Quantum Griffiths effects in itinerant Heisenberg magnets

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We study the influence of quenched disorder on quantum phase transitions in itinerant magnets with Heisenberg spin symmetry, paying particular attention to rare disorder fluctuations. In contrast to the Ising case where the Landau damping of the spin fluctuations suppresses the tunneling of the rare regions, the Heisenberg system displays strong power-law quantum Griffiths singularities in the vicinity of the quantum critical point. We discuss these phenomena based on general scaling arguments, and we illustrate them by an explicit calculation for $O(N)$ spin symmetry in the large-N limit. We also discuss broad implications for the classification of quantum phase transitions in the presence of quenched disorder.

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I. INTRODUCTION

At low temperatures, strongly correlated materials can display a surprising sensitivity to small amounts of imperfections and disorder. This effect is particularly pronounced close to a quantum phase transition, where large fluctuations in space and time become fundamentally connected. Thermodynamic and response properties of a material are then affected much more dramatically than close to a classical phase transition, permitting exotic phenomena like infinite-randomness critical points with activated rather than power-law dynamical scaling,[1–8] smeared transitions,[9] or non-universal exponents at certain impurity quantum phase transitions.[10]

One interesting aspect of phase transitions in disordered systems is the Griffiths effects.[11] They are caused by large spatial regions that are devoid of impurities and can show local order even if the bulk system is in the disordered phase. The fluctuations of these regions are very slow because they require changing the order parameter in a large volume. Griffiths[11] showed that this leads to a singular free energy in a whole parameter region in the vicinity of the critical point which is now known as the Griffiths phase. In generic classical systems, the contribution of the rare regions to thermodynamic observables is very weak since the singularity in the free energy is only an essential one.[11–13] The consequences for the dynamics are more severe with the rare regions dominating the long-time behavior.[13–15]

Due to the perfect disorder correlations in (imaginary) time, Griffiths phenomena at zero temperature quantum phase transitions are enhanced compared to their classical counterparts. In random quantum Ising systems[1–5] and quantum Ising spin glasses,[6–8] thermodynamic quantities display power-law singularities with continuously varying exponents in the Griffiths phase, with the average susceptibility actually diverging inside this region.

The systems in which these quantum Griffiths phenomena have been shown unambiguously all have undamped dynamics (a dynamical exponent $z=1$ in the corresponding clean system). However, many systems of experimental importance involve magnetic[16–19] or superconducting[20] degrees of freedom coupled to (gapless) conduction electrons which leads to overdamped dynamics characterized by a clean dynamical exponent $z>1$. Studying the effects of rare regions in this case is therefore an important issue. In recent years, there has been an intense debate on the theory of quantum Griffiths effects in itinerant *Ising* magnets. It has been suggested[21] that overdamped systems show quantum Griffiths phenomena similar to that of undamped systems. However, recently it has been shown[9,22] that the overdamping prevents the rare regions from tunneling leading to static rare regions displaying superparamagnetic rather than quantum Griffiths behavior, at least for sufficiently low temperatures. In Ref. 22, it was also pointed out that different behavior is expected for systems with continuous spin symmetry.

In this paper, we examine quantum Griffiths effects in itinerant *Heisenberg* magnets in detail. Our results can be summarized as follows: In contrast to the Ising case, itinerant magnets with Heisenberg symmetry [in general, $O(N)$ symmetry with $N>1$] do display power-law quantum Griffiths singularities. The locally ordered rare regions are not static but retain their quantum dynamics. Their low-energy density of states follows a power law, $\rho(\epsilon)\sim\epsilon^{d/z'-1}$ where d is the space dimensionality and z' is a continuously varying dynamical exponent. This leads to power-law dependencies of several observables on the temperature T, including the specific heat, $C\sim T^{d/z'}$, and the magnetic susceptibility, $\chi\sim T^{d/z'-1}$. Our results are not limited to Heisenberg magnets, they generally apply to $O(N)$ order parameters with $N>1$ including the recently investigated superconductor-metal transition[23] in thin nanowires.[20]

The paper is organized as follows: In Sec. II we derive quantum Griffiths effects from general scaling arguments based on the observation that a rare region in an itinerant Heisenberg magnet is at its lower critical dimension. These arguments suggest a general classification of disordered quantum phase transitions in terms of the dimensionality of the rare regions. In Sec. III we then present an explicit calculation for $O(N)$ spin symmetry in the large-N limit. We conclude in Sec. IV by discussing the importance of the spin symmetry, the relation to Kondo physics, as well as possible experimental realizations of our predictions.

