

Asymptotic freedom of elastic strings and barriers

Peter Orland^{1,2,3,*} and Jing Xiao^{2,3,†}

¹Kavli Institute for Theoretical Physics, The University of California, Santa Barbara, California 93106, USA

²Physics Program, The Graduate School and University Center, The City University of New York, 365 Fifth Avenue, New York, New York 10016, USA

³Department of Natural Sciences, Baruch College, The City University of New York, 17 Lexington Avenue, New York, New York 10010, USA

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We study the problem of a quantized elastic string in the presence of an impenetrable wall. This is a two-dimensional field theory of an N -component real scalar field ϕ which becomes interacting through the restriction $\phi^T \phi \leq \phi_{\max}^2$, for a spherical wall of radius ϕ_{\max} . The $N=1$ case is a string vibrating in a plane between two straight walls. We review a simple nonperturbative argument that there is a mass gap in the spectrum, with asymptotically free behavior in the coupling $g = \phi_{\max}^{-1}$ for $N \geq 1$. This scaling behavior of the mass gap has been disputed in some of the recent literature. We find, however, that perturbation theory and the $1/N$ expansion each confirm that these models are asymptotically free. The $N \rightarrow \infty$ limit coincides with that of the $O(N)$ nonlinear σ model. A θ parameter and instantons exist for the two-dimensional $N=2$ model, which describes a string confined to the interior of a cylinder of radius ϕ_{\max} .

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I. INTRODUCTION

We will make a few observations in this paper concerning a nonrelativistic elastic string in N transverse dimensions in the presence of barriers. This is a theory of an N -component scalar field

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{pmatrix},$$

satisfying the condition $\phi^T \phi \leq \phi_{\max}^2$. The quantum string has the Lagrangian $\mathcal{L} = \frac{1}{2} \partial_t \phi^T \partial_t \phi - \frac{1}{2} \partial_x \phi^T \partial_x \phi$, where $\partial_t = \partial / \partial t$, $\partial_x = \partial / \partial x$, and the superscript T denotes the transpose. This is not a free two-dimensional field theory, however, due to the constraint. Here ϕ_{\max} is the radius of an infinitely deep spherical well and we interpret its reciprocal $g = \phi_{\max}^{-1}$ as the coupling constant. We shall refer to these as the B^N models, since target space is the N -dimensional ball B^N .

Note the similarity of the B^N model to the $O(N)$ nonlinear sigma model, for which the Lagrangian is the same, but the constraint is $\phi^T \phi = 1/g^2$. Indeed for large dimension of the target space N , most points of an N -dimensional ball are concentrated near the boundary. For this reason, one expects the two models to coincide as $N \rightarrow \infty$. In the large- N limit, it makes little difference whether the constraint is $\phi^T \phi = g^{-2}$ or $\phi^T \phi \leq g^{-2}$.

The $N=1$ model is of particular interest. This is a quantum string constrained to move in a planar channel of width $2/g$. The model arises in the statistical mechanics of a two-dimensional membrane between planar walls separated by a distance $2/g$,^{1,2} as well as striped phases of copper oxide layers.^{3,4} Another motivation for studying fields restricted in this way has been given in Ref. 5, where it has been argued

that it has practical utility in perturbation expansions.

A simple argument repeated below shows that the spectrum has a mass gap M , with the behavior

$$M \simeq \exp -Ag^{-2}, \quad g \rightarrow 0, \quad (1.1)$$

where A is a constant. This result was known to but disputed by Zaanen *et al.*,⁶ who use an argument similar to that of Helfrich and Servuss⁷ to conclude that

$$M \simeq \exp -Ag^{-\alpha}, \quad g \rightarrow 0, \quad (1.2)$$

where $\alpha \approx 2/3$. Nishiyama studied the model with the density-matrix renormalization group and has also argued for (1.2).⁸

We show in this paper that (1.1) is correct according to standard analytic methods. We consider two such methods. The first of these is a simple one-loop renormalization group analysis for $N=1$. The other is the $1/N$ expansion. In any case, if one accepts (1.2), the inevitable conclusion is that the analytic part of the β function is zero—which we show is not the case.

