \text{tr} i_j(t-t_1) r^2 j_-(t_1), \end{aligned} \quad (4.13)$$

where we used Eq. (3.14) to express v in terms of the reflect-

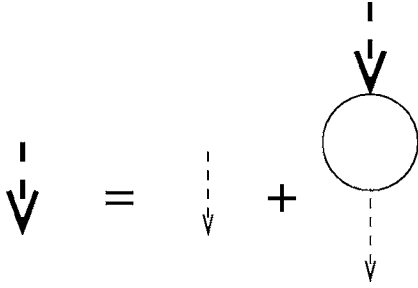


FIG. 6. Diagrammatic representation of the effective interaction κ (thick arrow) in terms of the bare interaction line $\ln f$ (thin arrow) and a bubble of single-fermion Green functions (solid loop).

tion matrix r . In Eq. (4.13), we abbreviated the contribution of the bubble by

$$j_-(t) = \frac{1}{2}iv_F \left(G_L^K(t,t) - \int d\tau d\sigma S(t,t-\tau) \times \tilde{G}_R^K(t-\tau,t-\sigma) S^\dagger(t-\sigma,t) \right), \quad (4.14)$$

where

$$\tilde{G}_R^K(t,s) = G_R^K(t,s) e^{-i[\tilde{\phi}(t) - \tilde{\phi}(s)]}. \quad (4.15)$$

The phase $\tilde{\phi}(t)$ satisfies the self-consistent equation

$$\tilde{\phi}(t) = 2 \int dt_1 \kappa_0(t_1 - t) \text{tr} r^2 j_-(t_1). \quad (4.16)$$

The quantity j_- is nothing but the Landauer expression for the difference of the current flowing into the leads coming from the quantum dot [second term in Eq. (4.14)] and coming from the fictitious reservoir [first term in Eq. (4.14)].

The overall phase factor that appears in $\tilde{G}_{mR}^K(t,s)$ can be written in terms of the electrochemical potential μ_c ,

$$\phi_{mR}(t) + \tilde{\phi}(t) - \phi_{mR}(s) + \tilde{\phi}(s) = \int_s^t dt' \mu_{cm}(t').$$

Here $\mu_{cm} = \mu_m + eV_d$ [the plus sign arises because the interaction is in the leads, not the dot, cf. Eq. (3.21)] and the dot potential is

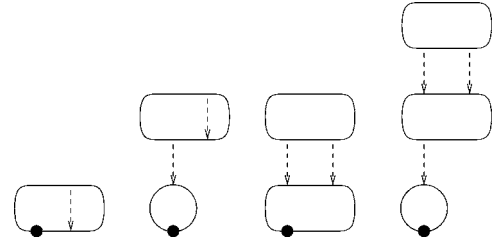


FIG. 7. Diagrammatic representation of subleading-in- $1/N$ corrections to the current.

$$CV_d(t) = \int_{k=j}^t \sum dt' I_j(t'), \quad (4.17)$$

where C is the dot's geometrical capacitance and we used the specific form of $\kappa_0(t'-t)$ and $i_{jk}(\tau)$. Note that in the limit $r_j \rightarrow 1$ the $\mu_{mL}(t)$ drop out of Eq. (4.13). Equation (4.17) is just the same as Eq. (1.1), written in terms of the currents. Hence, to leading order in $1/N$, our formalism recovers the self-consistent theory for the effect of the Coulomb interaction on transport through quantum dots.

To *subleading order* in $1/N$, one has to solve diagrams with one loop. The “interaction lines” in these diagrams are replaced by an effective interaction, which is an RPA-like series; see Fig. 6. Denoting the effective interaction by $\kappa(t,s)$, one finds that $\kappa(t,s)$ satisfies the self-consistent equation

$$\begin{aligned} \kappa(t,s) &= \kappa_0(t,s) - 2v_F \int dt_1 d\tau_1 d\sigma_1 \\ &\times [\kappa_0(t,t_1 - \tau_1) - \kappa_0(t,t_1 - \sigma_1)] \\ &\times \text{tr} r^2 S(t_1, t_1 - \tau_1) \tilde{G}_R^K(t_1 - \tau_1, t_1 - \sigma_1) \\ &\times S^\dagger(t_1 - \sigma_1, t_1) \kappa(t_1, s). \end{aligned} \quad (4.18)$$

Note that $\kappa(t,s) = 0$ if $t \geq s$ and that $\kappa(t,s)$ is real.

The one-loop interaction correction to the current can be represented by four diagrams; see Fig. 7. The first and second diagram in Fig. 7 represent a correction to j_- ,

$$\begin{aligned} j_-(t) &= \frac{1}{2}iv_F \left(G_L^K(t,t) - \int d\tau d\sigma S(t,t-\tau) \tilde{G}_R^K(t-\tau,t-\sigma) S^\dagger(t-\sigma,t) \right) + v_F^2 \text{Im} \int d \\ &\times s_1 d\tau_1 d\sigma_1 \kappa(s_1, t - \tau_1) S(t, t - \tau_1) \tilde{G}_R^K(t - \tau_1, s_1 - \sigma_1) S^\dagger(s_1 - \sigma_1, s_1) \\ &\times \left((1 - r^2) G_L^K(s_1, t) + r^2 \int d\tau_2 d\sigma_2 S(s_1, s_1 - \tau_2) \tilde{G}_R^K(s_1 - \tau_2, t - \sigma_2) S^\dagger(t - \sigma_2, t) \right), \end{aligned} \quad (4.19)$$

whereas the third and fourth diagram represent an additional renormalization of the distribution function for right-moving fermions, which can be represented by a change in the relation between to \tilde{G}_R and G_R ,

$$\tilde{G}_R^K(t,s) = G_R^K(t,s)e^{-i[\tilde{\phi}(t)-\tilde{\phi}(s)]+\alpha(t,s)}. \quad (4.20)$$

For the calculation of $\alpha(t,s)$, the regularization of the fermion operators at the impurity site is important, which makes the actual calculation rather cumbersome. The final result reads

$$\begin{aligned} \alpha(t,s) = & -\frac{1}{2}v_F^2 \int dt_1 dt_2 [\kappa(t_1-t) - \kappa(t_1-s)] [\kappa(t_2-t) - \kappa(t_2-s)] \text{tr} \left[G_L^K(t_1,t_2) G_L^K(t_2,t_1) \right. \\ & - \left. \left((1-r^2) G_L^K(t_1,t_2) + r^2 \int d\tau_1 d\sigma_2 S(t_1,t_1-\tau_1) \tilde{G}_R^K(t_1-\tau_1,t_2-\sigma_2) S^\dagger(t_2-\sigma_2,t_2) \right) \right. \\ & \left. \times \left((1-r^2) G_L^K(t_2,t_1) + r^2 \int d\tau_1 d\sigma_2 S(t_2,t_2-\tau_2) \tilde{G}_R^K(t_2-\tau_2,t_1-\sigma_1) S^\dagger(t_1-\sigma_1,t_1) \right) \right]. \end{aligned} \quad (4.21)$$

Equations (4.13), (4.16), and (4.18)–(4.21) give the solution of the problem up to subleading order in $1/N$.

Note that the leading corrections to the theory are in the form of a modified expression for the current j_- ; Eq. (4.16) and hence the relation (4.17) between $I_j(t)$ and the self-consistent potential V_d remains unchanged.

The interaction correction to j_- , contained in the second and third line of Eq. (4.19), consists of a term proportional to r^2 and a term proportional to $1-r^2$. When expanding in the action and truncating the expansion at second order, as is done in Sec. IV A and in Refs. 7 and 8, one recovers the first term only, with prefactor unity instead of $1-r^2$. However, upon setting $r^2 \rightarrow 1$, which is the relevant limit for a coherent quantum dot with a unitary scattering matrix, the prefactor of that first term vanishes, and the interaction correction to j_- is given by the second term instead. It is this replacement of the first term by the second term in the interaction correction that is the essential difference between the theories of Refs. 7 and 8 on the one hand and that of this work and Ref. 24 on the other hand.

V. STEADY-STATE TRANSPORT

In this section we analyze the results of the previous section for the case of steady-state transport through the quantum dot. We assume that the chemical potentials in the reservoirs and the gate voltages defining the shape of the dot are time independent. This implies that the scattering matrix $S(t,t-\tau)$ is independent of t and the Green functions $G_L(t,t-\tau)$ and $G_R(t,t-\tau)$ are independent of t . Current conservation then implies that there is no net current into the dot, $\sum_j I_j = 0$; see Eq. (3.27). This allows us to focus our calculation on the weighted difference of currents in the left and right leads,

$$I = \sum_{j=1}^N \Lambda_j I_j, \quad (5.1)$$

where Λ was defined in Eq. (2.4) above. We will be using the Fourier transform of the scattering matrix and the Green functions,

$$S(\varepsilon) = \int_0^\infty dt S(t) e^{i\varepsilon t}, \quad (5.2)$$

$$S^\dagger(\varepsilon) = \int_{-\infty}^0 dt S^\dagger(t) e^{i\varepsilon t}, \quad (5.3)$$

$$G_{mnR,L}^K(\varepsilon) = \int dt G_{mnR,L}^K(t) e^{i\varepsilon t} = \frac{i}{v_F} \delta_{mn} [2f_{R,L}(\varepsilon) - 1], \quad (5.4)$$

where f is the distribution function.

A. Fully coherent dot

We first consider the case $r^2=1$, corresponding to a fully coherent dot. We first consider the function $\tilde{\phi}$. By Eq. (4.16), $\tilde{\phi}$ depends on $\text{tr } j_-$ only. For the time-independent case, one has

$$\text{tr } j_- = -\frac{N}{2\pi} \frac{\partial \tilde{\phi}}{\partial t}. \quad (5.5)$$

Calculating the derivative $\partial \tilde{\phi} / \partial t$ from Eq. (4.16) one then finds

$$\frac{\partial \tilde{\phi}}{\partial t} = -\frac{N}{\pi} \int dt_1 \frac{\partial \kappa_0(t-t_1)}{\partial t} \frac{\partial \tilde{\phi}}{\partial t} = \frac{\partial \tilde{\phi}}{\partial t}, \quad (5.6)$$

which implies that the derivative $\partial \tilde{\phi} / \partial t$ is left undetermined by the self-consistency equation (4.16); any time-independent value for $\partial \tilde{\phi} / \partial t$ is a solution of the self-consistency equation. This is no problem for steady-state transport; in the previous section we have seen that $\tilde{\phi}$ is determined once the time dependence of bias and gate voltages is taken into account. In this section, we will set $\tilde{\phi}=0$ from now on.

