Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance

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We define for quantum many-body systems a quasiadiabatic continuation of quantum states. The continuation is valid when the Hamiltonian has a gap, or else has a sufficiently small low-energy density of states, and thus is away from a quantum phase transition. This continuation takes local operators into local operators, while approximately preserving the ground-state expectation values. We apply this continuation to the problem of gauge theories coupled to matter, and propose the distinction of perimeter law versus "zero law" to identify confinement. We also apply the continuation to local bosonic models with emergent gauge theories. We show that local gauge invariance is topological and cannot be broken by any local perturbations in the bosonic models in either continuous or discrete gauge groups. We show that the ground-state degeneracy in emergent discrete gauge theories is a robust property of the bosonic model, and we argue that the robustness of local gauge invariance in the continuous case protects the gapless gauge boson.

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I. INTRODUCTION

Traditionally, gauge theory was described as a theory of a vector field a_μ that has a local gauge symmetry. In a Lagrangian framework, this means $L(a_\mu+\partial_\mu\phi)=L(a_\mu)$. This gauge symmetry was believed to be the defining property of a gauge theory. It was believed that the gauge symmetry protects the gapless gauge boson for continuous gauge groups and the topological ground-state degeneracy on compact space for discrete gauge groups. Even the slightest gauge symmetry breaking, such as a gauge potential term $(a_\mu)^2$ in the Lagrangian, gives a finite mass to the gauge boson or lifts the topological degeneracy. In this case we can no longer regard the theory as a gauge theory at low energies.

However, the above standard picture for gauge symmetry protecting gapless gauge bosons and ground-state degeneracy is very formal, since the gauge symmetry is not really a symmetry. Within the Hamiltonian formulation of gauge theories, the gauge transformation is simply a transformation between different labels that label the *same* physical state. It is a do-nothing transformation. It is very different from the usual symmetry transformation that transforms a physical state to a *different* physical state. Therefore, it is not clear what is the essence of gauge theory and gauge symmetry.

In last 15 years, it was shown, with increasing rigor, that *deconfined* gauge theories can emerge from certain local bosonic lattice models.[1–16] If the emergent gauge theory has a continuous gauge group, the local bosonic model will have gapless excitations that behave just like gauge bosons.[8,11,14–16] If the emergent gauge theory has a discrete gauge group or has a Chern-Simons term, the local bosonic model will have degenerate ground states on compact space.[4,17,18,34] This raises a physical question: What protects the gapless gauge bosons and the ground-state degeneracy? According to the standard picture for gauge theory, those properties are protected by gauge symmetry, but from the point of view of the local bosonic model, what is this "gauge symmetry"? How does gauge symmetry emerge at low energies?

A close examination of those local bosonic models with emergent gauge theory reveals that the emergence of gauge theory is intimately connected to string condensation.[11,13] The gauge symmetry is related to the integrity of the strings. If the strings are unbreakable, one can show that the low-energy states are gauge invariant. However, in general, the strings in the boson model are not perfectly well defined. Strings may break up momentarily and rejoin. One may wonder if this means that the gauge symmetry becomes approximate. We know that a theory that loses its gauge symmetry even slightly no longer behaves like a gauge theory at low energies. This seems to suggest that breakable strings will give gauge bosons a mass gap or lift the ground-state degeneracy.

On the other hand, it was believed that the gaplessness of the gauge bosons and the degeneracy of the ground states in those bosonic models are topological and are robust against any local perturbations.[4,17,18] A formal argument goes as the follows (see, for example, Ref. 19, p. 393 and p. 435). We first derive the low-energy effective gauge theory of the bosonic model. We then argue that any generic perturbation of the bosonic model cannot generate any terms that break the gauge symmetry in the low energy effective gauge theory. Therefore, all the properties protected by the gauge symmetry are robust against arbitrary perturbations of the bosonic model.

We see that to understand why gauge symmetry remains exact even for a generic boson model with virtually breakable strings is vital in our understanding why the degeneracy of the ground states is protected even when the original boson model has no symmetry, and why the gaplessness of the gauge bosons is protected even when the original boson model has only translation symmetry. We would like to address some of these issues in this paper.

In the next section, we start by introducing some bosonic models with emergent gauge theories, and raise the question of the ability of topological order and gauge symmetry to survive local perturbations of the Hamiltonian, as well as proposing the zero law to identify deconfined gauge theories. We then introduce in the following section a quasiadiabatic continuation which enables us to identify appropriately dressed operators for the perturbed Hamiltonian, such that the dressed operator has almost the same ground-state expectation value for the perturbed Hamiltonian as the original operator had for the original Hamiltonian. The final section illustrates the application of the continuation via a series of examples, beginning with local models without emergent gauge structure, such as quantum Ising models, and then going on to emergent gauge theories. Many of these systems are theories for which the unperturbed Hamiltonian has a ground-state degeneracy and then a gap to the rest of the spectrum, with no local operators connecting the degenerate ground states. We are then able to show that, so long as the gap to the rest of the spectrum remains open, the splitting between the low-energy states of the perturbed Hamiltonian remains exponentially small. This is illustrated in the case of the ferromagnetic quantum Ising model; in the case of the fractional quantum Hall effect where we are able to extend results on the insensitivity of the topological degeneracy to disorder;[17,18] and in the case of an emergent Z_2 gauge theory. The case of gapless theories is also discussed; it will turn out that gauge symmetry is much more robust in compact than in noncompact theories.