II. QUANTUM GRIFFITHS EFFECTS FROM SCALING ARGUMENTS

Our starting point is a quantum Landau-Ginzburg-Wilson free energy functional for an N-component ($N>1$) order parameter field $\phi=(\phi_1,\ldots,\phi_N)$. For definiteness, we consider the itinerant antiferromagnetic transition. The action of the clean system reads[24–26]

$$S=\int dxdy\,\phi(x)\Gamma(x,y)\phi(y)+\frac{u}{2N}\int dx\,\phi^4(x). \qquad (1)$$

Here, $x\equiv(\mathbf{x},\tau)$ comprises position $\mathbf{x}$ and imaginary time τ, and $\int dx\equiv\int d\mathbf{x}\int_0^{1/T}d\tau$. The imaginary time direction which characterizes the quantum dynamics formally appears like an additional spatial dimension of a classical system. The perfect disorder correlations in this imaginary time direction are ultimately responsible for the enhancement of the Griffiths effects at zero temperature. $\Gamma(x,y)$ is the bare inverse propagator (bare two-point vertex), whose Fourier transform is

$$\Gamma(\mathbf{q},\omega_n)=(r_0+\mathbf{q}^2+\gamma|\omega_n|^{2/z}) \qquad (2)$$

and r_0 is the bare energy gap, i.e., the bare distance from the clean critical point.

We are interested in the case of overdamped spin dynamics ($z=2$) with $\gamma\simeq(J\rho_F)^2/(E_F a_0^2)$ where J is the coupling constant between spin degrees of freedom and conduction electrons, with density of states, ρ_F, and Fermi energy, E_F, respectively. a_0 is the lattice constant. In order to demonstrate the special behavior of Heisenberg systems with $z=2$, where overdamping is caused by the particle-hole continuum of itinerant electrons, we frequently discuss variable z and compare the behavior with ballistic spin systems with $z=1$. We will use a system of units with $\gamma=1$. The clean system undergoes the quantum phase transition when the renormalized gap r vanishes. To introduce quenched disorder, we dilute the system with nonmagnetic impurities of spatial density p, i.e., we add a random potential, $\delta r(\mathbf{x})=\Sigma_i V[\mathbf{x}-\mathbf{x}(i)]$, to r_0. Here, $\mathbf{x}(i)$ are the positions of the impurities, and $V(\mathbf{x})$ is a positive short-ranged impurity potential.

We first present the general scaling arguments leading to quantum Griffiths behavior in this system. Despite the dilution, there are statistically rare large spatial regions devoid of impurities and thus unaffected by the disorder. The probability for finding such a region of volume L^d, frequently referred to as an instanton, is

$$w\sim(1-p)^{(L/a_0)^d}=\exp(-cL^d), \qquad (3)$$

with $c=-a_0^{-d}\ln(1-p)$. Below the clean critical point, the rare regions can be locally in the ordered phase even though the bulk system is not. At zero temperature, each rare region is equivalent to a one-dimensional classical $O(N)$ model in a rodlike geometry: finite in the d space dimensions but infinite in imaginary time. For overdamped dynamics, $z=2$, the interaction in imaginary time direction is of the form $(\tau-\tau')^{-2}$. One-dimensional continuous-symmetry $O(N)$ models with $1/\mathbf{x}^2$ interaction are known to be exactly *at* their lower critical dimension.[27–29] Therefore an isolated rare region of linear size L cannot independently undergo a phase transition. Its energy gap depends exponentially on its volume (i.e., the effective spin of the droplet)

$$\epsilon_L\sim\exp(-bL^d). \qquad (4)$$

Equivalently, the susceptibility of such a region diverges exponentially with its volume. Combining Eqs. (3) and (4) gives a power-law density of states for the energy gap ϵ (to leading exponential accuracy),