We also consider the interesting case of $N=2$. This field theory describes a string allowed to vibrate inside a cylinder. We find that there is a topological term that can be included in the action and that there are instantons. We determine the instanton solutions; they are similar to those of the circular brane model.⁹

Before concluding this Introduction, we repeat the argument given in Refs. 1, 2, 6, and 8 that a mass gap appears and depends on the coupling as (1.1). Though referred to as a mean-field argument in Ref. 8, it is closer in spirit to the theorems of Peierls and of Mermin and Wagner forbidding continuous symmetry breaking in two dimensions. If we ignore the constraint and use our massless Lagrangian, the two-point equal-time correlation function behaves as

$$\langle \phi^T(x)\phi(0) \rangle = \frac{N}{2\pi} \ln \frac{|x|}{a}, \quad (1.3)$$

where a is a short-distance cutoff. But we also have the strict inequality

$$\langle \phi^T(x)\phi(0) \rangle \leq g^{-2}.$$

Thus, Eq. (1.3) should be valid for $|x|$ approximately in the range $a < |x| < ae^{2\pi/g^2N}$, but not for $|x| > ae^{2\pi/g^2N}$. This limiting value of $|x|$ should be the correlation length of the theory, which is the inverse of the mass gap. Thus we obtain

$$M \approx a^{-1} \exp - \frac{2\pi}{g^2N}, \quad g \rightarrow 0, \quad (1.4)$$

which is just (1.1) with $A=2\pi/N$. We shall verify (1.4) in the next section. This argument is very suggestive—perhaps it will point the way to a rigorous proof of the mass gap, which is lacking for many interesting field theories. The reader should take the argument with a grain of salt, however. It does not really establish that the two-point function falls off exponentially, but only shows that it cannot behave logarithmically. For example, power-law decay of the two-point function cannot easily be ruled out.

The β function to lowest order follows simply from (1.4). It is negative and vanishes at $g=0$:

$$\beta(g) = \left. \frac{\partial g^2}{\partial \ln a^{-1}} \right|_{M \text{ fixed}} = -\frac{g^4N}{2\pi}.$$

II. EXPANSION METHODS

The Lagrangian of the $N=1$ model, Wick rotated to Euclidean space, is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi,$$

where $\partial_\mu = \partial / \partial x^\mu$ with $x^0=t$, $x^1=x$. Since $\phi^2 \leq g^{-2}$, we parametrize ϕ by a new field ψ , through $\phi(x) = g^{-1} \sin \psi(x)$. This choice of parametrization is not unique (other choices of parametrization, such as $\phi = g^{-1} \tanh \psi$, would give the same result). This mapping from ψ to ϕ is many to one, instead of one to one, but this fact will not make any difference as far as perturbation theory is concerned.

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2g^2} \cos^2 \psi \partial_\mu \psi \partial^\mu \psi = \frac{1}{2g^2} \left(1 - \psi^2 + \frac{1}{3} \psi^4 \right) \partial_\mu \psi \partial^\mu \psi.$$

The functional integral is

$$W[J] = \int [d\psi] \exp - \frac{1}{\hbar} \int d^2x [\mathcal{L} - J\psi - \hbar a^{-2} \ln(\cos \psi)],$$

where a is a short-distance cutoff with dimensions of centimeters. The third term in the exponent, which comes from the Jacobian in the functional measure, is of one higher order of \hbar than the Lagrangian; though this term must be considered at two loops, we may ignore it in our one-loop calculation.

The leading term of the effective action is

$$\frac{1}{2g^2} \left(1 - \frac{g^2}{4\pi} \ln(\mu^2 a^2) \right) \partial_\mu \phi \partial^\mu \phi,$$

where μ is an infrared cutoff, with dimensions of inverse centimeters, and where a quadratically divergent contribution is canceled by a counterterm. From this expression we find that the mass gap scales as

$$M \approx a^{-1} \exp - \frac{2\pi}{g^2}.$$

This agrees with the result (1.4) discussed in the Introduction.