The effective interaction function κ can be solved from Eq. (4.18) using Fourier transform,

$$\kappa(\omega) = \kappa_0(\omega) \left(\kappa_0(\omega) - \frac{i}{\pi} \int d\varepsilon \text{tr} S(\varepsilon - \omega) \times [f_R(\varepsilon - \omega) - f_R(\varepsilon)] S^\dagger(\varepsilon) \right)^{-1}.$$

Here $\kappa_0(\omega)$ is the Fourier transform of $\kappa_0(\tau)$,

$$\kappa_0(\omega) = \int d\tau \kappa_0(\tau) e^{i\omega\tau}. \quad (5.7)$$

This effective interaction is the same as that obtained by Golubev and Zaikin.²⁴ We further need to calculate the func-

tion $\alpha(t-s)$; see Eq. (4.20). Fourier transforming Eq. (4.21) one has

$$\alpha(\tau) = -\frac{1}{2\pi^2} \int d\omega [1 - \cos(\omega\tau)] |\kappa(\omega)|^2 \times \left(\omega N \coth(\omega/2T) - \sum_{\pm} \int d\varepsilon \text{tr} S(\varepsilon \pm \omega) f_R(\varepsilon \pm \omega) \times S^\dagger(\varepsilon \pm \omega) S(\varepsilon) [1 - f_R(\varepsilon)] S^\dagger(\varepsilon) \right). \quad (5.8)$$

Finally, with these results the current through the dot is

$$I = -\frac{e}{2\pi\hbar} \text{tr} \Lambda \mu_R + \frac{e}{2\pi\hbar} \int d\varepsilon \text{tr} \Lambda S(\varepsilon) f_R(\varepsilon) S^\dagger(\varepsilon) + \frac{e}{2\pi^2\hbar} \text{Im} \int d\omega d\varepsilon \kappa(\omega) \text{tr} \Lambda \times \{ S(\varepsilon) f_R(\varepsilon + \omega) S^\dagger(\varepsilon + \omega) S(\varepsilon) [1 - f_R(\varepsilon)] S^\dagger(\varepsilon) + S(\varepsilon) [1 - f_R(\varepsilon + \omega)] S^\dagger(\varepsilon + \omega) S(\varepsilon) f_R(\varepsilon) S^\dagger(\varepsilon) \} + \frac{e}{16\pi^4\hbar} \int d\omega d\varepsilon |\kappa(\omega)|^2 \text{tr} \Lambda S(\varepsilon) [2f_R(\varepsilon) - f_R(\varepsilon - \omega) - f_R(\varepsilon + \omega)] S^\dagger(\varepsilon) \times \left(\omega N \coth \frac{\omega}{2T} - \sum_{\pm} \int d\varepsilon' \text{tr} S(\varepsilon' \pm \omega) f_R(\varepsilon' \pm \omega) S^\dagger(\varepsilon' \pm \omega) S(\varepsilon) [1 - f_R(\varepsilon')] S^\dagger(\varepsilon') \right). \quad (5.9)$$

In order to find the linear conductance, we set $\mu_R = -\Lambda eV$, $\mu_L = 0$, expand in V , and find $I = GV$, with

$$G = \frac{e^2}{2\pi\hbar} \text{tr} \Lambda^2 - \frac{e^2}{2\pi\hbar} \int d\varepsilon \left(-\frac{\partial f(\varepsilon)}{\partial \varepsilon} \right) \text{tr} \Lambda S(\varepsilon) \Lambda S^\dagger(\varepsilon) + \frac{e^2}{2\pi^2\hbar} \text{Im} \int d\varepsilon d\omega \left(-\frac{\partial f(\varepsilon)}{\partial \varepsilon} \right) \kappa(\omega) \text{tr} [\Lambda S(\varepsilon) \Lambda S^\dagger(\varepsilon + \omega) - \Lambda S(\varepsilon) S^\dagger(\varepsilon + \omega) S(\varepsilon) \Lambda S^\dagger(\varepsilon)] [1 - 2f(\varepsilon + \omega)]. \quad (5.10)$$

The same equation was derived previously by Golubev and Zaikin using a different method.²⁴

Equation (5.9) gives the current for one particular realization of the scattering matrix $S(\varepsilon)$. In order to find the average and variance of the current or the conductance for an ensemble of quantum dots, we need to average $S(\varepsilon)$ over the appropriate ensemble of scattering matrices. Details on the averaging procedure can be found in Appendix B. For large N , the average of a product of traces may be calculated as the product of the averages. This means that the interaction kernel $\kappa(\omega)$ may be averaged separately, with the result

$$\kappa(\omega) = \frac{\pi}{g i \omega \tau_d [1 + (\pi/E_c g \tau_d)(1 + i\omega\tau_d)]}. \quad (5.11)$$

In Eq. (5.11), τ_d is the mean dwell time for electrons entering the dot. In terms of the dot's mean level spacing Δ and the total conductance $g = g_1 + g_2$ of the point contacts, one has

$$\tau_d = \frac{2\pi}{g\Delta}. \quad (5.12)$$

(If levels are spin degenerate, Δ is half the mean spacing between spin-degenerate levels.) Performing the ensemble average for the current, one then finds, to leading and sub-leading order in N and in the absence of time-reversal symmetry,

$$\langle I \rangle = -\frac{e}{2\pi\hbar} \text{tr} \Lambda \mu_R + \frac{e}{2\pi\hbar g} \int d\varepsilon [g \text{tr} \Lambda r_c f(\varepsilon) r_c^\dagger + \text{tr} \Lambda (1 - r_c r_c^\dagger) \text{tr} f(\varepsilon) (1 - r_c^\dagger r_c)] + \frac{e}{2\pi^2\hbar g^2} \text{Im} \int d\varepsilon d\omega \kappa(\omega) \frac{(i\omega\tau_d)^2}{(1 + i\omega\tau_d)^2} \times \{ g \text{tr} \Lambda r_c f(\varepsilon + \omega) r_c^\dagger (1 - r_c r_c^\dagger) \text{tr} [1 - f(\varepsilon)] (1 - r_c^\dagger r_c) + g \text{tr} \Lambda r_c [1 - f(\varepsilon + \omega)] r_c^\dagger (1 - r_c r_c^\dagger) \text{tr} f(\varepsilon) (1 - r_c^\dagger r_c) - \text{tr} \Lambda (1 - r_c r_c^\dagger) \text{tr} f(\varepsilon + \omega) (1 - r_c^\dagger r_c) r_c^\dagger r_c \text{tr} [1 - f(\varepsilon)] \times (1 - r_c^\dagger r_c) - \text{tr} \Lambda (1 - r_c r_c^\dagger) \text{tr} [1 - f(\varepsilon + \omega)] \times (1 - r_c^\dagger r_c) r_c^\dagger r_c \text{tr} f(\varepsilon) (1 - r_c^\dagger r_c) \}, \quad (5.13)$$

From Eq. (5.13) we conclude that there is no interaction correction to the average current for ideal point contacts. Setting $\mu_R = -eV\Lambda$ and expanding Eq. (5.13) to first order in the bias voltage V , we find the ensemble average of the linear dc conductance,

$$\langle G \rangle = \frac{e^2}{2\pi\hbar} \frac{g_1 g_2}{g} \left(1 - \text{Im} \int d\omega \frac{(g_2 s_1 + g_1 s_2) \tau_d [(\omega/T) - \sinh(\omega/T)]}{2g^2(1+i\omega\tau_d)[1+(\pi/E_c g \tau_d)(1+i\omega\tau_d)] \sinh^2(\omega/2T)} \right). \quad (5.14)$$

Performing the frequency integral in the limit $\hbar/\tau_d \ll T \ll E_c g$ one finds²⁴

$$\langle G \rangle = \frac{e^2}{2\pi\hbar} \frac{g_1 g_2}{g} \left[1 - \frac{2(s_1 g_2 + s_2 g_1)}{g^2} \ln \frac{E_c g e^{1+C}}{2\pi^2 T} \right], \quad (5.15)$$

which is the same result as for a quantum dot without coherent scattering from inside the dot; see Eq. (2.8). In the low-temperature limit $T \ll \hbar/\tau_d \ll E_c g$ one finds a saturation of the interaction correction,²⁴

$$\langle G \rangle = \frac{e^2}{2\pi\hbar} \frac{g_1 g_2}{g} \left(1 - \frac{2(g_2 s_1 + g_1 s_2)}{g^2} \ln \frac{E_c g \tau_d}{\pi} \right). \quad (5.16)$$

For the quantum interference corrections we have performed calculations for the linear conductance with ideal point contacts only. Weak localization is a small difference δG of the ensemble averaged conductance with and without magnetic field. We find that the interaction correction to weak localization is zero, so that δG is given by the noninteracting theory,^{56,57}

$$\delta G = - \frac{\nu_s e^2 N_1 N_2}{2\pi\hbar N^2}, \quad (5.17)$$

where ν_s denotes the spin degeneracy.

Closed-form expressions for the interaction correction to the conductance fluctuations could be obtained for the limiting cases $T \ll \hbar/\tau_d \ll NE_c$ and $\hbar/\tau_d \ll T \ll NE_c$, for which we find, in the absence of time-reversal symmetry,

$$\text{var } G = \left(\frac{\nu_s e^2 N_1 N_2}{2\pi\hbar N^2} \right)^2 \left[1 - \frac{2}{3} (\pi T \tau_d)^2 \times \left(1 + \frac{4\pi}{E_c N^2 \tau_d} \ln \frac{E_c N \tau_d}{\pi} \right) \right]. \quad (5.18)$$

and

$$\text{var } G = \left(\frac{\nu_s e^2 N_1 N_2}{2\pi\hbar N^2} \right)^2 \frac{\pi}{6T\tau_d} \left(1 - \frac{2\pi}{5TN\tau_d} \right), \quad (5.19)$$

respectively. For intermediate temperatures, the final result still contains two energy integrations, which are easily integrated numerically. Figure 8 shows the interaction correction to the variance as a function of temperature for the value $\pi/E_c N \tau_d = 0.1$. In the presence of time-reversal symmetry, the variance of the conductance is a factor 2 higher than in the absence of time-reversal symmetry. We conclude that, at zero temperature, there is no interaction correction to the conductance fluctuations, to leading order in $1/N$. At small but finite temperatures, interactions lead to a small decrease of the conductance fluctuations.

It is legitimate to ask how one can distinguish between the interaction correction to weak localization and the conductance fluctuations and the second-order interaction corrections to the classical (i.e., of order N) conductance or the second-order quantum interference corrections in the noninteracting theory, as both are of order $1/N$. With respect to the comparison to the second-order interaction correction to the classical conductance, an answer is readily given: this interaction correction does not depend on the magnetic field and has no mesoscopic fluctuations as a function of, e.g., Fermi energy or magnetic field. A formal method to distinguish the interaction corrections discussed above from second-order quantum interference corrections follows after introducing the number of ‘‘orbital channels’’ $N^o = N/\nu_s$. Then one finds that the interaction correction is a $1/N$ correction, while the second-order quantum interference correction is of order $N/(N^o)^2$. For the conductance fluctuations in the absence of time-reversal symmetry this problem does not arise, as there are no second-order quantum interference corrections in this case.^{40,56,57}

The absence of an interaction correction to weak localization, and to the conductance fluctuations at zero temperature, is perhaps one of the most striking predictions of our theory. In Sec. VII, we will give an argument to explain why such a renormalization leaves these ensemble averaged quantities unchanged, using the observation that the form of the conductance correction derives from the renormalized scattering matrix Eq. (3.6).

B. Incoherent dot

The opposite limit of a quantum dot without coherent reflection from inside the dot can be obtained from the general formalism of Sec. IV by setting $S(t, t-\tau) \rightarrow \delta(\tau)$ and interpreting r as the reflection matrix r_c of the point contacts. In

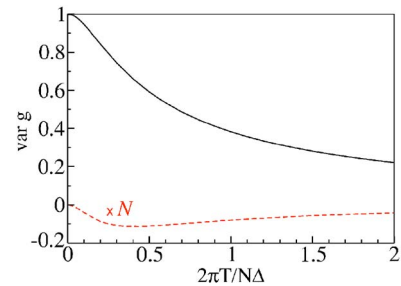


FIG. 8. (Color online) The interaction correction to the variance of the conductance, made dimensionless by writing $G = (\nu_s e^2 / 2\pi\hbar) (N_1 N_2 / N^2) g$. The solid curve shows the noninteracting contribution to the conductance fluctuations; the dashed curve shows the interaction correction for $\pi/E_c N \tau_d = 0.1$, multiplied by N to make it fit on the same scale as the noninteracting contribution.

this case, the distribution function of the left-moving fermions represents the distribution function of electrons leaving the quantum dot. Self-consistent determination of the corresponding chemical potential μ_L is problematic, however, because that would require knowledge of the average dwell time inside the dot, which is not contained in this model.

