Measuring the Kondo effect in the Aharonov-Bohm interferometer

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The conductance $\mathcal{G}$ of an Aharonov-Bohm interferometer (ABI), with a strongly correlated quantum dot on one arm, is expressed in terms of the dot Green function, G_{dd}, the magnetic flux ϕ and the noninteracting parameters of the ABI $\{J_{\mu\nu}\}$. At $T=0$, Fermi liquid theory yields the exact ϕ dependence of $\mathcal{G}$. We show that one can extract G_{dd} from the observed $\mathcal{G}(\phi)$, for both closed and open ABI's. In the latter case, the phase shift β deduced from $\mathcal{G}\approx A+B\cos(\phi+\beta)$ depends strongly on the $\{J_{\mu\nu}\}$'s, and usually $\beta\neq\pi/2$. The $\{J_{\mu\nu}\}$'s may also reduce the Kondo temperature.

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The recent observation of the Kondo effect in quantum dots (QD's), with tunable parameters[1] has generated much theoretical and experimental activity. For temperatures T below the Kondo temperature T_K, the spin of an electron localized on the QD is dynamically screened by the electrons in the Fermi sea, yielding a large conductance $\mathcal{G}$ through the QD, close to the unitary value $2e^2/h$, and a transmission phase α equal to $\pi/2$.[2,3] These predictions have been tested by embedding the strongly-correlated QD on one arm of an Aharonov-Bohm interferometer (ABI), for both a closed (two-terminal) ABI (Ref. 4) and an open (multiterminal) ABI.[5] Both experiments exhibited the Aharonov-Bohm oscillations with the normalized flux $\phi=e\Phi/\hbar c$. The former experiments exhibited the expected "phase rigidity," with $\mathcal{G}(\phi)=\mathcal{G}(-\phi)$.[6] The latter experiments attempted to measure α, and found a variety of behaviors which were inconsistent with the expected value of $\pi/2$. As a result, Ji *et al.*[5] stated that "the full explanation of the Kondo effect may go beyond the framework of the Anderson model." Relevant theoretical papers have concentrated on the dot alone[3,7] (when it is detached from the ABI) or applied various techniques[8–10] to the QD on simple models of the closed ABI. However, it is not very clear how to make *quantitative* comparisons of theory and experiment.

Most of the theoretical discussions of QD's concentrate on the retarded Green function for electrons with energy ω on the QD, $G_{dd}(\omega)$ (we ignore the spin index, since we assume no magnetic asymmetry). For a simple QD, connected to a broad electronic band, the $T=0$ transmission amplitude for electrons going through the QD is proportional to $G_{dd}(\epsilon_F)$, where ϵ_F is the Fermi energy (taken as zero below).[2,11] The ABI experiments were intended to measure *both* the magnitude and the phase of G_{dd}, and compare with theory. In this paper we concentrate on the following question: given experimental data on the flux-dependent conductance of the ABI $\mathcal{G}(\phi)$ how can we deduce the "intrinsic" Green function G_{dd}? An earlier paper[12] answered this question for noninteracting electrons on a simple model for a closed ABI, and made some speculations on the interacting case far above T_K. Another paper[13] showed that for noninteracting electrons, the phase shift measured in the open ABI depends on details of the opening. Here we calculate $\mathcal{G}(\phi)$ exactly deep in the Kondo regime, give explicit instructions for extracting G_{dd} from the measured $\mathcal{G}(\phi)$, and show that the puzzling measured data in the open ABI (Ref. 5) can be imitated by appropriate choices of the ABI parameters.