$$\rho(\epsilon)\propto\epsilon^{c/b-1}=\epsilon^{d/z'-1}, \qquad (5)$$

where the second equality defines the customarily used dynamical exponent z'.[30] It continuously varies with disorder strength or distance from the clean critical point. Many results follow from this. For instance, a region with a local energy gap ϵ has a local spin susceptibility that decays exponentially in imaginary time, $\chi_{\text{loc}}(\tau\to\infty)\propto\exp(-\epsilon\tau)$. Averaging by means of ρ yields

$$\chi_{\text{loc}}^{\text{av}}(\tau\to\infty)\propto\tau^{-d/z'}. \qquad (6)$$

The temperature dependence of the static average susceptibility is then

$$\chi_{\text{loc}}^{\text{av}}(T)=\int_0^{1/T}d\tau\,\chi_{\text{loc}}^{\text{av}}(\tau)\propto T^{d/z'-1}. \qquad (7)$$

If $d<z'$, the local zero-temperature susceptibility diverges, even though the system is globally still in the disordered phase. Analogously, the contribution of the rare regions to the specific heat C can be obtained from

$$\Delta E=\int d\epsilon\,\rho(\epsilon)\epsilon e^{-\epsilon/T}/(1+e^{-\epsilon/T})\propto T^{d/z'+1}, \qquad (8)$$

which gives $\Delta C\propto T^{d/z'}$. Other observables can be determined in a similar fashion. The power-law density of states (5) in the Griffiths phase of a disordered itinerant $O(N)$ magnet and the resulting quantum Griffiths singularities (6)–(8) are the central results of this paper. They take the same form as the quantum Griffiths singularities in undamped (clean $z=1$) random quantum Ising models[1–5] and quantum Ising spin glasses.[6–8]

These scaling arguments suggest a general classification of Griffiths phenomena in the vicinity of bulk phase transitions (at least those described by Landau-Ginzburg-Wilson theories) with weak, random-T_c or random-mass type disorder. It is based on the effective dimensionality of the rare regions. Three cases can be distinguished.

(i) If the rare regions are *below* the lower critical dimensionality d_c^- of the problem, their energy gap depends on their size via a power law, $\epsilon_L\sim L^{-\psi}$. Since the probability for finding a rare region is exponentially small in L, the low-energy density of states in this first case is exponentially small. This leads to weak "classical" Griffiths singularities characterized by an essential singularity in the free energy. This case is realized in generic classical systems (where the rare regions are finite in all directions and thus effectively zero-dimensional). It also occurs in quantum rotor systems with Heisenberg symmetry and undamped ($z=1$) dynamics.[31,32] Here, the rare regions are equivalent to one-dimensional classical Heisenberg models which are also below $d_c^-=2$.

(ii) In the second class, the rare regions are exactly *at* the lower critical dimension and their energy gap shows an exponential dependence [like Eq. (4)] on L. As shown above, this leads to a power-law density of states and strong power-law quantum Griffiths singularities. This second case is realized, e.g., in classical Ising models with linear defects[1,33] and random quantum Ising models (each rare region corresponds to a one-dimensional classical Ising model)[2,3] as well as in the disordered itinerant quantum Heisenberg magnets studied here (the rare regions are equivalent to classical one-dimensional Heisenberg models with $1/\mathbf{x}^2$ interaction).

(iii) Finally, in the third class, the rare regions can undergo the phase transition independently from the bulk system, i.e., they are *above* the lower critical dimension. In this case, the locally ordered rare regions become truly static which leads to a smeared phase transition. This happens, e.g., for classical Ising magnets with planar defects[34] (the rare regions are effectively two-dimensional) or for itinerant quantum Ising magnets[9,22] where the rare regions are equivalent to classical one-dimensional Ising models with $1/\mathbf{x}^2$ interaction.[35]