For small r_c , the results of Sec. IV A can be used. In this case one finds

$$I = -\frac{e}{2\pi\hbar} \text{tr} \Lambda(1 - r_c r_c^\dagger) \mu_R + \frac{1}{2} e i v_F^2 \int_0^\infty d\tau \sin[\kappa_0(-\tau)] \left| \frac{f(0)^2}{f(\tau)f(-\tau)} \right|^{1/N} \times \text{tr} \Lambda r_c r_c^\dagger [G_R^K(\tau)G_L^K(-\tau) - G_L^K(\tau)G_R^K(-\tau)]. \quad (5.20)$$

For $N=2$ there is an additional oscillating interaction correction to the current. Equation (5.20) is the nonequilibrium generalization of the original linear response theory of Refs. 1, 3, and 4. For the linear conductance one then finds

$$G = \frac{e^2}{2\pi\hbar} \left[\frac{N_1 N_2}{N} - \text{tr} \Lambda^2 r_c r_c^\dagger \left(1 + \frac{2}{N} \ln \frac{E_c N e^{1+C}}{2\pi^2 T} \right) \right] \quad (5.21)$$

if $\ln(E_c/\pi T) \ll N/2$ but $T \ll E_c N$ and

$$G = \frac{e^2}{2\pi\hbar} \left[\frac{N_1 N_2}{N} - \frac{1}{2\pi^{1/2}} \Gamma\left(1 - \frac{1}{N}\right) \Gamma\left(\frac{1}{N} - \frac{1}{2}\right) \times \left(\frac{E_c N e^C}{\pi^2 T} \right)^{2/N} \text{tr} \Lambda^2 r_c r_c^\dagger \cos \frac{\pi}{N} \right] \quad (5.22)$$

if $\ln(E_c/\pi T) \gg N/2$ (but temperature is larger than a suitable lower limit^{3,4,8}).

For large N , one recovers the same expressions as Golubev and Zaikin, see Eq. (2.8) and Ref. 23, starting from the results of Sec. IV B.

VI. TIME-DEPENDENT TRANSPORT: ADIABATIC APPROXIMATION AND LINEAR RESPONSE

In this section we will consider the case of a fully coherent dot only. We will limit ourselves to a theory of adiabatic and linear response transport—the response to either a slowly varying internal potential or to a small bias voltage, but not both—and consider the average and variance of the current only.

We will present our final results in terms of the Fourier transform of the scattering matrix,

$$S(t; \varepsilon) = \int d\tau e^{i\varepsilon\tau} S(t + \tau/2, t - \tau/2), \\ S^\dagger(t; \varepsilon) = \int d\tau e^{i\varepsilon\tau} S^\dagger(t + \tau/2, t - \tau/2). \quad (6.1)$$

In the adiabatic approximation, the Fourier transform $S(t; \varepsilon)$ satisfies the unitarity condition

$$[1 - (i/2)D_{t,\varepsilon}]S(t; \varepsilon)S^\dagger(t; \varepsilon) = 1, \quad (6.2)$$

where $D_{t,\varepsilon}AB = (\partial A/\partial t)(\partial B/\partial \varepsilon) - (\partial A/\partial \varepsilon)(\partial B/\partial t)$. To leading order in the adiabatic approximation, the Fourier transform

$S(t; \varepsilon)$ is equal to the “frozen” scattering matrix, taken by fixing the internal potentials at the value they have at time t . However, the frozen scattering matrix is unitary, whereas $S(t; \varepsilon)$ is not. Since our final expression for the current will be manifestly of first order in combined linear response and adiabatic approximation, we can neglect the difference between $S(t; \varepsilon)$ and the frozen scattering matrix in our final expressions.

We will set the chemical potential μ_L of the electrons in the fictitious reservoir equal to zero for all times. In the adiabatic approximation and for linear transport, the Keldysh Green function $G_{mR}^K(t, s)$ depends on the time difference $t-s$ only, $m=1, \dots, N$, and reads

$$G_{mnR}^K(t + \tau/2, t - \tau/2) = -\frac{T}{v_F} \delta_{mn} P \frac{e^{-i[\mu_{mR}(t) + eV_d(t)]\tau}}{\sinh(\pi T \tau)}, \quad (6.3)$$

where V_d is the dot potential; see Eq. (4.17).

Calculating the interaction function κ and the interaction correction α in the adiabatic limit is problematic. The reason is that both quantities involve long time scales during which one cannot assume that the derivative of the scattering matrix is constant (as one does in the adiabatic approximation). However, since κ and the interaction correction α contain traces, they are self-averaging, and we can replace them by their ensemble averages. Taking the ensemble average in the defining equation (4.18) for $\kappa(t, s)$, using the known probability distributions of time-dependent scattering matrices,⁴⁸ the equation for $\kappa(t, s)$ becomes

$$\kappa(t, s) = \kappa_0(t-s) + \frac{E_c N}{\pi} \int_t^{t_2} dt_2 \int_t^{t_1} dt_1 \kappa(t_2, s) \times e^{-E_c N(t_2-t)/\pi} e^{-(t_2-t_1)/\tau_d}. \quad (6.4)$$

This equation has no reference to the distribution functions in the lead or the time dependence of the scattering matrix. Hence we find that the solution is of the equilibrium form $\kappa(t-s)$, the Fourier transform of which is given in Eq. (5.11). For the interaction correction $\alpha(t + \tau/2, t - \tau/2)$, we note that it is an even function of τ . This implies its ensemble average must vanish if we consider the adiabatic approximation and linear response only.

We are now in a position to calculate the self-consistent potential V_d and the current I_j . The self-consistent equation for the potential V_d reads

$$e \frac{\partial V_d}{\partial t} = -\frac{E_c}{\pi} \int d\varepsilon \left(-\frac{\partial f(\varepsilon - \mu_c)}{\partial \varepsilon} \right) \text{tr} \left[i \frac{\partial \mu_c}{\partial t} S(t; \varepsilon) \frac{\partial S^\dagger(t; \varepsilon)}{\partial \varepsilon} + i S(t; \varepsilon) \frac{\partial S^\dagger(t; \varepsilon)}{\partial t} + \frac{1}{\pi} \text{Re} \int d\omega \kappa(\omega) [2f \times (\varepsilon + \omega - \mu_c) - 1] \left(\frac{\partial \mu_c}{\partial t} \frac{\partial}{\partial \varepsilon} + \frac{\partial}{\partial t} \right) \times [S(t; \varepsilon) S^\dagger(\varepsilon + \omega)] \right], \quad (6.5)$$

where the distribution function f is evaluated with respect to the electrochemical potential

$$\mu_c = eV_d + \frac{1}{N} \sum_k \mu_{kR}. \quad (6.6)$$

The first terms give the mean-field contribution to V_d . The last term gives the interaction correction. Upon taking the ensemble average, which is appropriate if V_d enters as a variable in the expression for the current, the last term vanishes.

For the current we find

$$I_j(t) = \frac{C}{N} \frac{\partial V_d}{\partial t} + \frac{e}{2\pi} \int d\varepsilon \left(-\frac{\partial f(\varepsilon - \mu_c)}{\partial \varepsilon} \right) \text{tr} \left(\delta_{jk} - \frac{1}{N} \right) \\ \times \left\{ j(t; \varepsilon) + \frac{1}{\pi} \text{Im} \int d\omega \kappa(\omega) [2f(\varepsilon + \omega - \mu_c) - 1] \right. \\ \left. \times S(t; \varepsilon) S^\dagger(t; \varepsilon + \omega) [j(t; \varepsilon + \omega) - j(t; \varepsilon)] \right\}, \quad (6.7)$$

where we abbreviated

$$j(t; \varepsilon) = \mu_R - S(t; \varepsilon) \left(\mu_R - i \frac{\partial \mu_c}{\partial t} \frac{\partial}{\partial \varepsilon} - i \frac{\partial}{\partial t} \right) S^\dagger(t; \varepsilon). \quad (6.8)$$

In the expressions for $I_j(t)$ and $\partial V_d(t)/\partial t$ one may replace the Fourier transformed scattering matrix $S(t; \varepsilon)$ by the frozen scattering matrix. Then a time derivative $\partial/\partial t$ is replaced by $(\partial X/\partial t)(\partial/\partial X)$, where X is a (set of) parameter(s) that characterizes the potentials in the dot.

In Eq. (6.7) the terms proportional to μ_R give the dc conductance of the dot, cf. Eq. (5.10). The first line of Eq. (6.7) is nothing but the Landauer formula with a self-consistent potential in the dot; the last lines give the correction from interactions beyond the mean-field level. The self-consistent potential V_d was omitted from Eq. (5.10) because it cannot be determined from the steady-state formulation of the problem only. The terms proportional to the time derivative give the emissivity $-dQ/dX$ of the quantum dot. Again, the first two lines of Eq. (6.7) correspond to the emissivity according to the self-consistent (Hartree) theory of Büttiker and co-workers²⁸ whereas the remaining lines of that equation give the correction from interactions beyond the mean-field level. The governing equation (6.5) for the self-consistent potential $V_d(t)$ is the same as that of the Hartree theory of Ref. 28. It depends both on (changes in) the dot potential and the chemical potential in the lead. After taking the ensemble average there are no interaction corrections to the Hartree theory for V_d .

Although the above expression for the current was derived in linear response and in the adiabatic approximation, it can be used to calculate the electrochemical capacitance C_c of the quantum dot, the derivative of the charge on the dot to a uniform shift of chemical potential μ_R . This is possible because, for the Hamiltonian H_0 , slowly increasing the chemical potential of all channels simultaneously corresponds to a time-independent small shift of the electrochemical potential; see Eq. (3.32). Hence the time derivatives of the phases carried by the bare Green functions G_R and G_L differ only by a small amount, and linear response is mean-

ingful. Considering the rate of change of the charge on the dot for the case in which there is no explicit time dependence of the dot scattering matrix, one finds from Eq. (6.7)

$$\frac{dQ}{dt} = -C \frac{\partial V_d}{\partial t}, \quad (6.9)$$

where $\partial V_d/\partial t$ can be calculated from Eq. (6.5). Upon changing all chemical potentials for incoming electrons simultaneously by an amount $-eV(t)$, we find²⁶

$$C_c = \frac{dQ}{dV} = [C^{-1} + (e^2 dn/d\varepsilon)^{-1}]^{-1}, \quad (6.10)$$

where $dn/d\varepsilon$ is the density of states of the dot,

$$\frac{dn}{d\varepsilon} = \frac{1}{2\pi} \int d\varepsilon \left(-\frac{\partial f(\varepsilon - \mu_c)}{\partial \varepsilon} \right) \text{tr} \left(iS(\varepsilon) \frac{\partial S^\dagger(\varepsilon)}{\partial \varepsilon} \right. \\ \left. + \frac{1}{\pi} \text{Re} \int d\omega \kappa(\omega) [2f(\varepsilon + \omega - \mu_c) - 1] \right. \\ \left. \times \frac{\partial}{\partial \varepsilon} S(\varepsilon) S^\dagger(\varepsilon + \omega) \right). \quad (6.11)$$

Here, the first term is the Hartree expression for the density of states of an open quantum dot;^{26,27} the second term is the interaction correction.