Our qualitative results should apply for a large class of ABI's. For simplicity, we demonstrate them for the specific Anderson model shown in Fig. 1, which captures the important ingredients. The conductance $\mathcal{G}$ is measured between the two leads which are attached to sites L and R on the ABI ring. The QD (denoted "D") is connected to L (R) via n_l (n_r) sites. The lower "reference" branch contains n_0 sites. Except for the QD, we use a tight binding model, with the real hopping matrix elements as indicated in the figure. Site energies are ϵ_l, ϵ_r, and ϵ_0 on the respective branches, ϵ_L and ϵ_R, on sites L and R and zero on the leads. Using gauge invariance, we introduce the normalized flux ϕ as a phase factor in $J_{D1}=J^*_{1D}=j_l e^{i\phi}$. The Hamiltonian on the dot is $\mathcal{H}_d=\epsilon_d\Sigma_\sigma n_{d\sigma}+Un_{d\uparrow}n_{d\downarrow}$, with obvious notations. Here we assume that the transport is dominated by the level ϵ_d on the QD (distant full or empty levels would add a smooth potential-scattering term, which would not change our conclusions near the resonance of interest). We also assume that U is very large, and ignore the resonance at $2\epsilon_d+U$. Figure 1 generalizes earlier models,[8,9] by adding the internal structure on the links between D, L, and R. Some such structure always exists in experiments, and may have important effects on the observed conductance (see below). For the open ABI, each dashed line represents an additional lead, with a hopping matrix element $-J_X$ on its first bond.[13]

For $T\ll T_K$, it is sufficient to calculate $\mathcal{G}$ at $T=0$. Irrespective of the above details, one has

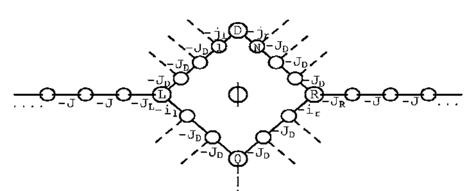

FIG. 1. The model for the ABI, with $n_l=n_r=3$, $n_0=5$.

$$\mathcal{G} = \frac{2e^2}{h} 4\Gamma_L(0)\Gamma_R(0)|G_{LR}(\omega=0)|^2, \tag{1}$$

where $G_{LR}(\omega)$ is the Fourier transform of the retarded Green function between sites L and R.[14] Also, $\Gamma_{L,R}(\omega) = -\mathrm{Im}\,\Sigma^0_{L,R}(\omega)$, where $\Sigma^0_{L,R}$ is the self-energy generated at sites L or R due to the leads. Since interactions exist only on the dot, one can use the equations of motion to express all the retarded Green functions $G_{\mu\nu}$ in terms of G_{dd}. If $g_{\mu\nu}$ denotes the Green functions of the whole network without the QD and the two bonds connected to it, then it is straightforward to obtain[15] the relation

$$G_{LR} = g_{LR} + \sum_{\mu\nu} g_{L\mu} J_{\mu D} J_{D\nu} g_{\nu R} G_{dd}, \tag{2}$$

where in our model the only contributions to the sum come from $J_{D1}=J^*_{1D}=j_l e^{i\phi}$ and $J_{DR}=J_{RD}=j_r$ (see Fig. 1).

For the noninteracting case, one has

$$G_{dd} = 1/[\omega - \epsilon_d - \Sigma_0(\omega)], \tag{3}$$

with the noninteracting self-energy

$$\Sigma_0(\omega) = \sum_{\mu\nu} J_{D\mu} g_{\mu\nu}(\omega) J_{\nu D} \equiv \delta\epsilon_d(\omega) - i\Delta_0(\omega). \tag{4}$$

Since the phase ϕ is only contained in $J_{D1}=J^*_{1D}$, the complex matrix $g_{\mu\nu}$ is independent of ϕ and obeys $g_{\mu\nu}=g_{\nu\mu}$. Therefore, $\Sigma_0(\omega)$ has a simple explicit dependence on ϕ,

$$\Sigma_0(\omega) = j_l^2 g_{11} + j_r^2 g_{NN} + 2 j_l j_r g_{1N} \cos\phi. \tag{5}$$

Except for the very special case $n_l=n_r=n_0=0$ (discussed, e.g., in Refs. 8 and 9), g_{1N} is not a real number, and therefore *both* $\delta\epsilon_d(\omega)$ *and* $\Delta_0(\omega)$ oscillate with ϕ,

$$\delta\epsilon_d = a_1 + b_1\cos\phi; \quad \Delta_0 = a_2 + b_2\cos\phi. \tag{6}$$