III. RARE REGIONS IN THE LARGE-*N* LIMIT

To complement the general scaling arguments and to obtain quantitative estimates for the exponent z' we now perform an explicit calculation of the Griffiths effects in the model (1) in the large-N limit. The approach is a generalization of Bray's treatment[14] of the classical case. In the large-N limit, a clean system undergoes a quantum phase transition for $g=g_c \propto \Lambda^{2-d-z}$ with coupling constant $g=u/|r_0|$ and upper momentum cutoff Λ. For $g<g_c$, the clean system is in the ordered state with the order parameter

$$\phi_{0,\text{clean}} = [N(g_c-g)/(g_c g)]^{1/2} \tag{9}$$

and vanishing gap. In the random system, we consider a droplet of size L^d, devoid of impurities, and determine its size dependent energy gap, ϵ. It is determined by the equation of state $\epsilon\phi_0=h$, where

$$\epsilon = r_0 + u\langle\phi^2\rangle + \frac{u\phi_0^2}{N}. \tag{10}$$

h is the field conjugate to the order parameter, $\phi_0=\langle\phi\rangle$, of the droplet and

$$\langle\phi^2\rangle = \sum_{\mathbf{q},\omega_n} \frac{TL^{-d}}{\epsilon + \mathbf{q}^2 + |\omega_n|^{2/z}}. \tag{11}$$

For $T>0$, both the $\mathbf{q}$ and ω sums are discrete. Consequently, the order parameter $\phi_0=h/\epsilon$ vanishes for $h\to 0$ since ϵ must remain positive to avoid a divergence of the $\mathbf{q}=\mathbf{0}$, $\omega_n=0$ contribution to $\langle\phi^2\rangle$. Thus classical droplets are below d_c^-. At

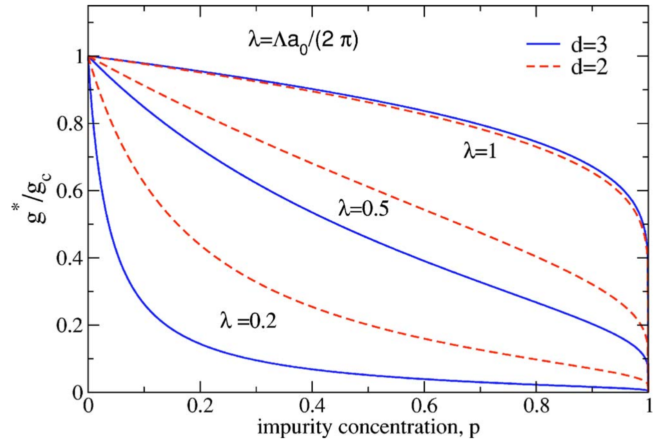

FIG. 1. (Color online) Coupling constant g^*/g_c below which quantum Griffiths effects cause a diverging low energy density of states, as a function of disorder concentration, p, in two and three dimensions for various values of $\lambda=\Lambda a_0/2\pi$.

$T=0$, a frequency integration must be performed and the $\epsilon\to 0$ limit becomes less singular. Yet, for $z<2$ droplets remain below d_c^- since the $\mathbf{q}=\mathbf{0}$ contribution to $\langle\phi^2\rangle$ still diverges as $L^{-d}\epsilon^{(z-2)/2}$. For $z=2$ this term diverges only as $\ln(\epsilon L^2)$ and, as expected, droplets with $z=2$ are marginal and located at their lower critical dimension.

To quantify these arguments and to determine the dependence of ϵ on L for $T=0$, we apply the finite size analysis of the large-N theory[36] to the quantum limit. As shown in the Appendix, we obtain for $\varepsilon L^2 \ll 1$ and $z=2$:

$$\langle\phi^2\rangle = \frac{1}{g_c} - \frac{L^{-d}}{\pi}\ln(\epsilon L^2). \tag{12}$$

Inserting this into Eq. (10) gives for small ϵ:

$$\epsilon = L^{-2}\exp(-bL^d), \tag{13}$$

with $b=\pi(g_c-g)/(g_c g)=\pi/N\phi_{0,\text{clean}}^2$. This explicitly verifies Eq. (4) in the large-N limit. For a given distance, b, to the clean critical point, only droplets larger than a certain value contribute to quantum Griffiths behavior. For $z<2$, the last term in Eq. (12) is proportional to $L^{-d}\epsilon^{(z-2)/2}$ and we obtain $\epsilon \propto L^{-\psi}$ with $\psi=2d/(2-z)$. For $z>2$, $\phi_0\neq 0$, i.e., even a finite droplet is allowed to order at $T=0$. Since the onset of order depends on the size of the droplet a smearing of the transition occurs. All this is in agreement with our general expectation, discussed above.

