Renormalization-group approach to strong-coupled superconductors

S.-W. Tsai,[1] A. H. Castro Neto,[1] R. Shankar,[2] and D. K. Campbell[1]
[1]*Department of Physics, Boston University, Boston, Massachusetts 02215, USA*
[2]*Sloane Laboratory of Physics, Yale University, New Haven, Connecticut 06520, USA*
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We develop an asymptotically exact renormalization-group approach that treats electron-electron and electron-phonon interactions on an equal footing. The approach allows an unbiased study of the instabilities of Fermi liquids without the assumption of a broken symmetry. We apply our method to the problem of strongly coupled superconductors and find the temperature T^* below which the high-temperature Fermi liquid state becomes unstable towards Cooper pairing. We show that T^* is the same as the critical temperature T_c obtained in Eliashberg's strong coupling theory starting from the low-temperature superconducting phase. A $1/N$-expansion shows that the method is asymptotically exact and Migdal's theorem follows as a consequence. Finally, our results lead to a novel way to calculate numerically, from microscopic parameters, the transition temperature of superconductors.

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I. INTRODUCTION

There is a renewed interest in understanding the interplay between electron-electron and electron-phonon interactions in strongly correlated electronic systems.[1–5] While the experimental data indicate the important role played by both electron-electron and electron-phonon interactions, theoretical progress has been limited due to the complexity associated with treating these two interactions on an equal footing. Most approaches rely on mean-field treatments where a "favorite" order parameter and broken symmetry are introduced "by hand." It is highly desirable if instead one could predict the leading instability starting from the geometry of the Fermi surface, the strength of the couplings, and the bare energy scales of the problem, such as the Fermi energy, E_F, and the Debye frequency, ω_D.

The renormalization group (RG) provides such an approach,[6] one that has been successful in explaining the stability and instabilities of Landau Fermi liquids in more than one dimension. Let us recall the two-stage procedure advocated there. Firstly, given a microscopic theory defined in all of momentum space, one integrates out all modes except those within an energy cutoff Λ of E_F. In $d=2$, which is our focus in this work (the $d=3$ case can be treated in an analogous way[8]), the remaining phase space has the form of an annulus of radius k_F (the Fermi momentum) and width $2\Lambda/v_F$ (where v_F is the Fermi velocity). In the beginning of this process of integrating out high-energy modes, the corrections are treated perturbatively in the strength of the interactions. In the second stage, provided that mode eliminations have reached an energy cutoff $\Lambda \ll E_F$, a $1/N$ expansion emerges, with $N \simeq E_F/\Lambda$. More precisely, imagine dividing the annulus into N patches of size of order $(\Lambda/v_F)^2$. The momentum of each fermion $\mathbf{k}$ is a sum of a "large" part (order k_F) centered on a patch labeled by a patch index $i=1,\ldots,N$ and a "small" momentum (order Λ/v_F) within the patch.[6] It can then be verified that in all Feynman diagrams of this cutoff theory the patch index plays the role of a conserved isospin index exactly as in a theory with N fermionic species. The electron-electron interaction terms, written in this notation, come with a prefactor of $1/N$ ($\sim\Lambda/E_F$), and the RG corrections can be organized in terms of powers of $1/N$. Summing up the series of dominant corrections in order $1/N$ becomes asymptotically exact since $N\to\infty$ as the RG procedure decreases the cutoff $\Lambda\to 0$. This $1/N$ expansion has been developed in Ref. 6 and has been used to show that, provided one can start the RG flow at $\Lambda_0 \ll E_F$ (large N), Landau's Fermi liquid theory emerges as the asymptotically exact theory for repulsive fermions with a generic (nonnested and with no singularities) Fermi surface. In this way the RG expansion goes beyond perturbation theory and the end values of the couplings can be large. We make use of this expansion here and extend it for the case when phonons are present.

Electron-phonon interactions in the Wilson-like RG of Ref. 6, wherein momenta are rescaled to attain a fixed point, posed the following problem:[7] electronic momentum scales differently parallel and perpendicular to the Fermi surface while phonon momentum scales isotropically. We circumvent this problem by using the quantum field theory version of the RG in which the cutoff dependence of couplings that preserves the physical quantities defines the flow, with no rescaling of momenta or frequencies.