The ensemble average of the interaction correction to the density of states, and hence of the interaction correction to the electrochemical capacitance, is zero. It has, however, mesoscopic fluctuations. Calculating the interaction correction to the variance of the density of states, we find, for zero temperature,

$$\text{var}(dn/d\varepsilon) = \frac{v_s^2 \tau_d^2}{2\pi^2} \left(1 + \frac{22}{3N} \right), \quad (6.12)$$

from which one concludes, for $\hbar/\tau_d \ll E_c N$,

$$\text{var} C_c = \frac{\pi^2 e^4 v_s^2}{2E_c^4 N^4 \tau_d^2} \left(1 + \frac{22}{3N} \right). \quad (6.13)$$

Equations (6.12) and (6.13) are valid in the absence of time-reversal symmetry. With time-reversal symmetry, the variance is a factor 2 larger. The first term is the variance of the mean-field contribution to the capacitance,^{36,58} the second term is the interaction correction. Figure 9 shows the temperature dependence of the variance of the electrochemical capacitance.

If the quantum dot is operated as a quantum pump, two shape-defining parameters X_1 and X_2 are varied periodically in time, whereas the bias voltages are kept zero. We consider a harmonic time dependence for the parameters X_1 and X_2 ,

$$X_1(t) = \delta X_1 \sin(\omega t),$$

$$X_2(t) = \delta X_2 \sin(\omega t + \phi). \quad (6.14)$$

The charge pumped in one cycle Q_{pump} is defined as the integral of the current in one of the point contacts. Integrating over one cycle, one then finds for small pumping amplitudes³⁰

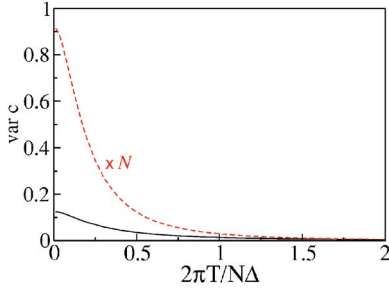


FIG. 9. (Color online) The variance of the mean-field contribution to the capacitance, made dimensionless by writing $C_c = 2\pi e^2 c / (E_c^2 N^2 \tau_d)$ (solid curve) and the interaction correction to the variance of the electrochemical capacitance (dashed), as a function of temperature. The interaction correction to the variance has been multiplied by N in order to fit on the same vertical scale.

$$Q_{\text{pump}} = (\pi \delta X_1 \delta X_2 \sin \phi)^2 \times \left(\frac{\partial}{\partial X_1} \frac{dQ_{\text{pump}}}{dX_2} - \frac{\partial}{\partial X_2} \frac{dQ_{\text{pump}}}{dX_1} \right). \quad (6.15)$$

[For large amplitudes δX_1 and δX_2 one has to average Eq. (6.15) over the area enclosed in the two-dimensional space spanned by the parameters X_1 and X_2 .]

Since the total current is conserved, we may calculate Q_{pump} as the integral of $I(t) = \sum_j I_j(t) \Lambda_{jj}$ over one period, i.e., (N_2/N) times the integral over the current through the first point contact minus (N_1/N) times the current in the second point contact. The ensemble average $\langle Q_{\text{pump}} \rangle$ is zero for symmetry reasons. When calculating the variance of the pumped charge, we note that, up to corrections of order $1/N^2$, one may replace the average of a product of traces by the product of the averages, with the exception of the two traces that contain the matrix Λ . Their average is zero, and one needs to consider the average of the product. This means that all terms proportional to $\partial V_d / \partial t$ can be dropped, and one may set

$$\begin{aligned} \frac{dQ_{\text{pump}}}{dX} = & -\frac{e}{2\pi} \int d\varepsilon \left(-\frac{\partial f(\varepsilon)}{\partial \varepsilon} \right) \text{tr} \left[i\Lambda S(\varepsilon) \frac{\partial S^\dagger(\varepsilon)}{\partial X} \right. \\ & + \frac{1}{\pi} \text{Re} \int d\omega \kappa(\omega) [2f(\varepsilon + \omega) - 1] \text{tr} \Lambda S(\varepsilon) \\ & \left. \times \left(\frac{\partial S^\dagger(\varepsilon + \omega)}{\partial X} - S^\dagger(\varepsilon + \omega) S(\varepsilon) \frac{\partial S^\dagger(\varepsilon)}{\partial X} \right) \right]. \end{aligned} \quad (6.16)$$

Using the results of Appendix B to perform the ensemble average, we find that, at zero temperature and for small pumping amplitudes, the variance of the pumped charge is given by

$$\text{var } Q_{\text{pump}} = \frac{4N_1 N_2 \nu_s^4}{N^4} \left(1 + \frac{2}{N} \right) (\delta X_1 \delta X_2 \sin \phi)^2, \quad (6.17)$$

irrespective of the presence or absence of a time-reversal symmetry breaking magnetic field. Here we have used the standard convention (within random matrix theory) to relate

the effect of the shape-defining parameters X_1 and X_2 to a parametric motion of the scattering matrix. The first term in Eq. (6.17) is the leading noninteracting contribution to the variance;^{30,59} the second term is the interaction correction. For temperatures $T \gg \hbar / \tau_d$ we find

$$\text{var } Q_{\text{pump}} = (\pi \delta X_1 \delta X_2 \sin \phi)^2 \frac{N_1 N_2 \nu_s^4}{12 T N^4 \tau_d} \left(1 + \frac{\pi}{T N \tau_d} \right), \quad (6.18)$$

where, again, the first term is the leading noninteracting contribution to the variance of the pumped current^{59,60} and the second term is the interaction correction.

VII. CONCLUSION

In this paper we have presented a systematic theory to evaluate the effects of electron-electron interactions on quantum transport through open quantum dots, accommodating arbitrary bias and time dependence of the dot's scattering matrix $S(t, t')$. Our result takes the form of an expression for the sample-specific current through a quantum dot in terms of $S(t, t')$. This expression has been obtained using a systematic expansion in $1/N$, where N is the total number of channels in the point contacts connecting the dot to the electron reservoirs. To leading order in $1/N$, our theory reproduces what one finds if the interactions were treated in the Hartree approximation. The subleading-in- $1/N$ corrections represent contributions to the current through the dot that cannot be described by means of a self-consistent potential.

Upon taking the average over an ensemble of chaotic quantum dots, we have been able to calculate the interaction corrections to weak localization, universal conductance fluctuations, capacitance fluctuations, and to the pumped current in case the quantum dot is operated as a quantum pump. For a quantum dot with ideal point contacts, we found that the interaction correction to the weak localization correction to the conductance vanishes, as well as the interaction correction to the conductance fluctuations at zero temperature. There is a negative interaction correction to the conductance fluctuations at finite temperature. There are positive interaction corrections to the capacitance fluctuations and to the fluctuations of the pumped current, but no corrections to their average. All interaction corrections are small as $1/N$ in comparison to the noninteracting (i.e., Hartree) contributions.

The problem of weak Coulomb blockade in open quantum dots has been addressed previously in the literature.^{7,8,24} Our findings for the quantum interference corrections, as well as our expression for the sample-specific dc conductance and capacitance, differ from those of Refs. 7 and 8. To explain the difference, we have shown that the formal expansion in the scattering matrix, used in Refs. 7 and 8, cannot be used to describe a coherent quantum dot. We believe that the expressions presented here, which were obtained using a systematic expansion in $1/N$, are correct. Our sample-specific expression agrees with that obtained in Ref. 24, although Ref. 24 does not consider the quantum corrections to the conductance.

The absence of an interaction correction to weak localization and to the conductance fluctuations at zero temperature can be understood using a simple argument: Since the charging interaction cannot lead to dephasing⁷—it corresponds to a time-dependent *uniform* shift of the dot’s potential—its only effect is a renormalization $S(\varepsilon) \rightarrow S'(\varepsilon)$ of the scattering matrix $S(\varepsilon)$ of the dot. To leading order in perturbation theory, such a renormalization was calculated explicitly in Sec. III B and in Ref. 24. Like the probability distribution functional $P[S(\varepsilon)]$ of the “bare” energy-dependent scattering matrix $S(\varepsilon)$, the probability distribution functional $P[S'(\varepsilon)]$ of the renormalized scattering matrix $S'(\varepsilon)$ is invariant under left and right multiplication of $S'(\varepsilon)$ with energy-independent unitary matrices U and V ,^{61–64}

$$P[S'(\varepsilon)] = P[US'(\varepsilon)V]. \quad (7.1)$$

In the presence of time-reversal symmetry, $U=V^T$. The statistical invariance (7.1), together with the same invariance for $P[S(\varepsilon)]$, implies that the distributions of both $S(\varepsilon)$ and $S'(\varepsilon)$ for a fixed energy ε are those of the circular ensemble from random matrix theory.^{40,65} Since the weak localization correction to the conductance and the conductance fluctuations at zero temperature do not depend on statistical correlations between scattering matrices $S(\varepsilon)$ at different energies, we conclude that these are given by the circular ensemble averages,^{56,57} and hence that there is no interaction correction. There is an interaction correction, however, to the conductance fluctuations at a finite temperature, which do involve such correlations. The above argument does not apply to a quantum dot with nonideal leads, for which the invariance property (7.1) does not hold, or to a quantum dot with relaxation, for which the scattering matrix is subunitary and the invariance property (7.1) is not sufficient to fix the distribution.

How reliable is the expansion in $1/N$ and how significant are the interaction corrections, as they are a factor $1/N$ smaller than the Hartree contributions? Recalling that N is the total number of channels in the point contacts connecting the dot and the electron reservoirs, not counting spin degeneracy, one has $N=4$ for a dot with two spin-degenerate single-mode point contacts. Although $N=4$ cannot be classified as “large,” we believe that for $N=4$ the $1/N$ expansion should give a reliable estimate of the interaction correction, whereas, on the other hand, the interaction corrections should still be significant (if they are nonzero). Measurements of the conductance distribution in single-mode quantum dots have shown excellent agreement with the (noninteracting) random matrix theory down to the lowest temperatures.⁶⁶ This experimental observation is in agreement with our theory since, to leading order in $1/N$ and at zero temperature, we find no interaction correction to the average and variance of the conductance. The argument following Eq. (7.1) is extremely suggestive and may hint that the agreement with random matrix theory is not restricted to the approximations of this paper, though we caution that higher orders in the interaction are presumably not simply described by an effective $S'(\varepsilon)$ due to inelastic processes. As long as the Fermi-liquid picture holds, however, one may

expect this description to be restored at zero temperature.

The interaction correction we calculated here is the result of the capacitive interaction only. For large N , it is a factor $\sim 1/N$ smaller than the leading noninteracting (Hartree) contribution to the transport properties and their quantum interference corrections. We have not considered other contributions to the electron-electron interaction. These were omitted from the Hamiltonian of the quantum dot because they are much smaller than the capacitive interaction if the dimensionless conductance of the dot is large.^{6,8,35} The interactions we omitted will give additional corrections to the dot’s transport properties, e.g., because they lead to dephasing. To the best of our knowledge, a microscopic theory of the effect of these residual interactions on the transport properties of a chaotic quantum dot still has to be developed.