Inserting our result for the coefficient b into Eq. (5), we obtain an explicit expression for the Griffiths exponent

$$z' = \frac{d\pi(g_c-g)a_0^d}{g_c g\ln(1-p)^{-1}}. \tag{14}$$

The density of states (5) diverges for $\epsilon\to 0$ if $z'>d$. z' vanishes as one approaches the clean critical point $g\to g_c$, but becomes larger as $(g_c-g)/g$ grows. In Fig. 1 we plot the coupling constant g^*, below which $z'>d$, as a function of the impurity concentration, p, for three different values of the nonuniversal number $a_0\Lambda$. Quantum Griffiths effects dominate the low energy excitations for $g<g^*$, provided that

droplets are still sufficiently diluted and the system is above the critical point, g_c^{dis}, of the random system. Observable quantum Griffiths effects exist unless Λa_0 becomes small.

At finite temperatures, a crossover occurs to weaker classical Griffiths effects. To estimate the characteristic crossover temperature for $z=2$, we decompose $\langle \phi^2 \rangle$, Eq. (11), into its zero-temperature part and the more singular classical ($\omega_n = \mathbf{q} = 0$) contribution:

$$\langle \phi^2 \rangle_T \simeq \langle \phi^2 \rangle_{T=0} + L^{-d} \frac{T}{\varepsilon_L}. \tag{15}$$

The crossover occurs when the classical term becomes comparable to $\langle \phi^2 \rangle_{T=0}$. We find that droplets with $L > L_0(T)$, determined by $T = (b/\pi) L_0^{d-2} \exp(-bL_0^d)$, behave classically and $\rho(\epsilon)$ is suppressed for $\epsilon < \epsilon_0 = L_0^{-2} \exp(-bL_0^d)$. Droplets smaller than $L_0(T)$ still follow the quantum dynamics. Quantum Griffiths behavior persists as long as $\epsilon_0(T) < T$. This is fulfilled for sufficiently low temperatures $T < T_0 = f_d b^{2/d}$ with $f_d = \pi^{-2/d} \exp(-\pi)$. Above the temperature T_0 the density of states for $\omega > T$ is then given as

$$\rho_{class}(\varepsilon) \propto \exp\left(-\frac{A}{\varepsilon}\right), \tag{16}$$

with $A = \pi T \ln(1-p)^{-1}/(ba_0^d)$, in agreement with the behavior for classical Heisenberg systems.[14] Instead of adding the purely classical contribution to $\langle \phi^2 \rangle_{T=0}$, the behavior at low T may be described by explicitly calculating the first low temperature corrections to $\langle \phi^2 \rangle_{T=0}$. We find that these low temperature corrections behave as

$$\langle \phi^2 \rangle_T = \langle \phi^2 \rangle_{T=0} + \frac{L^{-d}}{48\pi} \frac{T^2}{(4\pi\varepsilon_L)^2} + \cdots . \tag{17}$$

An argumentation identical to the one given below Eq. (15) yields that these corrections become relevant above a temperature which behaves identical to $T_0 \propto b^{2/d}$, only with a different numerical prefactor $f_d \simeq e^{-1}(16\pi\sqrt{3})^{4/d}$. The closer one approaches the clean quantum phase transition, where b vanishes, the narrower is the region of quantum Griffiths behavior. More importantly, very close to the dirty critical point, g_c^{dis}, the crossover temperature is exponentially suppressed because the droplets have a minimum size of the order of the bulk correlation length.

IV. DISCUSSION AND CONCLUSIONS

To summarize, we have studied quantum Griffiths effects in itinerant magnets with continuous order parameter symmetry, using the itinerant antiferromagnet as the primary example. We have shown that this system displays strong power-law quantum Griffiths singularities. In this section we will address a few open questions and important implications of the results.