Using these expressions, Eq. (2) can be written as

$$G_{LR} = g_{LR} G_{dd}\left([G_{dd}]^{-1} + \Sigma_0 + \sum_{\mu\nu} J_{D\mu} w_{\mu\nu} J_{\nu D}\right), \tag{7}$$

with $w_{\mu\nu} \equiv g_{\mu R} g_{L\nu}/g_{LR} - g_{\mu\nu}$. In calculating the necessary $g_{\mu\nu}$'s, it is convenient to first eliminate all the leads, replacing them by site self-energies. For example, at site L we replace the bare Green function $1/(\omega-\epsilon_L)$ by $1/[\omega-\epsilon_L-\Sigma^0_L(\omega)]$. For our one-dimensional leads one has $\Sigma^0_L(\omega) = -e^{i|q|a} J_L^2/J$, where $\omega=-2J\cos(qa)$ is the energy of the electron in the band of the leads (we assume all the leads to have the same J and the same lattice constant a).[15] The results remain valid also for more complex leads, as long as one uses the appropriate band-generated self-energies. For the remaining $N=n_l+n_r+n_0+2$ sites of the ring (without the QD), $g_{\mu\nu}$ is then the inverse of a tridiagonal $N\times N$ matrix, $\mathcal{M}_{\mu\nu}=J_{\mu\nu}+[\omega-\epsilon_\mu-\Sigma^0_\mu(\omega)]\delta_{\mu\nu}$. For such a matrix, it turns out that one always has $w_{1N}=0$. Thus,

$$\mathcal{G} = \mathcal{G}_{\mathrm{ref}} |G_{dd}|^2 |G^{-1}_{dd} + \Sigma_0 + x + y e^{-i\phi}|^2, \tag{8}$$

where $x=j_l^2 w_{11}+j_r^2 w_{NN}$, $y=j_l j_r w_{N1}$ and the "reference" conductance $\mathcal{G}_{\mathrm{ref}} \equiv (2e^2/h)4\Gamma_L(0)\Gamma_R(0)|g_{LR}|^2$ depend only on the parameters of the noninteracting parts of the ABI, and not on the QD parameters ϵ_d and U.

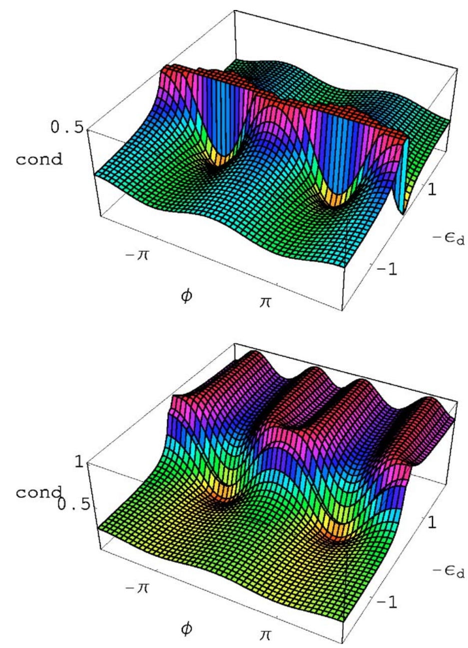

FIG. 2. (Color online) Conductance (in units of $2e^2/h$) through the closed ABI versus the normalized flux ϕ and the energy of the state on the dot ϵ_d (the gate voltage), without (top) and with interactions (bottom). $n_l=n_r=2$, $n_0=3$, $J_L=J_R=J_D=1$, $j_l=j_r=0.2$, $i_l=i_r=0.4$, $\epsilon_l=\epsilon_r=\epsilon_0=-0.3$, $\epsilon_L=\epsilon_R=0$. All energies are in units of J.