The paper is organized as follows: The RG approach for interacting fermions that are also coupled to phonons is described in Sec. II. In Sec. III we illustrate our method for a circular Fermi surface and an analytical solution is obtained. The RG indicates an instability in the Cooper channel when the electron-phonon coupling is strong enough to overcome the effective repulsive electron-electron interactions. In Sec. IV we present the finite temperature formalism. We obtain the temperature T^* at which the high-temperature Fermi liquid state becomes unstable in the Cooper channel. We demonstrate that T^* is the *same* as the superconducting temperature T_c that is obtained from the Eliashberg theory of strongly coupled superconductor, which approaches the transition from the ordered phase.[10,11] This is an alternative derivation of Eliashberg's equations starting from a Fermi liquid

state. Furthermore, by extending the $1/N$ analysis to the problem with phonons (Sec. V), we show that, as in the case of Landau's Fermi liquid theory for electron-electron interactions, the Eliashberg theory is the exact low energy effective theory obtained by using RG. Migdal's theorem is also derived from the $1/N$ analysis. Section VI contains further discussions and conclusion.

II. RG FOR INTERACTING ELECTRONS COUPLED TO PHONONS

We work in the path-integral representation and consider the general action

$$S(\psi,\phi) = S_e(\psi) + S_{ph}(\phi) + S_{e\text{-}ph}(\psi,\phi) + S_{e\text{-}e}(\psi), \tag{1}$$

where

$$S_e = \int_{\omega\mathbf{k}} \psi_k^{\dagger\sigma}(i\omega - \epsilon_{\mathbf{k}})\psi_{k\sigma} \tag{2}$$

is the free electron action and

$$S_{ph} = \int_{\Omega\mathbf{q}} \phi_q^{\dagger}(i\Omega - w_{\mathbf{q}})\phi_q \tag{3}$$

is the free phonon action ($k=\{\omega,\mathbf{k}\}$ and $q=\{\Omega,\mathbf{q}\}$, where ω,Ω are fermionic and bosonic Matsubara frequencies, respectively, and $\mathbf{k},\mathbf{q}$ are the momenta). The interaction terms are given by

$$S_{e\text{-}ph} = \int_{\omega\mathbf{k}} \int_{\Omega\mathbf{q}} g(q)\psi_{k+q}^{\dagger\sigma}\psi_{k\sigma}(\phi_q + \phi_{-q}^{\dagger}) \tag{4}$$

and

$$S_{e\text{-}e} = \frac{1}{2}\prod_{i=1}^{3}\int_{\omega_i\mathbf{k}_i} u(k_4,k_3,k_2,k_1)\psi_{k_4}^{\dagger\sigma}\psi_{k_2\sigma}\psi_{k_3}^{\dagger\sigma'}\psi_{k_1\sigma'}, \tag{5}$$

where $k_4=k_1+k_2-k_3$. (We use units such that $\hbar=1=k_B$.) The above action defines the input physics at a cutoff Λ_0 such that $\omega_D, g, u \ll \Lambda_0 \ll E_F$ (thus $N \simeq E_F/\Lambda_0 \gg 1$).

The phonons, being described by a Gaussian action, can be integrated out exactly, leading to an electron-electron problem with retarded interactions

$$\tilde{u}(k_4,k_3,k_2,k_1) = u(k_4,k_3,k_2,k_1) - 2g(k_1,k_3)g(k_2,k_4)D(k_1-k_3), \tag{6}$$

where

$$D(q) = \frac{\omega_{\mathbf{q}}}{\omega^2 + \omega_{\mathbf{q}}^2} \tag{7}$$

is the phonon propagator. Here we are considering fermions with spin and it is sufficient to focus on interactions involving two fermions with opposite spins.[9] For spinless fermions the initial condition must be antisymmetrized with respect to $1\leftrightarrow 2$ and $3\leftrightarrow 4$. The Feynman diagram associated with the initial vertex $\tilde{u}$ is shown in Fig. 1(a). The vertex is a function of the momenta and frequencies of the incoming and outgoing electrons, with momentum and frequency conservation. The dependence on the magnitude of the momenta is irrelevant and all $\mathbf{k}$'s are taken to be on the Fermi surface. For a generic non-nested Fermi surface, with no singularities and with time-reversal symmetry, only two types of scattering exist:[6] forward scattering with $k_1=k_3, k_2=k_4$ (these evolve into Landau parameters) and the scattering in the Cooper channel with $k_1=-k_2, k_3=-k_4$. The box vertex in the diagrams in Fig. 1 can represent either one of these scattering processes. The forward scattering channel (which does not flow under the RG as in the case of pure electron-electron interactions[6]) contributes to the electron self-energy $\Sigma(\mathbf{k},\omega)$ as shown in Fig. 1(b). We can write