In addition to the interaction corrections studied here, which are not periodic in the dimensionless gate voltage \mathcal{N} , there exist periodic-in- \mathcal{N} interaction corrections to the dot’s capacitance and conductance.^{6–8} These occur to N th order in perturbation theory, which rules out a large- N approach. Periodic-in-gate-voltage interaction corrections have been studied using an expansion in the scattering matrix.^{6,8} Although we have shown that such an expansion is not justified for aperiodic interaction corrections, we must leave the question of its validity for the periodic-in- \mathcal{N} corrections unanswered.

Before closing, we would like to make two remarks on the relation of our work to that of Golubev and Zaikin.²⁴ First, Golubev and Zaikin consider both the case of a “voltage biased” and a “current biased” quantum dot. The latter case corresponds to a quantum dot in series with a large ohmic resistor. In Ref. 24 it is shown that the interaction corrections in the two cases are qualitatively different. Our calculation, in which the chemical potentials of the electrons coming in through the point contacts are kept fixed, corresponds to the voltage biased case. Second, Golubev and Zaikin argue that the sole effect of interactions as a renormalization of the reflection matrix of the point contacts r_c only. While this is true for an incoherent dot,^{41,42,67,68} our calculation shows that this is not correct for a coherent dot: When quantum interference corrections are taken into account, the renormalization of the scattering matrix is more complicated, and involves more than one energy. This is manifested, e.g., in the existence of an interaction correction to the conductance fluctuations at finite temperature.

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APPENDIX A: BOSONIZATION

In this section we calculate the contour-ordered correlation functions of the fermion fields in the Hamiltonian \hat{H}_0 of Eq. (3.18). The bosonization method allows for an exact calculation of the fermion correlation functions for the Hamil-

tonian \hat{H}_0 , despite the fact that \hat{H}_0 is quadratic in the fermion operators⁶⁹: \hat{H}_0 is rewritten in terms boson fields $\hat{\varphi}_{jL}(x)$ and $\hat{\varphi}_{jR}(x)$. In the bosonized language \hat{H}_0 is diagonalized easily, and the fermion correlators can be found from the boson correlation functions.

The boson fields $\hat{\varphi}_{jL}(x)$ and $\hat{\varphi}_{jR}(x)$ are related to the fermion fields $\hat{\psi}_{jL}(x)$ and $\hat{\psi}_{jR}(x)$ as

$$\hat{\psi}_{jL} = \frac{\hat{\eta}_j}{\sqrt{2\pi\lambda v_F}} e^{-i\hat{\varphi}_{jL}}, \quad j = 1, \dots, N, \quad (\text{A1})$$

$$\hat{\psi}_{jR} = \frac{\hat{\eta}_j}{\sqrt{2\pi\lambda v_F}} e^{i\hat{\varphi}_{jR}}, \quad j = 1, \dots, N. \quad (\text{A2})$$

They obey the commutation rules

$$[\hat{\varphi}_{jL}(x), \hat{\varphi}_{iL}(y)] = -i\pi\delta_{ij}\text{sgn}(x-y),$$

$$[\hat{\varphi}_{jR}(x), \hat{\varphi}_{iR}(y)] = i\pi\delta_{ij}\text{sgn}(x-y),$$

$$[\hat{\varphi}_{jR}(x), \hat{\varphi}_{iL}(y)] = -i\pi\delta_{ij}. \quad (\text{A3})$$

In Eq. (A2) $\hat{\eta}_j = \hat{\eta}_j^\dagger$ is a Majorana fermion, $\{\hat{\eta}_j, \hat{\eta}_i\} = 2\delta_{ij}$, and λ is a cutoff time that is taken to zero at the end of the calculation. Since the Majorana fermions do not enter into the Hamiltonian, their average is given by

$$\langle T_c \hat{\eta}_j(t), \hat{\eta}_i(s) \rangle = \delta_{ij} \text{sgn}_c(t-s), \quad (\text{A4})$$

where the sign sgn_c is defined with respect to the ordering along the integration contour c .

The (normal-ordered) densities of left and right moving fermions are proportional to the derivatives of the boson fields,

$$:\hat{\psi}_{jL}^\dagger(x)\hat{\psi}_{jL}(x): = \frac{1}{2\pi} \frac{\partial}{\partial x} \hat{\varphi}_{jL}(x), \quad (\text{A5})$$

$$:\hat{\psi}_{jR}^\dagger(x)\hat{\psi}_{jR}(x): = \frac{1}{2\pi} \frac{\partial}{\partial x} \hat{\varphi}_{jR}(x). \quad (\text{A6})$$

Hence the number of electrons on the quantum dot is simply expressed as the sum of left-moving and right-moving boson fields at $x=0$,

$$\hat{N}_{\text{dot}} = -\frac{1}{2\pi} \sum_{j=1}^N [\hat{\varphi}_{jL}(0) + \hat{\varphi}_{jR}(0)]. \quad (\text{A7})$$

There are no boundary terms at $x=-\infty$ because the interaction extends over the entire lead. The reference electron number N_{ref} has been absorbed in the definition of normal ordering of the fermion operators. Then, once expressed in terms of the boson fields, the Hamiltonian \hat{H}_0 becomes quadratic,

$$\begin{aligned} \hat{H}_0 = & \frac{v_F}{4\pi} \sum_{j=1}^N \int_{-\infty}^{\infty} dx \left[\left(\frac{\partial \hat{\varphi}_{jL}}{\partial x} \right)^2 + \left(\frac{\partial \hat{\varphi}_{jR}}{\partial x} \right)^2 \right] \\ & + \frac{E_c}{4\pi^2} \left(\sum_{j=1}^N [\hat{\varphi}_{jL}(0) + \hat{\varphi}_{jR}(0) + 2\pi N] \right)^2. \end{aligned} \quad (\text{A8})$$

The Hamiltonian is supplemented with the condition that the chemical potential of left-moving fermions moving towards the dot-lead interface at $x=0$ is equal to $\mu_{jL}(t)/2\pi v_F$ and that the density of right-moving fermions is equal to $\mu_{jR}(t)/2\pi v_F$. We generate the electron densities corresponding to these chemical potentials by the inclusion of a time-dependent forward-scattering potential in the Hamiltonian. This potential is located at a distance a from the dot-lead interface at $x=0$. (The precise value of a is not important for the subsequent considerations, as long as $a>0$.) In boson language it has the form

$$\begin{aligned} \hat{H}'(t) = & \frac{v_F}{2\pi} \sum_{j=1}^N m_{jL}(t) \left. \frac{\partial \hat{\varphi}_{jL}(x)}{\partial x} \right|_{x \rightarrow a} \\ & - \frac{v_F}{2\pi} \sum_{j=1}^N m_{jR}(t) \left. \frac{\partial \hat{\varphi}_{jR}(x)}{\partial x} \right|_{x \rightarrow -a}, \end{aligned} \quad (\text{A9})$$

where m_{jL} and m_{jR} are the integrals of the chemical potentials μ_{jL} and μ_{jR} , respectively,

$$m_{jL}(t) = \int_{-\infty}^{t+a/v_F} dt' \mu_{jL}(t'), \quad j = 1, \dots, N,$$

$$m_{jR}(t) = \int_{-\infty}^{t+a/v_F} dt' \mu_{jR}(t'), \quad j = 1, \dots, N. \quad (\text{A10})$$

We assume that the system is in equilibrium, $\mu_{jL}(t) = \mu_{jR}(t) = 0, j=1, \dots, N$ for times t smaller than a reference time t_{ref} .

In order to find correlation functions of fermion creation and annihilation operators, expressions of the form $T_c e^a$ need to be averaged over the boson fields, where a is an arbitrary linear combination of the boson fields φ_{jL} and φ_{jR} , $j=1, \dots, N$, at different times. Since the Hamiltonian is quadratic in the boson fields, such an average can be calculated as

$$\langle \exp(\hat{a}) \rangle = \exp\left(\langle \hat{a} \rangle + \frac{1}{2} [\langle \hat{a}^2 \rangle - \langle \hat{a} \rangle^2] \right), \quad (\text{A11})$$

which means that only contour-ordered correlators of up to two boson fields at different times are needed. Hence it remains to calculate the correlation functions of the boson fields.

For this calculation, it is necessary that we transform the boson fields such that the charge mode is separated from modes that do not charge the quantum dot. The corresponding basis change reads

$$\varphi_{\mu L} = \sum_j o_{\mu j} \varphi_{jL}, \quad \varphi_{\mu R} = \sum_j o_{\mu j} \varphi_{jR}, \quad (\text{A12})$$

where the transformed fields carry a Greek index and the original fields carry a roman index. Here the $o_{\mu j}$ are real numbers satisfying

$$\sum_j o_{\mu j} o_{\nu j} = \delta_{\mu\nu}, \quad \sum_\mu o_{\mu j} o_{\mu i} = \delta_{ij}$$

and

$$o_{1j} = N^{-1/2}, \quad j = 1, \dots, N. \quad (\text{A13})$$

With this choice, the transformed fields $\hat{\phi}_{1L}$ and $\hat{\phi}_{1R}$ describe a mode that charges the quantum dot, whereas all other fields $\hat{\phi}_{\mu L}$ and $\hat{\phi}_{\mu R}$, $\mu=2, \dots, N$, do not involve charging of the dot. The transformed boson fields $\hat{\phi}_{\mu L}$ and $\hat{\phi}_{\mu R}$ satisfy the same boson commutation relations as the original fields $\hat{\phi}_{jL}$ and $\hat{\phi}_{jR}$,

$$\begin{aligned} [\hat{\phi}_{\mu L}(x), \hat{\phi}_{\nu L}(y)] &= -i\pi\delta_{\mu\nu}\text{sgn}(x-y), \\ [\hat{\phi}_{\mu R}(x), \hat{\phi}_{\nu R}(y)] &= i\pi\delta_{\mu\nu}\text{sgn}(x-y), \\ [\hat{\phi}_{\mu R}(x), \hat{\phi}_{\nu L}(y)] &= -i\pi\delta_{\mu\nu}. \end{aligned} \quad (\text{A14})$$

In terms of the transformed fields, the boson Hamiltonian \hat{H}_0 and the time-dependent perturbation \hat{H}' read

$$\begin{aligned} \hat{H}_0 &= \frac{v_F}{4\pi} \sum_{\mu=1}^N \int_{-\infty}^{\infty} dx \left[\left(\frac{\partial \hat{\phi}_{\mu L}}{\partial x} \right)^2 + \left(\frac{\partial \hat{\phi}_{\mu R}}{\partial x} \right)^2 \right] \\ &+ \frac{E_c N}{4\pi^2} \left(\hat{\phi}_{L,1}(0) + \hat{\phi}_{R,1}(0) + \frac{2\pi\mathcal{N}}{\sqrt{N}} \right)^2, \end{aligned} \quad (\text{A15})$$

$$\begin{aligned} \hat{H}' &= \frac{v_F}{2\pi} \sum_{\mu=1}^N m_{\mu L}(t) \left. \frac{\partial \hat{\phi}_{\mu L}(x)}{\partial x} \right|_{x \rightarrow a} \\ &- \frac{v_F}{2\pi} \sum_{\mu=1}^N m_{\mu R}(t) \left. \frac{\partial \hat{\phi}_{\mu R}(x)}{\partial x} \right|_{x \rightarrow -a}, \end{aligned} \quad (\text{A16})$$

where

$$m_{\mu L}(t) = \sum_j o_{\mu j} m_{jL}(t), \quad m_{\mu R}(t) = \sum_j o_{\mu j} m_{jR}(t). \quad (\text{A17})$$