The $1/N$ expansion for the B^N model is extremely simple. At leading order, all expressions coincide with those of the nonlinear $O(N)$ sigma model. There is some difference to first order in $1/N$, but we will not discuss this issue in detail.

Using a standard integral formula for the Heaviside function, the functional integral is

$$Z = \int [d\omega][d\phi] \exp - \int d^2x \left(\frac{1}{2} \partial_\mu \phi^T \partial^\mu \phi + i\omega(\phi^T \phi - G^{-2}N) + a^{-d} \ln(\omega - i\epsilon) \right),$$

where $G^2 = g^2/N$. Integration over ϕ yields

$$Z = \int [d\omega] \exp - N \left(\frac{1}{2} \text{Tr} \ln(-\partial^2/2 - i\omega) - iG^{-2} \int d^2x \omega + \frac{1}{N} a^{-d} \ln(\omega - i\epsilon) \right). \quad (2.1)$$

This integral is dominated by a saddle point on the imaginary axis for large N , which we write as $\omega_0 = -im^2/2$. The presence of ϵ in Eq. (2.1) assures that the logarithm is defined on the correct sheet in the vicinity of the saddle point. Except for the last term in Eq. (2.1) this is the same expression obtained for the σ model. The equation for the saddle point is

$$1 = G^2 \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2}. \quad (2.2)$$

If the momentum integral (2.2) is cut off by $|p| < a^{-1}$, we find the standard result

$$m = a^{-1} e^{-2\pi/G^2} (1 - e^{-4\pi/G^2})^{-1/2},$$

confirming that the model is asymptotically free.

The B^N model is different from the $O(N)\sigma$ model to order $1/N$. That is because the last term in Eq. (2.1) will contribute to this order.

III. THE TOPOLOGICAL TERM AND INSTANTONS FOR $N=2$

For the case of $N=2$ in two dimensions, a new term can be added to the action. We consider the B^2 model in Euclidean space-time. We take this space-time to be a two-dimensional ball, i.e., a disk, with radius R . The target space is also a disk with radius g^{-1} . A smooth field configuration is

a map from the first disk to the second. This map has a degree which is the number of images of the space-time disk in the target space disk. This degree is

$$\nu = \frac{g^2}{2\pi} \int_{|x| \leq R} d^2x (\partial_1 \phi^1 \partial_2 \phi^2 - \partial_1 \phi^2 \partial_2 \phi^1).$$

The action with such a term is

$$S = S_0 + i\theta\nu = \frac{1}{2} \int_{|x| \leq R} d^2x (\partial_1 \phi^T \partial_1 \phi + \partial_2 \phi^T \partial_2 \phi) + i\theta\nu,$$

where the (time-reversal-violating) parameter θ is defined modulo 2π .

A similar term can be defined for the circular-brane model.⁹ In this model, the field ϕ is unconstrained except at the boundary, where it is required that $\phi^T \phi = g^{-2}$. The instanton solutions are of the same form as those found below.

To show explicitly all the instantons and anti-instantons, we use complex coordinates. Let us define $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$, $\partial = \frac{1}{2} \partial_1 - (i/2) \partial_2 = \partial / \partial z$, $\bar{\partial} = \frac{1}{2} \partial_1 + (i/2) \partial_2 = \partial / \partial \bar{z}$, and $\phi = \phi^1 + i\phi^2$, $\bar{\phi} = \phi^1 - i\phi^2$. For instantons, ϕ is analytic and $\bar{\phi}$ is antianalytic,

$$\partial \bar{\phi} = 0, \quad \bar{\partial} \phi = 0,$$

and for anti-instantons, ϕ is antianalytic and $\bar{\phi}$ is analytic,

$$\partial \phi = 0, \quad \bar{\partial} \bar{\phi} = 0.$$

The general instanton solution must be an analytic map from the disk of radius R to the disk of radius g^{-1} of degree ν . This can only be of the form