Our first point concerns the importance of Kondo physics. In Eq. (1) we assumed, following Refs. 24–26, that spin degrees of freedom in disordered metals can be described by quantum rotors with overdamped dynamics. One might worry about the Kondo dynamics of the entire droplet which is not included in the rotor approach (in Ref. 37 it was shown that an extended magnetic structure in a metallic environment can behave at low T as an anisotropic multichannel Kondo problem). Our theory is valid only if the k-channel Kondo behavior, presumably with large $k \sim (L/a_0)^d$, emerges only for $T_K \ll \varepsilon(L)$. Using the results for the related problem of a large droplet induced by a single magnetic impurity[38,39] it indeed follows that the Kondo temperature $T_K \sim \varepsilon(L) \exp(-\text{const}\, L^d)$ is exponentially smaller than the crossover scale for quantum Griffiths behavior and thus negligible.

We emphasize the difference between continuous spin symmetry and Ising symmetry. For Ising symmetry, rare regions are *above* the lower critical dimension. They cease to tunnel and become static at sufficiently low temperatures, leading to superparamagnetic behavior[22] and, ultimately, to a smeared transition.[9] Quantum Griffiths behavior can at best occur in a transient temperature window.[21] In contrast, for continuous symmetry, the rare regions are *exactly at* the lower critical dimension and retain their dynamics, with a power-law low-energy density of states. Quantum Griffiths effects dominate the low-temperature physics for $g^* > g > g_c^{\text{dis}}$.

Griffiths phenomena in itinerant *ferro*magnets require separate attention because mode-coupling effects induce a long-range interaction of the spin fluctuations.[40] This can potentially change the conditions for locally ordered droplets and thus the form of the Griffiths effects.

Let us comment on measuring the predicted effects. Many of the heavy electron systems displaying magnetic quantum phase transitions have a strong spin anisotropy and are thus better described by Ising models. Gd-based intermetallics have a local Heisenberg symmetry, but the hybridization between magnetic and conduction electrons is very small. This yields a very low temperature, T_0, for the onset of quantum Griffiths behavior. Most promising are 3d-transition metal systems with weak spin-orbit interaction and strong hybridization. A candidate is the 3d heavy fermion system LiV_2O_4,[41] where recent experiments did show a broad distribution of relaxation rates.[42] In addition, the XY-version of our theory ($N=2$) directly predicts quantum Griffiths behavior in disordered thin nanowires[20] close to the metal-superconductor quantum phase transition,[23] with direct impact on the conductance fluctuations of these systems. Our results hopefully motivate further the search for new and unconventional behavior in dissipative quantum systems with continuous order parameter symmetry.

This paper has focused on the Griffiths region above the dirty critical point. There is, however, a possible connection to the properties of the quantum critical point itself. It is known that the quantum critical points of undamped random quantum Ising models, which also display power-law quantum Griffiths effects, are of exotic infinite-randomness type.[1–5] The underlying strong-disorder renormalization group[2,43] supports a close connection between the quantum Griffiths effects and the exotic critical properties. This suggests that the quantum critical point of disordered itinerant Heisenberg magnets may also be of infinite-randomness type.

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APPENDIX: FINITE SIZE ANALYSIS OF THE LARGE *N* THEORY

In this Appendix we summarize the derivation of Eq. (12) by generalizing the finite size analysis of Ref. 36 to the quantum case. The order parameter fluctuations

$$\langle \phi^2 \rangle = \sum_{\mathbf{q},\omega_n} \frac{TL^{-d}}{\epsilon + \mathbf{q}^2 + |\omega_n|^{2/z}} \tag{A1}$$

determine the value of $\epsilon(L)$ through $\epsilon = r_0 + u\langle\phi^2\rangle + u\phi_0^2/N$. We consider a single droplet of size L^d completely devoid of impurities. Such a droplet can be described as a clean, but finite system. Therefore the sum over the momenta $\mathbf{q}$ is discrete while the frequency summation becomes an integration in the zero temperature limit. The discrete momentum sum is analyzed using the Poisson summation formula, and we obtain $\langle\phi^2\rangle = \langle\phi^2\rangle_\infty + J$ with

$$\langle \phi^2 \rangle_\infty = \int \frac{d^D q}{(2\pi)^d} \int \frac{d\omega}{2\pi} \frac{1}{q^2 + |\omega|^{2/z} + \varepsilon} \tag{A2}$$

as well as

$$J = \sum_{\mathbf{n}\neq \mathbf{0}} S(\mathbf{n}) \tag{A3}$$

with

$$S(\mathbf{n}) = \int \frac{d^d q}{(2\pi)^d} \int \frac{d\omega}{2\pi} \frac{e^{i\mathbf{q}\cdot\mathbf{n}L}}{q^2 + |\omega|^{2/z} + \varepsilon}. \tag{A4}$$

Here $\mathbf{n} = (n_1, \ldots, n_d)$ is a d-component vector with integer components.