In the transformed basis there is no charging interaction for the modes $\mu=2, \dots, N$. Solving the Heisenberg evolution equation for the fields $\hat{\phi}_{\mu L}$ and $\hat{\phi}_{\mu R}$,

$$\frac{\partial \hat{\phi}_{\mu L}(x,t)}{\partial t} = v_F m_{\mu L}(t) \delta(x-a) + v_F \frac{\partial \hat{\phi}_{\mu L}(x,t)}{\partial x},$$

$$\frac{\partial \hat{\phi}_{\mu R}(x,t)}{\partial t} = v_F m_{\mu R}(t) \delta(x+a) - v_F \frac{\partial \hat{\phi}_{\mu R}(x,t)}{\partial x},$$

we obtain an expression for $\hat{\phi}(x,t)$ in terms of boson fields at the reference time t_{ref} ,

$$\begin{aligned} \hat{\phi}_{\mu L}(x,t) &= \hat{\phi}_{\mu L}(x + v_F(t - t_{\text{ref}}), t_{\text{ref}}) \\ &+ m_{\mu L}[t + (x-a)/v_F] \theta(a-x), \\ \hat{\phi}_{\mu R}(x,t) &= \hat{\phi}_{\mu R}(x - v_F(t - t_{\text{ref}}), t_{\text{ref}}) \\ &- m_{\mu R}[t - (x+a)/v_F] \theta(a+x), \end{aligned} \quad (\text{A18})$$

[recall that $m_{\mu L}(t) = m_{\mu R}(t) = 0$ for $t < t_{\text{ref}}$]. For the transformed mode $\mu=1$, the charging interaction is important. In this case, the Heisenberg equation of motion reads

$$\begin{aligned} \frac{\partial \hat{\phi}_{1L}(x,t)}{\partial t} &= v_F m_{1L}(t) \delta(x-a) + v_F \frac{\partial \hat{\phi}_{1L}(x,t)}{\partial x} \\ &+ \frac{E_c N}{\pi} \left(\hat{\phi}_{1L}(0) + \hat{\phi}_{1R}(0) + \frac{2\pi\mathcal{N}}{\sqrt{N}} \right) \theta(-x), \\ \frac{\partial \hat{\phi}_{1R}(x,t)}{\partial t} &= v_F m_{1R}(t) \delta(x+a) \\ &- v_F \frac{\partial \hat{\phi}_{1L}(x,t)}{\partial x} - \frac{E_c N}{\pi} \left(\hat{\phi}_{1L}(0) + \hat{\phi}_{1R}(0) \right. \\ &\left. + \frac{2\pi\mathcal{N}}{\sqrt{N}} \right) \theta(-x) \end{aligned}$$

and has the solution

$$\begin{aligned} \hat{\phi}_{1L}(x,t) &= \hat{\phi}_{1L}(x + v_F(t - t_{\text{ref}}), t_{\text{ref}}) \\ &+ m_{1L}[t + (x-a)/v_F] \theta(a-x) \\ &- \frac{E_c N}{\pi} \int_{t_{\text{ref}}}^t dt' \theta(-x) \theta(t' - t - x/v_F) e^{-E_c N(t-t')/\pi} \\ &\times \left(\hat{\phi}_{1R}(-v_F(t' - t_{\text{ref}}), t_{\text{ref}}) - m_{1R}(t' - a/v_F) \right. \\ &+ \hat{\phi}_{1L}(v_F(t - t_{\text{ref}}), t_{\text{ref}}) \\ &\left. + m_{1L}(t' - a/v_F) + \frac{2\pi\mathcal{N}}{\sqrt{N}} \right), \\ \hat{\phi}_{1R}(x,t) &= \hat{\phi}_{1R}(x - v_F(t - t_{\text{ref}}), t_{\text{ref}}) \\ &- m_{1R}[t + (x+a)/v_F] \theta(a+x) \\ &- \frac{E_c N}{\pi} \int_{t_{\text{ref}}}^t dt' [\theta(x) \theta(t - x/v_F - t') e^{-E_c N(t-x/v_F-t')/\pi} \\ &+ \theta(-x) e^{-E_c N(t-t')/\pi}] \\ &\times \left(\hat{\phi}_{1R}(-v_F(t' - t_{\text{ref}}), t_{\text{ref}}) - m_{1R}(t' - a/v_F) \right. \\ &\left. + \hat{\phi}_{1L}(v_F(t - t_{\text{ref}}), t_{\text{ref}}) + m_{1L}(t' - a/v_F) + \frac{2\pi\mathcal{N}}{\sqrt{N}} \right). \end{aligned} \quad (\text{A19})$$

According to Eqs. (A18) and (A19), the effect of the time-dependent potentials is to shift the boson fields by a real number. Such a shift affects the average of the boson fields,

$$\begin{aligned} \langle \hat{\phi}_{\mu L}(0,t) \rangle &= m_{\mu L}(t - a/v_F), \\ \langle \hat{\phi}_{\mu R}(0,t) \rangle &= -m_{\mu R}(t - a/v_F) - \frac{2\pi\mathcal{N}}{\sqrt{N}} \delta_{\mu,1} \\ &+ \frac{E_c N}{\pi} \delta_{\mu,1} \int_{t_{\text{ref}}}^t dt' e^{-E_c N(t-t')/\pi} \\ &\times [m_{1R}(t' - a/v_F) - m_{1L}(t' - a/v_F)], \end{aligned} \quad (\text{A20})$$

but not the connected two-point correlators. Hence the connected two-boson correlation functions are the same as in equilibrium. They can be obtained by analytical continuation from the imaginary-time correlators obtained in Refs. 6 and 8 or they can be calculated directly from the solution of the equation of motion. In the latter method one first calculates the spectral density $A(t, t')$. Since Eqs. (A18) and (A19) express all boson operators in terms of the operators at the same reference time t_{ref} , the spectral density can be found from the equal-time commutation relations (A14). We only need the spectral density at $x=0$,

$$\begin{aligned}
 A_{\mu L, \nu L}(t, t') &= \langle [\hat{\phi}_{\mu L}(0, t), \hat{\phi}_{\nu L}(0, t')]_- \rangle = -i\pi\delta_{\mu\nu}\text{sgn}(t - t'), \\
 A_{\mu L, \nu R}(t, t') &= \langle [\hat{\phi}_{\mu L}(0, t), \hat{\phi}_{\nu R}(0, t')]_- \rangle \\
 &= i\pi\delta_{\mu\nu} \left(1 - \frac{E_c N}{\pi} \delta_{\mu,1} \int_0^\infty d\zeta e^{-E_c N \zeta / \pi} \right. \\
 &\quad \left. \times \text{sgn}(t' - t - \zeta) \right), \\
 A_{\mu R, \nu L}(t, t') &= \langle [\hat{\phi}_{\mu R}(0, t), \hat{\phi}_{\nu L}(0, t')]_- \rangle \\
 &= -i\pi\delta_{\mu\nu} \left(1 - \frac{E_c N}{\pi} \delta_{\mu,1} \int_0^\infty d\zeta e^{-E_c N \zeta / \pi} \right. \\
 &\quad \left. \times \text{sgn}(t - \zeta - t') \right), \\
 A_{\mu R, \nu R}(t, t') &= \langle [\hat{\phi}_{\mu R}(0, t), \hat{\phi}_{\nu R}(0, t')]_- \rangle \\
 &= -i\pi\delta_{\mu\nu}\text{sgn}(t - t'). \tag{A21}
 \end{aligned}$$

The two-boson correlation functions are then found using the fluctuation-dissipation theorem, see, e.g., Ref. 70. The resulting expressions for the averages and the connected correlation functions of the original boson fields $\hat{\phi}_{Lj}$ and $\hat{\phi}_{Rj}$, $j=1, \dots, N$ are

$$\langle \hat{\phi}_{jL}(t) \rangle = \int^t dt' \mu_{jL}(t'), \tag{A22}$$

$$\begin{aligned}
 \langle \hat{\phi}_{jR}(t) \rangle &= - \int^t dt' \mu_{jL}(t') - \frac{2\pi N}{N} - \int^t dt' \sum_{k=1}^N i_{jk}(t - t') \\
 &\quad \times [\mu_{kR}(t'') - \mu_{kL}(t'')], \tag{A23}
 \end{aligned}$$

where i_{ik} was defined in Eq. (3.25), and

$$\begin{aligned}
 \langle T_c \hat{\phi}_{iL}(t') \hat{\phi}_{jL}(t'') \rangle &= \delta_{ij} \ln \left(\frac{\lambda \pi T / i}{\sinh\{\pi T [t' - t'' - i\lambda \text{sgn}_c(t' - t'')]\}} \right) \\
 &\quad + A + \left(\delta_{ij} - \frac{1}{N} \right) B \\
 &= \langle T_c \hat{\phi}_{iR}(t') \hat{\phi}_{jR}(t'') \rangle,
 \end{aligned}$$

$$\begin{aligned}
 \langle T_c \hat{\phi}_{iL}(t') \hat{\phi}_{jR}(t'') \rangle &= \delta_{ij} \frac{i\pi}{2} \text{sign}_c(t' - t'') - \frac{1}{N} \ln f(t'', t') - A \\
 &= \langle T_c \hat{\phi}_{iR}(t'') \hat{\phi}_{jL}(t') \rangle, \tag{A24}
 \end{aligned}$$

where A and B are positive constants that are taken to infinity at the end of the calculation and f is given in Eq. (3.36). Calculation of the fermion correlators of Sec. III E using the rule (A11) is straightforward now.

APPENDIX B: ENSEMBLE AVERAGE

In this section we present some material relevant for performing the averages over an ensemble of microscopically different but macroscopically equivalent quantum dots. We consider the statistical distribution of the scattering matrix for an ensemble of chaotic quantum dots. The dots are placed in a magnetic field. The magnetic field strength is described by the dimensionless parameter α , which is proportional to the magnetic flux Φ through the dot,

$$\alpha = \frac{e\Phi}{hc} \left(\frac{\kappa \hbar v_F l}{L^2 \Delta} \right)^{1/2}, \tag{B1}$$

where Δ is the mean spacing between spin-degenerate levels, L its linear size, l the elastic mean free path inside the dot, and κ a numerical constant of order unity. One has $\kappa = 4\pi/15$ for a diffusive sphere of radius L and $\kappa = \pi/2$ for a diffusive disk of the same radius.⁷¹ If the electron motion inside the dot is ballistic, but with diffusive boundary scattering, l is replaced by $5L/8$ and $\pi L/4$ for the cases of the sphere and the disk, respectively. The shape of the dot is controlled by two gate voltages, which are described by the dimensionless parameters X_1 and X_2 . The normalization of the parameters X_1 and X_2 is the same as in Ref. 48.

Two types of scattering processes contribute to the scattering matrix: direct reflection at the contacts, which is described by the energy-independent reflection matrix r_c , and scattering from inside the quantum dot. The matrix r_c is a property of the contacts and is not subject to mesoscopic fluctuations; it is only the contribution that involves scattering from inside the dot that has fluctuations. In order to manifestly separate the two contributions, S is parametrized as⁷²

$$S = r_c + (1 - r_c r_c^\dagger)^{1/2} S_0 (1 + r_c^\dagger S_0)^{-1} (1 - r_c^\dagger r_c)^{1/2}, \tag{B2}$$

where S_0 has the same statistics as the scattering matrix of a chaotic quantum dot ideal contacts.