$$\Sigma(\omega,\mathbf{k}) = \Sigma_0 + i[1 - Z(\omega,\mathbf{k})]\omega \tag{8}$$

with two types of contributions: a shift in the chemical potential ($\delta\mu \propto \Sigma_0$) and wave-function renormalization, $Z(\omega,\mathbf{k})$. The shift in the chemical potential can be reabsorbed in the theory by assuming a fixed number of electrons.[6] The wave-function renormalization, $Z(\omega,\mathbf{k})$, is of special interest in this problem.

The starting point of our RG is the interaction vertex $\Gamma^{(4)}[\tilde{u}]$ in the Cooper channel, as shown in Fig. 1(c):

$$\Gamma^{(4)}[\tilde{u}(-k_3,k_3,-k_1,k_1)] = \tilde{u}(-k_3,k_3,-k_1,k_1) - \int_{\omega\mathbf{k}} \frac{\tilde{u}(-k,k,-k_1,k_1)\tilde{u}(-k_3,k_3,-k,k)}{[i\omega - \epsilon_{\mathbf{k}} - \Sigma(\omega,\mathbf{k})][-i\omega - \epsilon_{\mathbf{k}} - \Sigma(-\omega,\mathbf{k})]}, \tag{9}$$

where the momentum integral is such that all internal energies lie between 0 and Λ.

III. EXACT ANALYTICAL SOLUTION FOR A CIRCULAR FERMI SURFACE

In order to establish the method and proceed with simple analytical manipulations, we focus on an isotropic Fermi surface with no singular regions, and Einstein phonons with frequency ω_E (the generalization for an arbitrary phonon spectrum is straightforward[8]). Since $E_F \gg \omega_D, g, u$ we ignore radial excursions away from the Fermi surface in the coupling constants. In this case the external electron momentum can be put on k_F and $\tilde{u}(-k_3,k_3,-k_1,k_1)$ depends on the angles only via the difference $\theta_1-\theta_3$. Let us define

$$\tilde{v}(\omega_1,\omega_3) = N(0)\int \frac{d\theta_1}{2\pi}\int \frac{d\theta_3}{2\pi}\tilde{u}(-k_3,k_3,-k_1,k_1),$$

where $N(0)$ is the Fermi surface density of states.

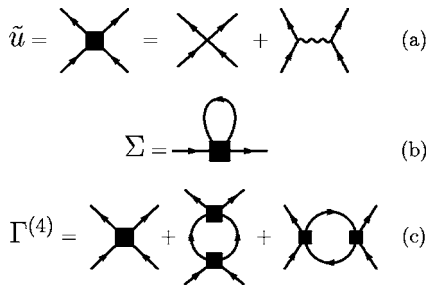

FIG. 1. (a) The retarded interaction $\tilde{u}$. (b) The self-energy correction. (c) The interaction vertex.

The RG equations are obtained from (9) by imposing the condition of cutoff independence, namely,

$$\frac{d\Gamma^{(4)}}{d\ell} = 0, \tag{10}$$

where $\ell = \ln(\Lambda_0/\Lambda)$ is the RG scale. From (9) we obtain

$$\frac{d}{d\ell}\tilde{v}(\omega_1,\omega_3,\ell) = -\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\Lambda_\ell \tilde{v}(\omega_1,\omega,\ell)\tilde{v}(\omega,\omega_3,\ell)}{\Lambda_\ell^2 + Z_\ell^2(\omega)\omega^2}, \tag{11}$$

with the initial RG condition

$$\tilde{v}(\omega_1,\omega_3,\ell=0) = u_0 - \lambda\omega_E D(\omega_1-\omega_3),$$

where u_0 is the bare electron-electron interaction[12] and

$$\lambda = \frac{2N(0)g^2}{\omega_E} \tag{12}$$

is the electron-phonon coupling constant. Notice that this is a *functional* equation, because retardation introduced by the phonons leads to a mixing of the couplings at low and high frequencies.