At $T=0$ the electrons must obey the Fermi liquid relations.[16] Specifically, at the Fermi energy one expects

$$\mathrm{Im}[G^{-1}_{dd}](\omega=0) \equiv \Delta_0(\omega=0). \tag{9}$$

This condition should hold for *any* network in which the QD is embedded, and is therefore true for both the closed and the open ABI. In the limit of a very large negative ϵ_d, when $\langle n_d\rangle \equiv -2\int d\omega f(\omega)\mathrm{Im}[G_{dd}(\omega)]/\pi \to 1$ ($f(\omega)$ is the Fermi distribution function), one also expects that $\mathrm{Re}[G^{-1}_{dd}](\omega=0) \to 0$. For the simple QD with two leads, one has $\Sigma_0(\omega) = \Sigma^0_L(\omega)+\Sigma^0_R(\omega)$, and in the symmetric case $\Gamma_L=\Gamma_R=\Delta_0/2$ the conductance reaches its unitary limit $2e^2/h$ [Eq. (1), with $G_{LR}\to G_{dd}$]. In this limit the phase of G_{dd} becomes $\pi/2$.[2]

We now discuss the closed ABI. In this case, w_{11}, w_{NN} and w_{N1} and therefore also x and y are all real numbers. Using the Fermi liquid result (9) and Eq. (6), Eq. (8) assumes the exact ϕ dependence

$$\frac{\mathcal{G}_{\mathrm{closed}}}{\mathcal{G}_{\mathrm{ref}}} \equiv \mathcal{F}(\phi) \equiv \frac{(\zeta + r_a + r_b\cos\phi)^2 + r_y^2\sin^2\phi}{\zeta^2 + (1 + r_d\cos\phi)^2}, \tag{10}$$

with the dimensionless function $\zeta(\phi)=\mathrm{Re}[G^{-1}_{dd}]/a_2$ and constants $r_a=(a_1+x)/a_2$, $r_b=(b_1+y)/a_2$, $r_y=y/a_2$ and $r_d=b_2/a_2$. For the noninteracting case, $\zeta=[-\epsilon_d-\delta\epsilon_d(0)]/a_2$, and Eq. (10) generalizes the results of Ref. 12. An example is shown on the top of Fig. 2. In the strongly correlated case and in the unitary limit,$\zeta=0$ and $\mathcal{G}_{\mathrm{closed}}\to\mathcal{G}_{\mathrm{ref}}\mathcal{F}_0(\phi)$. All the features in the ϕ dependence of $\mathcal{F}_0$ arise only due to the noninteracting parts of the ABI. Usually, Eq. (10) contains many harmonics. Except in special cases,[8] it is *not* dominated by the second harmonic, and the period of $\mathcal{F}_0(\phi)$ is not simply doubled. An example of this dependence is seen (for large negative ϵ_d) on the bottom of Fig. 2: except for the minima at $\phi=0$ and π, the maxima are *not* at $\pi/2$. Experimentally, one knows that one has reached this limit once the function $\mathcal{G}_{\text{closed}}(\phi)$ no longer changes with the gate voltage which governs ϵ_d. The reference conductance $\mathcal{G}_{\text{ref}}$ can be measured by disconnecting the QD, i.e., setting $j_l=j_r=0$. Alternatively, $\mathcal{G}_{\text{ref}}$ can be absorbed in the scales of the parameters in the numerator of Eq. (10). Having determined $\mathcal{G}_{\text{ref}}$, one can determine the four real parameters r_a, r_b, r_y, and r_d by a fit to $\mathcal{F}_0(\phi)$. (In practice, one only needs four values of the function).[17] Having found these parameters, one can now move away from the unitary limit, and measure $\mathcal{G}_{\text{closed}}=\mathcal{G}_{\text{ref}}\mathcal{F}(\phi)$. The unknown function $\zeta(\phi)$ can now be found from the quadratic equation

$$\zeta^2-2\zeta\frac{r_a+r_b\cos\phi}{\mathcal{F}-1}+\frac{\mathcal{F}-\mathcal{F}_0}{\mathcal{F}-1}(1+r_d\cos\phi)^2=0. \quad (11)$$

The solution should be chosen so that it decreases to zero at large negative ϵ_d and increases linearly with large positive ϵ_d. Having found the solution, the phase α of G_{dd} is then defined via

$$\cot\alpha=-\frac{\text{Re}[G_{dd}^{-1}](\omega=0)}{\Delta_0(\omega=0)}\equiv-\frac{\zeta}{1+r_d\cos\phi}. \quad (12)$$

This phase, or equivalently $\text{Re}[G_{dd}^{-1}]$, are the quantities obtained from theories.