As long as $d+z>2$, which we assume throughout this paper, it holds for small ε that $\langle\phi^2\rangle_\infty \simeq 1/g_c$, where g_c is the critical coupling constant of the clean bulk quantum phase transition, i.e., for $L\to\infty$. The corrections behave as $\varepsilon^{d+z-2/2}$ for $d+z<4$, as $\varepsilon\Lambda^{d+z-4}$ for $d+z>4$, and as $\varepsilon \log(\Lambda^2/\varepsilon)$ for $d+z=4$, i.e., they vanish as $\varepsilon\to 0$.

Next we analyze the sum over $\mathbf{n}$ in Eq. (A3) which takes into account the finite size of the droplet. Using $1/x = \int_0^\infty d\alpha \exp(-\alpha x)$ and performing the momentum and frequency integrations, it follows

$$S(\mathbf{n}) = \int_0^\infty d\alpha e^{-\alpha\varepsilon} \int \frac{d\omega d^d q}{(2\pi)^{d+1}} e^{i\mathbf{q}\cdot\mathbf{n}L} e^{-\alpha(q^2+|\omega|^{2/z})}$$
$$= \frac{\Gamma\left(1+\frac{z}{2}\right)}{2^d \pi^{(d+2)/2}} \int_0^\infty \frac{d\alpha e^{-\alpha\varepsilon}}{\alpha^{(d+z)/2}} \prod_{\mu=1,\ldots,d} e^{-n_\mu^2 L^2/4\alpha}.$$

Substituting $t = 4\pi\alpha L^{-2}$, we obtain for J:

$$J = \frac{\Gamma\left(1+\frac{z}{2}\right) L^{2-(d+z)}}{2^{2-z}\pi^{(4-z)/2}} A(\varepsilon L^2) \tag{A5}$$

with

$$A(\rho) = \int_0^\infty \frac{dt}{t^{(d+z)/2}} e^{-\rho t/4\pi} \left[B\left(\frac{1}{t}\right)^d - 1 \right] \tag{A6}$$

and

$$B(t) = \sum_{n=-\infty}^{\infty} \exp(-\pi t n^2). \tag{A7}$$

$B(t)$ fulfills the relation $B(t) = (1/t)^{1/2} B(1/t)$ which leads to

$$A(\rho) = \int_1^\infty \frac{dt}{t^{z/2}} e^{-\rho t/4\pi}(1 - t^{-D/2}) + \int_1^\infty dt [B(t)^d - 1]$$
$$\times [t^{(d+z-4)/2} e^{-\rho/4\pi t} + t^{-z/2} e^{-\rho t/4\pi}]. \tag{A8}$$

For $z \leq 2$, the dominant contribution to $A(\rho)$ comes from the first integral in Eq. (A8) which can be evaluated explicitly. The second integral is finite as $\rho = \varepsilon L^2 \to 0$ for all z. For small ρ and $z<2$ the leading contributions read

$$A(\rho) = \frac{\Gamma\left(1-\frac{z}{2}\right)}{(4\pi)^{z/2-1}} \rho^{-(2-z)/2} + \mathcal{O}(\rho^0), \tag{A9}$$

while for $z=2$, we obtain

$$A(\rho) = -\log\left(\frac{\rho}{4\pi}\right) + \mathcal{O}(\rho^0). \tag{A10}$$

If $z>2$, $A(\rho\to 0)$ remains finite.

Collecting the various contributions to $\langle\phi^2\rangle$ yields for $z=2$:

$$\langle \phi^2 \rangle = \frac{1}{g_c} - \frac{L^{-d}}{\pi} \log(\varepsilon L^2) \tag{A11}$$

which is the result given in Eq. (12) above.

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