The RG equation for $Z_\ell(\omega)$, which for a circular Fermi surface does not depend on the momentum direction, can be likewise derived and formally integrated to give

$$Z_\ell(\omega) = 1 + \frac{\lambda}{\pi\omega}\int_{\omega'}\int_{\Lambda_\ell}^{\infty}\frac{Z_\Lambda \omega' D(\omega-\omega')}{Z_\Lambda^2(\omega')\omega'^2 + \Lambda^2}. \tag{13}$$

The large-N considerations of Ref. 6 are fully applicable here since we are dealing with nonsingular interactions (albeit retarded). Thus the one-loop flows derived above are exact as $N\to\infty$. More detailed analysis is presented in Sec. V.

Notice that (11) can be seen as a matrix problem:

$$\frac{d\mathbf{U}}{d\ell} = -\mathbf{U}\cdot\mathbf{M}\cdot\mathbf{U}, \tag{14}$$

where

$$U_{ij}(\ell) = \tilde{v}(\omega_i,\omega_j,\ell),$$

and

$$M_{ij}(\ell) = \frac{\Lambda_\ell \delta_{ij}}{\pi(\Lambda_\ell^2 + Z_\ell^2(\omega_i)\omega_i^2)}.$$

The solution of (14) is

$$\mathbf{U}(\ell) = [1 + \mathbf{U}(0)\cdot\mathbf{P}(\ell)]^{-1}\mathbf{U}(0),$$

with $\mathbf{P}(\ell)$ defined as

$$\mathbf{P}(\ell) = \int_0^\ell d\ell' \mathbf{M}(\ell').$$

Therefore, the condition for the instability of the RG at a certain scale $\ell = \ell_c$ is given by

$$\det[1 + \mathbf{U}(0)\cdot\mathbf{P}(\ell_c)] = 0. \tag{15}$$

Equivalently, we could search for the eigenvector $\mathbf{f}$ of the matrix $\mathbf{U}^{-1}(\ell_c)$ with zero eigenvalue:

$$[1 + \mathbf{U}(0)\cdot\mathbf{P}(\ell_c)]\cdot\mathbf{f} = 0,$$

that is,

$$f(\omega) = -\frac{1}{\pi}\int_{\omega'}\int_{\Lambda_c}^{\infty}\frac{[u_0 - \lambda\omega_E D(\omega-\omega')]}{Z_\Lambda^2(\omega')\omega'^2 + \Lambda^2}f(\omega'), \tag{16}$$

which is an integral equation for $f(\omega)$.

For a given value of input parameters $(u_0, \lambda, \omega_E, \Lambda_0)$ the set of equations (16) and (13) can be solved for a critical cutoff energy scale, Λ_c, at which the running couplings diverge and the Fermi liquid description breaks down.

IV. FINITE TEMPERATURE FORMALISM AND DERIVATION OF ELIASHBERG'S EQUATIONS

The $T=0$ formalism presented in the previous section can be readily extended to $T>0$, providing us with a more experimentally accessible quantity, namely a critical temperature. We seek the temperature T^* below which the Fermi liquid description ceases to exist as one scales towards the Fermi surface, that is, as $\Lambda_c \to 0$. In this case we replace the integrals in (13) and (16) by Matsubara sums and extend the integrals in Λ from 0 to ∞ to obtain

$$Z(\omega_n)\phi(\omega_n) = -\pi T^*\sum_m [u_0 - \lambda\omega_E D(\omega_n-\omega_m)]\frac{\phi(\omega_m)}{|\omega_m|}, \tag{17}$$

where we have defined $\phi(\omega_n) = f(\omega_n)/Z(\omega_n)$ and the Matsubara frequency $\omega_n = \pi T^*(2n+1)$, where n is an integer. Moreover, from (13) we find

$$Z(\omega_n) = 1 + \lambda\omega_E\frac{\pi T^*}{\omega_n}\sum_m \mathrm{sgn}(\omega_m)D(\omega_n-\omega_m). \tag{18}$$

The solution of (17) and (18) gives the value of T^* as a function of the input parameters.

We now relate our approach to Eliashberg's self-consistent mean-field theory, which assumes a broken symmetry with a superconducting order parameter $\Delta(\omega_n)$, in contrast to ours, which starts from the Fermi liquid phase.