Up to this point, we have given an exact prescription on how to extract α from the experimental data. For demonstrating the *qualitative* dependence of $\mathcal{G}$ on ϕ and on the other parameters, we have used an approximate analytic solution of the equations of motion, truncated via decoupling of higher order Green functions.[15] In the limits $T=\omega=0$ and $U\to\infty$, this solution assumes the simple analytic form

$$\cot\alpha=-\frac{z\delta n-\frac{3}{4}\left[z+\sqrt{z^2+\delta n\left(\frac{3}{2}-\delta n\right)}\right]}{2\left\{\frac{3}{4}\delta n+z\left[z+\sqrt{z^2+\delta n\left(\frac{3}{2}-\delta n\right)}\right]\right\}}, \quad (13)$$

where z represents the value at $\omega=0$ of the noninteracting ratio

$$z(\omega)=[\omega-\epsilon_d-\delta\epsilon_d(\omega)]/[2\Delta_0(\omega)], \quad (14)$$

while δn is related to the electron occupation on the dot via $\langle n_d\rangle=2(1-\delta n)$ (which should be determined self-consistently). In practice, δn varies smoothly between $1/2$ (at $z\gg1$) and 1 (at $z\ll-1$), and the results of calculations are not very sensitive to the details of this variation. Equation (13) interpolates between $\alpha=\pi/2$ at $z\to\infty$ and $\alpha=\pi$ for $z\to-\infty$. In the latter limit, G_{dd} approaches the noninteracting form (3). Using Eq. (13) in Eq. (10) for a specific set of parameters yields the bottom of Fig. 2. One clearly sees the transition from the noninteracting behavior at large positive ϵ_d (compare with the top) to the unitary limit at large negative ϵ_d. For different sets of parameters one reproduces qualitatively all the earlier results, including the Fano-Kondo effect.[8] We have used these results to imitate real experimental "data," and were able to use the above algorithm to extract $\cot\alpha$ as in Eq. (12).

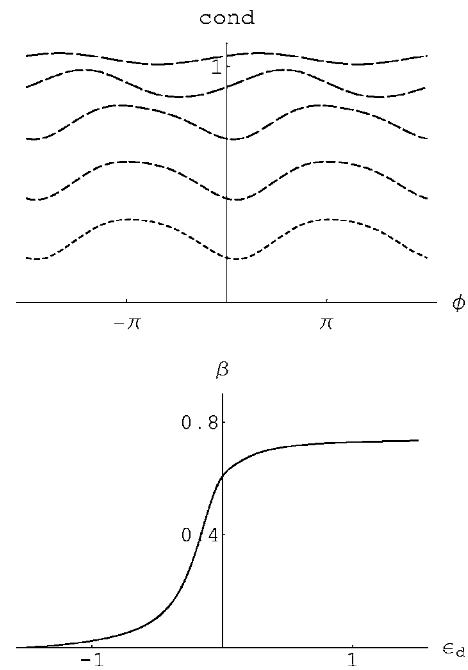

FIG. 3. Top: Conductance through the open ABI versus ϕ, at $\epsilon_d=(-1.5,-1,-0.5,0,0.5,1,1.5)J$, with infinite interactions. Graphs are shifted up with increasing ϵ_d. Parameters are the same as in Fig. 2, but with $J_X=0.5$. Bottom: The "measured" phase shift β (in units of π) versus ϵ_d.