For electrons with spin (and without spin-orbit scattering), the scattering matrix takes the form

$$S = S^o \otimes \mathbb{1}_2, \tag{B3}$$

where $\mathbb{1}_2$ is the 2×2 unit matrix (acting on the spin degrees of freedom) and S^o is a unitary matrix of size $N^o = N/2$ representing scattering of the orbital degrees of freedom. A similar decomposition applies to the matrices r_c and S_0 . The results below are for the distribution $P(S_0^o)$ of the orbital scattering matrix S_0^o , in the limit of large N . For simplicity of notation we will drop the subscript “0” and the superscript “o.”

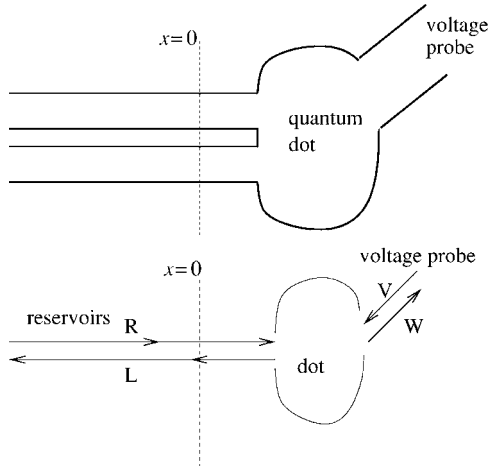


FIG. 10. Top panel: schematic drawing of a quantum dot with a voltage probe. Bottom panel: layout of labeling of fermion fields R, L, V , and W .

For large N , only moments of the distribution that contain as many factors of S as of its complex conjugate S^* and in which the indices of factors S and S^* are pairwise equal are nonzero.⁷³ This implies that all odd moments of $P(S)$ are zero. Since the probability of S is close to Gaussian for large N , with small non-Gaussian corrections, $P(S)$ is well characterized by its moments. The second moment characterizes the Gaussian part of the distribution; higher moments represent the non-Gaussian corrections to $P(S)$.

In order to specify $P(S)$, it is sufficient to consider generic moments in which all (pairs of) indices are different. Moments in which two or more pairs of indices coincide are fully determined by the invariance of the ensemble under basis changes in the leads.⁷³ For the general time-dependent case, we need the second moment $\langle S_{ij}(t, t-\tau)S_{ij}(s, s-\sigma)^* \rangle$ for $i \neq j$ only, which was calculated in Ref. 48,

$$\begin{aligned} & \langle S_{ij}(t, t-\tau)S_{ij}(s, s-\sigma)^* \rangle \\ &= \frac{1}{N\tau_d} \delta(\tau-\sigma) \theta(\tau) \\ & \times \exp \left[- \int_0^\tau d\tau' \left(\frac{1}{\tau_d} + [\alpha(t-\xi) - \alpha(s-\xi)]^2 \right. \right. \\ & \left. \left. + 2 \sum_j [X_j(t-\xi) - X_j(s-\xi)]^2 \right) \right]. \end{aligned} \quad (\text{B4})$$

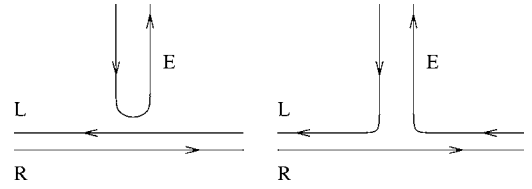


FIG. 11. Left diagram: Schematic drawing of the fictitious lead (E) coupling of which to the left-moving fermions through the Hamiltonian (C8) gives the second term of the effective action (C7). Right diagram: Definition of the fermion fields R, E , and L after the change of variables of Eq. (C9).

For the time-independent case, we need to go calculate the corrections to the Gaussian distribution. Using the shorthand notation $S(1)=S[\varepsilon(1), \alpha(1), X_1(1), X_2(1)]$, the relevant moments are

$$\langle S_{ij}(1)S_{ij}(2)^* \rangle = W_1(1,2), \quad (\text{B5})$$

$$\langle S_{ij}(1)S_{kj}(2)^*S_{kl}(3)S_{il}(4)^* \rangle = W_2(1,2,3,4), \quad (\text{B6})$$

$$\begin{aligned} \langle S_{ij}(1)S_{ij}(2)^*S_{kl}(3)S_{kl}(4)^* \rangle &= W_1(1,2)W_1(3,4) \\ &+ W_{1,1}(1,2,3,4), \end{aligned} \quad (\text{B7})$$

$$\langle S_{ij}(1)S_{kj}(2)^*S_{kl}(3)S_{ml}(4)^*S_{mn}(5)S_{in}(6)^* \rangle = W_3(1,2,3,4,5,6), \quad (\text{B8})$$

where all indices i, j, k, l, m , and n are different. In addition to the moments listed above, we need moments in which the indices of a factor S_{ij} or S_{ij}^* are interchanged. These follow from the above moments using the identity $S_{ij}(\varepsilon, X_1, X_2, \alpha) = S_{ji}(\varepsilon, X_1, X_2, -\alpha)$. The function W_1 was calculated in Refs. 74 and 75; the function W_2 was calculated in Ref. 48. Repeating the calculations of Ref. 48 for $W_{1,1}$ and W_3 , one finds, for large N ,

$$W_1(1,2) = \frac{1}{F_1(1,2)}, \quad (\text{B9})$$

$$W_2(1,2,3,4) = - \frac{F_2(1,2,3,4)}{F_1(1,2)F_1(3,2)F_1(3,4)F_1(1,4)}, \quad (\text{B10})$$

$$\begin{aligned} W_{1,1}(1,2,3,4) &= \frac{1}{F_1(1,2)^2 F_1(3,4)^2} \left(\frac{F_2(1,2,3,4)F_2(1,4,3,2)}{F_1(1,4)F_1(3,2)} + \frac{F_2(1,2,1,4)F_2(1,4,3,4)}{F_1(1,4)^2} + \frac{F_2(1,2,3,2)F_2(3,2,3,4)}{F_1(3,2)^2} \right. \\ & \left. - \frac{F_3(1,2,1,4,3,4)}{F_1(1,4)} - \frac{F_3(1,2,3,4,3,2)}{F_1(3,2)} \right), \end{aligned} \quad (\text{B11})$$

$$\begin{aligned}
 W_3(1,2,3,4,5,6) = & \frac{1}{F_1(1,2)F_1(3,2)F_1(3,4)F_1(5,4)F_1(5,6)F_1(1,6)} \left(\frac{F_2(1,2,3,4)F_2(1,4,5,6)}{F_1(1,4)} \right. \\
 & \left. + \frac{F_2(1,2,5,6)F_2(3,4,5,2)}{F_1(5,2)} + \frac{F_2(3,4,5,6)F_2(1,2,3,6)}{F_1(3,6)} - F_3(1,2,3,4,5,6) \right), \quad (\text{B12})
 \end{aligned}$$

where

$$\begin{aligned}
 F_n(1, \dots, n) = & N \left(1 - i \sum_{m=1}^n (-1)^m \varepsilon(m) \tau_d \right) + \left(\sum_{m=1}^n (-1)^m \alpha(m) \right)^2 \\
 & + 2 \sum_{j=1}^2 \left(\sum_{m=1}^n (-1)^m X_j(m) \right)^2. \quad (\text{B13})
 \end{aligned}$$

With the help of these moments, the ensemble averages of Secs. V and VI can be done using the diagrammatic technique of Ref. 73.

APPENDIX C: QUANTUM DOT WITH RELAXATION

A highly simplified model for relaxation is to consider the model of Sec. III C for $r < 1$. In this appendix, we describe a more realistic (but still phenomenological) model for relaxation in the quantum dot.

In this model, the quantum dot itself (rather than the point contact) is coupled to a fictitious lead; see Fig. 10. This fictitious lead is connected to a reservoir, the chemical potential of which is chosen such that no net current flows through this lead. For that reason, the fictitious lead is referred to as ‘‘voltage probe.’’ This model was originally proposed by Büttiker,^{50,51} and has been applied by various authors to describe relaxation effects in chaotic quantum dots.^{44,53–55,66} In order to model spatially distributed relaxation, the voltage probe has $M \gg 1$ channels, each coupled to the quantum dot via a tunnel barrier with transmission probability $\Gamma_V \ll 1$.⁴⁴ The relaxation rate is set by the product $M\Gamma_V$.

Electrons in the physical point contact are labeled ‘‘R’’ and ‘‘L’’ as before. We label electrons entering the dot from the voltage probe by ‘‘V,’’ and electrons exiting the dot towards the voltage probe by ‘‘W’’; see Fig. 11. The scattering matrix of the dot now has dimension $(N+M)$, and relates the fields ψ_R and ψ_V to the fields ψ_E and ψ_W ,

$$\begin{aligned}
 \hat{\psi}_{jL}(0, t) = & \sum_{k=1}^N \int d\tau S_{jk}(t, t - \tau) \hat{\psi}_{kR}(0, t - \tau) \\
 & + \sum_{k=1}^M \int d\tau \tilde{S}_{jk}(t, t - \tau) \hat{\psi}_{kV}(t - \tau), \\
 \hat{\psi}_{jW}(t) = & \sum_{k=1}^M \int d\tau \tilde{S}_{jk}(t, t - \tau) \hat{\psi}_{kV}(t - \tau) \\
 & + \sum_{k=1}^N \int d\tau \tilde{S}_{jk}(t, t - \tau) \hat{\psi}_{kR}(0, t - \tau). \quad (\text{C1})
 \end{aligned}$$

The fields ψ_V and ψ_W are evaluated at the interface between dot and the voltage probe reservoir.

In order to derive an effective action for the quantum dot with voltage probe attached to it, we again attach a fictitious lead at the location of the point contact, as described in Sec. III C. The Hamiltonian is written as the sum $\hat{H} = \hat{H}_0 + \hat{H}_1$, where \hat{H}_0 and \hat{H}_1 represent ideal coupling of the fictitious lead and an impurity at the contact to the fictitious lead, respectively. For \hat{H}_1 we take Eq. (3.13) with $r_j = 1, j = 1, \dots, N$. We then use Eq. (C1) to eliminate the fields ψ_L from the expression for \hat{H}_1 and arrive at the effective action

$$\begin{aligned}
 \mathcal{S} = & 2v_F \int_c dt_1 \int d\tau \sum_{mn} [\hat{\psi}_{mL}^\dagger(0, t_1) S_{mn}(t_1, t_1 - \tau) \\
 & \times \hat{\psi}_{nR}(0, t_1 - \tau) + \text{H. c.}] + 2v_F \int_c dt_1 \int d\tau \sum_{mp} \\
 & \times [\hat{\psi}_{mL}^\dagger(0, t_1) s_{mp}(t_1, t_1 - \tau) \hat{\psi}_{Vp}(t_1 - \tau) + \text{H. c.}]. \quad (\text{C2})
 \end{aligned}$$