Remarkably (17) and (18) coincide with the Eliashberg equations at $T=T_c$ if we replace $\phi(\omega_n)$ by $\Delta(\omega_n)$.[11] This result is striking since $\phi(\omega_n)$ is not an order parameter and no symmetry breaking was assumed in our calculation. By the same token, we can show that Λ_c plays the role of the zero temperature superconducting gap, Δ_0 since the RG procedure, approaching the instability from high temperatures, leads to an instability of the Fermi liquid state at a temperature T^* that is equal to the critical temperature T_c produced by the Eliashberg theory. Thus, it is no surprise that we can show that in the weak/intermediate coupling regime ($\mu^* < \lambda < 1$) we recover the McMillan formula:[13]

$$T^* \approx 1.13\omega_E \exp\left(-\frac{1+\lambda}{\lambda-\mu^*(1+\lambda)}\right), \tag{19}$$

where

$$\mu^* = \frac{u_0}{1+u_0 \ln(\Lambda_0/\omega_E)} \tag{20}$$

is the effective electron-electron interaction at the scale of ω_E (Anderson-Morel potential) and the Allen-Dynes expression[14]

$$T^* \approx 0.16\sqrt{\lambda}\omega_E \tag{21}$$

at strong coupling.

V. EXPANSION IN $1/N$ AND DERIVATION OF MIGDAL'S THEOREM

From the $1/N$ analysis of Ref. 6, we know that the solution above for the one-loop RG equations contains the sum of all dominant corrections in $1/N$. The terms that have been left out go to zero as $N\to\infty$. We have therefore demonstrated that, provided that one can start the RG flow at $\Lambda_0 \ll E_F$, Eliashberg's theory is the asymptotically exact description of the effective low-energy physics obtained by RG thanks to the small parameter $1/N$. While this is an important result, the main significance of our methods is that it can be used to study the competition between charge density wave (and other instabilities) and superconductivity in the strong coupling ($\lambda \ll 1$) regime of phonons, something that cannot easily be achieved in a mean-field approach. This issue arises, for example, for Fermi surfaces with regions of nesting. For arbitrary shapes of the Fermi surface the solution of the RG equations can be obtained numerically by discretizing the Fermi surfaces into patches.[9,15]

Since Eliashberg's theory is based on Migdal's theorem,[16] which states that electron-phonon vertex corrections to the electron self-energy vanish as $\omega_D/E_F \to 0$, it must be that this theorem is built into our approach. We show here how this result arises and that the small parameter ω_D/E_F of Migdal's theorem is replaced here by $1/N$.

Let us go back to the problem before we traced the phonons and discuss the $1/N$ hierarchy. The patch index notation carries very naturally to phonons. While it is obvious that when a fermion in patch i scatters to patch j, it emits a phonon of momentum $\mathbf{k}_j - \mathbf{k}_i$, we can also go the other way: a phonon of "large" momentum can be resolved, up to a twofold ambiguity, into a difference $\mathbf{k}_i - \mathbf{k}_j$ associated with patches i and j, due to the fact that the electron momenta lie in a thin annulus around the Fermi surface. This fact allows us to describe phonons with the double-index notation employed by 't Hooft for gluons in QCD:[17] the phonon line is seen as made out of two counter propagating electron lines, as depicted in Fig. 2(a). The feature of the large N approach, that given a sum or difference of momenta, one can uniquely reconstruct the parts as $\Lambda \to 0$, was pointed out in Ref. 6. Since integrating out the phonons produces a four-fermion interaction of size $1/N$ it is clear the electron-phonon vertex will be of size $1/\sqrt{N}$ in our notation.

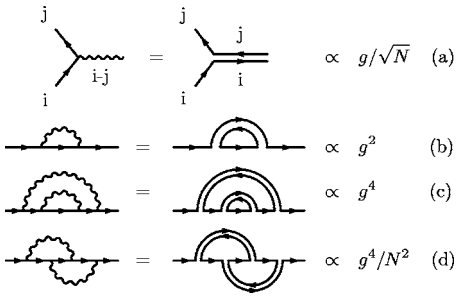

FIG. 2. (a) The electron-phonon vertex; (b)–(d) are self-energy corrections.