Note that the above analysis yields G_{dd} for the QD on the ABI, where this function (and thus also the phase α) depends explicitly on the flux ϕ, via z. At $T=\omega=0$, we expect α to depend only on the ratio z also for other theories. In our case, z can be extracted from the experimental data via

$$z=-[\tilde{\epsilon}_d+r_a+(r_b-r_y)\cos\phi]/(1+r_d\cos\phi), \quad (15)$$

where $\tilde{\epsilon}_d=(\epsilon_d-x)/a_2$ is just a shifted rescaled gate voltage. Having deduced the dependence of both z and α on ϕ, a parametric plot can yield α versus z, for comparison with single dot calculations. Alternatively, one can experimentally study the results as function of the coupling to the reference branch, i_l and i_r. Extrapolation to $i_l,i_r\to0$ would give the dependence of α on $\Sigma_0(0)$ for the upper branch alone. However, $\Sigma_0(\omega)$ still depends on the finite chains connecting D with L and R.[18]

We now turn to the open ABI, with $J_X\neq0$. Equation (8) remains correct, but now x and y become *complex*. Interestingly, Σ_0 is still given by Eq. (5). In the unitary limit, $\mathcal{G}(\phi)$ has the exact form

$$\mathcal{G}_{\text{open}}\to\mathcal{G}_{\text{ref}}\frac{A+B\cos(\phi+\tilde{\beta})+C\cos(2\phi+\gamma)}{(1+r_d\cos\phi)^2}, \quad (16)$$

and we need six parameters to fit it. Note that all the ABI parameters (including $\mathcal{G}_{\text{ref}}$) now also depend on J_X. The two lower curves in the top panel of Fig. 3 show results in this limit, where Eq. (16) is *exact*. Note that the graphs are not sinusoidal, mainly due to the second term in the numerator and to the denominator in Eq. (16). Since one remains close to the Kondo resonance, the denominator continues to be important, modifying the two-slit-like numerator. The asymmetric shape of each oscillation seems similar to that reported in Ref. 5. The other curves in the same panel were derived using the approximate Eq. (13). Again, one observes the crossover to the noninteracting sinusoidal shape at large positive ϵ_d. To extract a "transmission phase" from these curves, one can, e.g., follow the maxima as function of ϵ_d, or enforce a fit to the two-slit formula $\mathcal{G}_{\rm open} \approx A + B\cos(\phi+\beta)$. Since now there is no well-defined zero to ϕ, one can only deduce the relative change in the phase β. Setting $\beta = 0$ at $\epsilon_d \to -\infty$, the bottom of Fig. 3 shows this relative phase versus ϵ_d. For the parameters we used, the total change is about 0.8π, far away from the expected change in α, equal to $\pi/2$. The actual values depend on details of the ABI. This may explain the nontrivial values of the phases observed in Ref. 5: they result from the experimental setup, and *not* from a breakdown of the Anderson theory.

Finally, a few words about nonzero T or ω. Generally, T and ω enter into G_{dd} similarly. In the approximate solution of Ref. 15, one ends up with a competition between the variable $z(\omega)$ of Eq. (14) and $\ln(D/T)$ or $\ln(D/\omega)$, where $2D = 4J$ is the width of the band in the leads. This competition yields estimates of T_K,

$$\ln(T_K/D) = \pi[\epsilon_d + \delta\epsilon_d(0)]/\Delta_0(0). \tag{17}$$

Although more accurate theories end up with different expressions, all of them end up with a strong dependence on the ratio which appears on the RHS. In our case, this ratio oscillates strongly with ϕ, opening the possibility that for different fluxes the QD is below or above T_K. We emphasize the appearance of $\delta\epsilon_d$ in the numerator, ignored in some papers.

At nonzero T, the "intrinsic" phase of the QD is expected to start at 0 for large negative ϵ_d [where $T > T_K(\epsilon_d)$], then grow to $\pi/2$ for intermediate negative ϵ_d's (the unitary region), and finally grow to π at positive ϵ_d.[3] As mentioned, both $\delta\epsilon_d$ and Δ_0 depend on the opening parameter J_X. Using the approximation of Ref. 15 also for $T > 0$, we found that large values of J_X may completely eliminate the intermediate plateau in α, and give a direct increase of α from 0 to π. Unlike the noninteracting case,[13] where changing J_X only slightly modified the quantitative shape of the function $\beta(\epsilon_d)$, the effects here are *qualitative*: opening may lower T_K and completely eliminate the observability of the Kondo behavior. We expect similar qualitative results to follow other approximate or exact calculations. Again, these effects could have happened in Ref. 5.

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[16] N. E. Bickers, Rev. Mod. Phys. **59**, 845 (1987), Table IX.

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