The following formal manipulations closely follow those of Ref. 6. Since the current $I_j, j = 1, \dots, N$, does not depend on the fields $\hat{\psi}_{Vp}, p = 1, \dots, M$, we can integrate these out. This results in a new effective action,

$$\begin{aligned}
 \mathcal{S}' = & 2v_F \int_c dt_1 \int d\tau_1 \sum_{mn} [\hat{\psi}_{mL}^\dagger(t_1) S_{mn}(t_1, t_1 - \tau_1) \hat{\psi}_{nR}(t_1 - \tau_1) \\
 & + \text{H. c.}] + 4v_F^2 \int_c dt_1 dt_2 \int d\tau_1 d\tau_2 \sum_{mnp} \hat{\psi}_{mL}^\dagger(t_1) s_{mp} \\
 & \times (t_1, t_1 - \tau_1) G_V(t_1 - \tau_1, t_2 - \tau_2) (s^\dagger)_{pn}(t_2', t_2) \hat{\psi}_{nL}(t_2), \quad (\text{C3})
 \end{aligned}$$

where

$$G_V(t_1 - \tau_1, t_2 - \tau_2) = -i \langle T_c \hat{\psi}_{Vp}(t_1 - \tau_1) \hat{\psi}_{Vp}^\dagger(t_2 - \tau_2) \rangle \quad (\text{C4})$$

is the contour-ordered Green functions for fermions coming in from the voltage probe. (Note that the contour ordering is with respect to the contour times t_1 and t_2 .) We assume that the change of potentials (either external or internal) is slow in comparison to the *minimum* of the relaxation time and the

escape rate into the physical leads. Then using the gradient expansion we find

$$\begin{aligned} & \int d\tau_1 d\tau_2' s(t_1, t_1 - \tau_1) G_V(t_1 - \tau_1, t_2 - \tau_2) s^\dagger(t_2 - \tau_2, t_2) \\ & = G_V(t_1, t_2) + \delta G_V(t_1, t_2), \end{aligned} \quad (C5)$$

where $\delta G_V(t_1, t_2)$ is defined as

$$\begin{aligned} \delta G_V(t_1, t_2) = & - \int d\tau_1 d\tau_2 S(t_1, t_1 - \tau_1) G_V(t_1 - \tau_1, t_2 - \tau_1) \\ & \times S^\dagger(t_2 - \tau_2, t_2) + \frac{1}{2\pi v_F} \int d\varepsilon e^{-i\varepsilon(t_1 - t_2)} \frac{\partial f_V(\varepsilon)}{\partial \varepsilon} \\ & \times \text{tr} \left[\left(\frac{\partial S(t; \varepsilon)}{\partial t} + \frac{\partial \mu_V}{\partial t} \frac{\partial S(t; \varepsilon)}{\partial \varepsilon} \right) S^\dagger(t; \varepsilon) \right. \\ & \left. + \left(\frac{\partial s(t; \varepsilon)}{\partial t} + \frac{\partial \mu_V}{\partial t} \frac{\partial s(t; \varepsilon)}{\partial \varepsilon} \right) s^\dagger(t; \varepsilon) \right]. \end{aligned} \quad (C6)$$

Here $f_V(\varepsilon)$ is the distribution function in the voltage probe, and $S(t; \varepsilon)$ and $s(t; \varepsilon)$ are the Fourier transforms of the scattering matrices $S(t, t - \tau)$ and $s(t, t - \tau)$, respectively; see Eq. (6.1). The second term in Eq. (C6) does not depend on the contour positions of t_1 and t_2 . Upon substitution of Eq. (C5) into the effective action (C3), one finds

$$\begin{aligned} S' = & 2v_F \int_c dt_1 \int d\tau_1 \sum_{mn} \left[\hat{\psi}_{mL}^\dagger(t_1) S_{mn}(t_1, t_1 - \tau_1) \hat{\psi}_{nR}(t_1 - \tau_1) \right. \\ & \left. + \text{H. c.} \right] + 4v_F^2 \int_c dt_1 dt_2 \sum_{mn} \hat{\psi}_{mL}^\dagger(t_1) G_V(t_1, t_2) \hat{\psi}_{nL}(t_2) \\ & + 4v_F^2 \int_c t_1 dt_2 \sum_{mn} \hat{\psi}_{mL}^\dagger(t_1) \delta G_V(t_1, t_2) \hat{\psi}_{nL}(t_2). \end{aligned} \quad (C7)$$

At this point, we recognize that the second term in Eq. (C7) is the same one as the effective action that one would have obtained by coupling the left-moving fermions to a fictitious lead with N channels, with chemical potential μ_V . This situation is shown schematically in Fig. 11, where the fermions in that fictitious lead are labeled with the subscript “E”. The coupling that corresponds to the second term in Eq. (C7) is that of the Hamiltonian

$$\hat{H}_2 = 2v_F \sum_m v_m \left[\hat{\psi}_{mL}^\dagger(0) \hat{\psi}_{mE}(0) + \hat{\psi}_{mE}^\dagger(0) \hat{\psi}_{mL}(0) \right]. \quad (C8)$$

In order to eliminate the Hamiltonian (C8) we introduce new fermion fields according to⁸ (see Fig. 11)

$$\begin{aligned} \text{old} & & \text{new} \\ \hat{\psi}_{mL}(x) & \rightarrow \hat{\psi}_{mL}(x) \theta(-x) + \hat{\psi}_{mE}(-x) \theta(x), \\ \hat{\psi}_{mE}(x) & \rightarrow i[\hat{\psi}_{mE}(-x) \theta(-x) - \hat{\psi}_{mL}(x) \theta(x)], \end{aligned} \quad (C9)$$

where the Heaviside function $\theta(x)$ is defined such that $\theta(0) = 1/2$. For the new fermion fields, the “L” fields have the chemical potential μ_V of the voltage probe, whereas the “E” fields have the chemical potential μ_L of the fictitious lead. Substituting Eq. (C9) into the remaining terms of Eq. (C7), one finds the effective action

$$\begin{aligned} S = & v_F \int_c dt_1 \int d\tau_1 \sum_{mn} \left[[\hat{\psi}_{mL}^\dagger(t_1) + \hat{\psi}_{mE}^\dagger(t_1)] S_{mn}(t_1, t_1 - \tau_1) \right. \\ & \times \hat{\psi}_{nR}(t_1 - \tau_1) + \text{H. c.} \left. \right] + v_F^2 \int_c dt_1 dt_2 \sum_{mn} [\hat{\psi}_{mL}^\dagger(t_1) \\ & + \hat{\psi}_{mE}^\dagger(t_1)] \delta G_{Vmn}(t_1, t_2) [\hat{\psi}_{nL}(t_2) + \hat{\psi}_{nE}(t_2)], \end{aligned} \quad (C10)$$

where, as before, all fermion fields are evaluated at the position $x=0^+$. Although the field $\psi_{Ej}, j=1, \dots, N$ could, in principle, be integrated out of the effective action, it is more convenient to keep them explicit and deal with them at the level of the perturbation theory in S . No physical observables depend on the distribution function f_L of the field ψ_E .

The chemical potential μ_V of the voltage probe reservoir must be chosen such as to have no net current flowing into that reservoir at any time. For the current I_V in the voltage probe we find, in the adiabatic approximation,

$$\begin{aligned} I_V(t) = & \frac{eiv_F}{4\pi} \int d\varepsilon \text{tr} \left[G_V^K(t; \varepsilon) - \left(1 - \frac{i}{2} D_{t,\varepsilon} \right) \right. \\ & \times [\tilde{S}(t; \varepsilon) G_V^K(t; \varepsilon) \tilde{S}^\dagger(t; \varepsilon) - \tilde{s}(t; \varepsilon) G_{RR}^K(t; \varepsilon) \tilde{s}^\dagger(t; \varepsilon) \\ & \left. - \tilde{s}(t; \varepsilon) G_{RV}^K(t; \varepsilon) \tilde{S}^\dagger(t; \varepsilon) - \tilde{S}(t; \varepsilon) G_{VR}^K(t; \varepsilon) \tilde{s}^\dagger(t; \varepsilon) \right], \end{aligned} \quad (C11)$$

where $D_{t,\varepsilon} AB = (\partial A / \partial t)(\partial B / \partial \varepsilon) - (\partial A / \partial \varepsilon)(\partial B / \partial t)$, the superscript “K” refers to the Keldysh component, and

$$\begin{aligned} G_{RR}(t', s') & = -i \langle T_c \hat{\psi}_R(t') \hat{\psi}_R^\dagger(s') \rangle, \\ G_{RV}(t', s') & = -i \langle T_c \hat{\psi}_R(t') \hat{\psi}_V^\dagger(s') \rangle, \\ G_{VR}(t', s') & = -i \langle T_c \hat{\psi}_V(t') \hat{\psi}_R^\dagger(s') \rangle. \end{aligned} \quad (C12)$$

Calculation of the Green function $G_{RR}(t', s')$ can be done with help of the effective action (C10) derived above. For the Green functions $G_{RV}(t', s')$ and $G_{VR}(t', s')$, which keep explicit reference to field ψ_V , a slightly more complicated calculation is necessary. After some algebra, we then find

$$\begin{aligned} (\tilde{G}_{RV})_{ij}(t', s') = & -v_F i \left\langle \int_c dt_1 \int d\tau_1 \sum_m \hat{\psi}_{Ri}(t) [\hat{\psi}_{mL}^\dagger(t_1) \right. \\ & \left. + \hat{\psi}_{mE}^\dagger(t_1)] s_{mj}(t_1, t_1 - \tau_1) G_V(t_1 - \tau_1, s) \right\rangle, \\ (\tilde{G}_{VR})_{ij}(t, s) = & -v_F i \left\langle \int_c dt_1 \int d\tau_1 \sum_m G_V(t, t_1 - \tau_1) s_{im}^\dagger \right. \\ & \left. \times (t_1 - \tau_1, t_1) [\hat{\psi}_{mL}(t_1) + \hat{\psi}_{mE}(t_1)] \hat{\psi}_{jR}^\dagger(s) \right\rangle, \end{aligned} \quad (C13)$$

where the average is taken with respect to the effective action (C10).

This concludes the presentation of the formal framework of the current calculation in the presence of relaxation. The remainder of the calculation proceeds along the same lines as in Sec. IV. The backscattering term in the effective action (C10) is equal to the original effective action (3.17), but with a subunitary scattering matrix S instead of a unitary scattering matrix S ; the remaining terms in the effective action do not contain the fields ψ_R and hence do not contribute to the interaction correction to the current. As a consequence, the expression for the interaction correction to the current one obtains in an expansion in the action (C10) is formally equal to the expressions obtained in an expansion in the original action (3.17), but with a subunitary scattering matrix $S(t, t-\tau)$ instead of a unitary scattering matrix.

There is a crucial difference with the calculation of Sec. IV, however: The fact that the action (C10) features a subunitary scattering matrix means that, for strong relaxation, the effective action is small, and an expansion in the action is justified. This is in contrast to the case of a fully coherent quantum dot, where we argued that there was no small parameter that justified an expansion in the action. We may thus conclude that the interaction correction to the dc conductance and the capacitance calculated in Refs. 7 and 8

describe the case of a quantum dot with a relaxation time much smaller than the mean dwell time τ_d . In this sense, although they do not apply to the case of a fully coherent quantum dot, Refs. 7 and 8 do capture the first effect of coherent scattering inside the dot at temperatures where scattering inside the dot is still predominantly incoherent.

In light of this, the general trend of the results obtained in Refs. 7 and 8 is that the charging interaction *reduces* the effective dephasing rate: the weight of scattering processes with a short delay time is increased,⁸ leading to a smaller probability to dephase.⁷⁶ (Since the average delay time must remain constant, interactions must also increase the weight of scattering processes with a long delay time. If the dephasing time is shorter than the average delay time, the net effect of the interactions is to suppress dephasing.) A smaller dephasing rate corresponds to a larger weak localization correction and larger conductance fluctuations, as was found in Refs. 7 and 8. A detailed analysis of the expression for the current in the voltage probe model for the various limiting cases, as we did for the fully coherent quantum dot, is of little practical use, since the noninteracting (Hartree) contribution to the current contains the dephasing rate, which is an unknown parameter in this model.

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