Any given diagram made of n_u vertices u, n_g vertices g, and n_L internal loops is of order $(1/N)^n$ where $n=-n_u - n_g/2 + n_L$. This way we can organize the diagrammatic expansion in powers of $1/N$. The number of loops in each diagram, n_L, can be easily obtained using Fig. 2(a). Consider the problem of the RG for $\Sigma(\mathbf{k},\omega)$ shown in Figs. 2(b)–2(d). It is clear from Fig. 2(b) that the correction of order g^2 should be taken into account. While g^2 comes with a factor $1/N$, there is an internal closed loop giving an extra factor of N resulting in a correction of $O(N^0)$ for the self-energy. At the same order in perturbation expansion in $1/N$, there are diagrams that are made from a simple repetition of Fig. 2(b), which are obviously of order $O(N^0)$ but higher order in g. Diagram Fig. 2(c) is of order $O(N^0)$, like Fig. 2(b), and so are all the other "rainbow" diagrams that are automatically included into the theory. Thus, by solving the RG equations for a "running" self-energy at one loop, we in fact take into account *all* corrections of order $O(N^0)$ to *all* loops. The infinite series of diagrams being summed is not arbitrary and arises naturally from the RG equations, as shown by our $1/N$ analysis. The final diagram of order g^4 is shown in Fig. 2(d): this is the famous vertex correction to the electron self-energy due to electron-phonon interactions studied by Migdal.[16] Notice that Fig. 2(d) does not contain any internal loops and is of order $O(N^{-2})$ and thus is vanishingly small as $N\to\infty$. It is easy to show that these corrections to the self-energy lead to (13). Thus, Migdal's theorem is built into the large N approach.[18] We can also show that the phonon self-energy has only contributions of order $1/N^2$[8].

VI. DISCUSSION AND CONCLUSION

Our method provides a novel way to calculate T_c of superconductors, starting from the normal state. At the super-

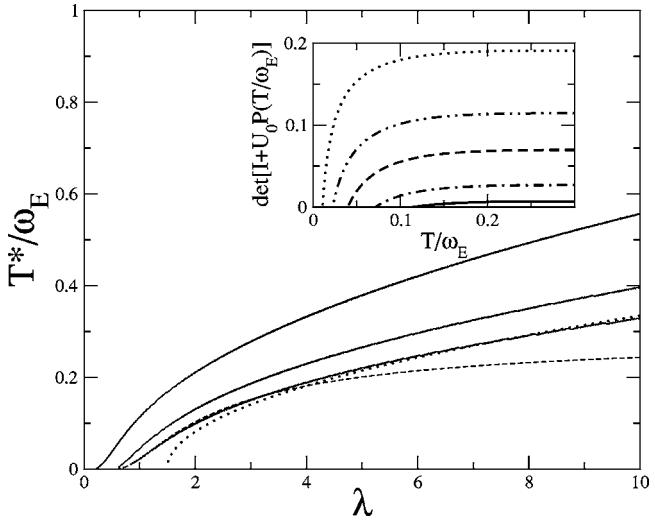

FIG. 3. The numerical solution of (15) (continuous line) for $\mu^*=0$, 0.17, and 0.24 (from top to bottom) as a function of λ. Dashed line: $T^*/\omega_E=1.1\exp[-(1+\lambda)/(\lambda+\mu^*(1+\lambda)\,]$ for $\mu^*=0.24$; dotted line: $T^*/\omega_E=0.11\sqrt{\lambda-1.49}$ for $\mu^*=0.24$. The inset shows the determinant in (15) as a function of T for $\mu^*=0$ and $\lambda=0.3$, 0.4, 0.5, 0.7, and 1 (from top to bottom).

conducting instability, Eliashberg's equations are derived from the RG flow equations. These self-consistent integral equations (17) are equivalent to the condition of zero determinant expressed in (15). In our RG approach, once the bare values of the couplings are given, we simply calculate the flow of $\det[1+\mathbf{U}(0)\cdot\mathbf{P}(T)]$ by solving the differential RG flow equations and determine T^* as the point when this quantity becomes zero (see inset in Fig. 3). This numerical calculation is much simpler than solving (17) directly, since it does not involve self-consistency conditions. In Fig. 3 we show the calculation of T^* as a function of λ with this new method together with the approximate asymptotic expressions for weak and strong coupling.

In summary, we have developed an asymptotically exact RG method that takes into account electron-electron and electron-phonon interactions in an unbiased way and reproduces all of Eliashberg's theory and also provides a framework (large N) for understanding it as well as Migdal's theorem that goes into it. Our procedure can be used for any Fermi surface geometry and for any number of scattering channels (forward, charge and spin density wave, etc.) and therefore allows for the study of the competition between scattering channels.[8] Finally, our procedure allows for a new numerical way to investigate superconductivity in metals.

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