$$\phi = g^{-1} \prod_{j=1}^{\nu} \frac{a_j z + b_j R}{\bar{b}_j z + \bar{a}_j R}, \quad (3.1)$$

with $\bar{\phi}$ given by complex conjugation. The complex moduli a_1, \dots, a_ν and b_1, \dots, b_ν satisfy $|a_j|^2 - |b_j|^2 = 1$. There are no poles in z in the disk. To see this, note that the poles of Eq. (3.1) lie at $z = -Ra_j/b_j$, $j = 1, \dots, \nu$. Since $|a_j/b_j|^2 = 1 + 1/|b_j|^2 \geq 1$, the function ϕ is completely analytic in the interior of the disk. Furthermore, there is no singularity at the boundary at which $|\phi| = g^{-1}$. By the maximum-modulus theorem, our constraint is satisfied; $|\phi|$ cannot exceed g^{-1} anywhere in the disk of radius R .

The anti-instanton solutions are very similar to Eq. (3.1). They are

$$\phi = g^{-1} \prod_{j=1}^{\nu} \frac{a_j \bar{z} + b_j R}{\bar{b}_j \bar{z} + \bar{a}_j R}, \quad (3.2)$$

with $\bar{\phi}$ again given by complex conjugation. As before, the complex moduli a_1, \dots, a_ν and b_1, \dots, b_ν satisfy $|a_j|^2 - |b_j|^2 = 1$.

The semiclassical expansion about instantons is insufficient to understand the exponential decay of correlation functions. The fluctuation determinant is completely insensitive to both ν and the moduli. An interesting question is whether fractional-topological charges can account for the correct exponentially decaying behavior of correlation functions.

IV. CONCLUSION

We have shown by several elementary methods that the energy gap in a quantum elastic string with barriers decays exponentially with the square of the barrier width. For the special case of a string in the interior of a cylindrical barrier, there are instantons, which are holomorphic maps from the two-dimensional disk to itself.

The large- N limit coincides with the spherical model in any dimension. The latter model has an ultraviolet-stable fixed point in three Euclidean dimensions (or two space and one time dimensions). We expect that such a fixed point exists for the three-dimensional B^N model for finite N , separating a spontaneously broken phase from a massive, strong-coupling phase. In this case, the model describes a quantum two-dimensional membrane, moving transversely, in the presence of a barrier.

An additional feature of the two-dimensional B^N model is that the β function is proportional to N ; in the case of the σ model, the β function is proportional to $N-2$.¹⁰

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*Electronic address: orland@gursey.baruch.cuny.edu; giantswing@gursey.baruch.cuny.edu

†Electronic address: jxiao@gursey.baruch.cuny.edu

¹W. Janke and H. Kleinert, Phys. Lett. A **117**, 353 (1986).

²D. Sornette, Europhys. Lett. **2**, 715 (1986).

³J. Zaanen and O. Gunnarsson, Phys. Rev. B **40**, R7391 (1989).

⁴V. J. Emory, S. Kivelson, and J. M. Tranquada, Proc. Natl. Acad. Sci. U.S.A. **96**, 8814 (1999).

⁵Y. Meurice, Phys. Rev. Lett. **88**, 141601 (2002).

⁶S. I. Mukhin, W. van Saarloos, and J. Zaanen, Phys. Rev. B **64**, 115105 (2001).

⁷W. Helfrich and R. M. Servuss, Nuovo Cimento B **D3**, 137 (1984).

⁸Y. Nishiyama, Phys. Rev. B **64**, 064510 (2001); **66**, 184501 (2002).

⁹S. L. Lukyanov and A. B. Zamolodchikov, J. Stat. Phys., P05003 (2004).

¹⁰A. M. Polyakov, Phys. Lett. **B59**, 87 (1975).