II. SIMPLE LOCAL BOSONIC MODELS WITH EMERGENT GAUGE THEORIES

To make the above discussion more concrete, in this section we are going to discuss two simple bosonic models that have emergent Z_2 and $U(1)$ gauge theories respectively.

A. A bosonic model with emergent Z_2 gauge theory

The first bosonic model is a spin-1/2 model on a d-dimensional cubic lattice.[20–22] The spins live on the links labeled by $\boldsymbol{i}$. The Hamiltonian is given by

$$H_0 = U\sum_{\boldsymbol{I}} (1 - W_{\boldsymbol{I}}) - g\sum_{\boldsymbol{p}} \left(\prod_{\text{edges of } \boldsymbol{p}} \sigma_{\boldsymbol{i}}^x \right) \tag{1}$$

where $\boldsymbol{I}$ labels the vertices and $\boldsymbol{p}$ the squares of the lattice, and $W_{\boldsymbol{I}}$ is given by

$$W_{\boldsymbol{I}} = \prod_{\text{legs of } \boldsymbol{I}} \sigma_{\boldsymbol{i}}^z. \tag{2}$$

The legs of a vertex are the links that connect to the vertex and the edges of a square are the four links around the square. $\sigma^{x,y,z}$ are the Pauli matrices.

When U is very large, the low-energy sector of the model is formed by closed-string states. What are the closed-string states? First, the state with all spins up is defined as the no-string state. A closed-string state is a state where the down spins form closed loops (see Fig. 1). We note that the closed-string states are the states that satisfy

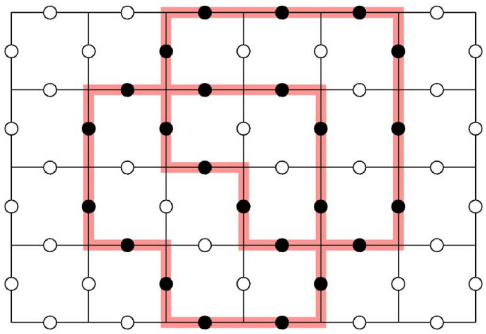

FIG. 1. (Color online) A closed-string state. The up spins are represented by open dots and the down spin by filled dots.

$W_{\boldsymbol{I}}|\text{closed strings}\rangle = |\text{closed strings}\rangle$. Since $[W_{\boldsymbol{I}}, H_0] = 0$, the closed-string states and the open-string states[33] do not mix. This allows us to plot the spectrum of H_0 separately for closed-string states and open-string states in Fig. 2.

When $g > 0$, the ground state $|\Phi_0\rangle$ of the model is the equal-weight superposition of all closed-string states. Such a state is called a closed-string condensed state since the closed-string creation operator $S(C_{\text{closed}})$ has a nonzero expectation value[23]

$$\langle \Phi_0 | S(C_{\text{closed}}) | \Phi_0 \rangle = 1 \tag{3}$$

regardless of the size of the string C_{closed}. Here a string creation operator that creates a string C is given by

$$S(C) = \prod_{\boldsymbol{i} \text{ on } C} \sigma_{\boldsymbol{i}}^x.$$

We can show Eq. (3) to be true by noting that

$$[S(C_{\text{closed}}), H_0] = 0. \tag{4}$$

What is the physical character of the closed-string condensed state? It is believed that the closed-string condensed state contain a new kind of order—topological order—which cannot be characterized by symmetry breaking and long-range order.[13,17] So we need a new way to characterize such an order. One way to characterize the topological order is through the robust ground-state degeneracy on a torus.[4,18]

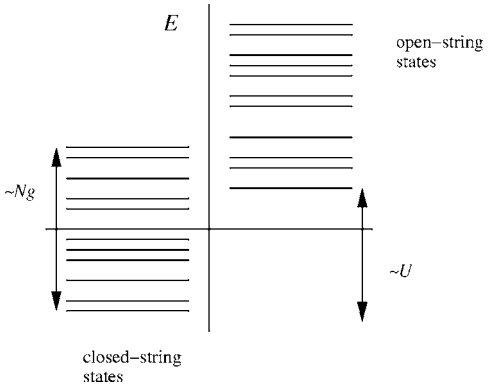

FIG. 2. The energy levels of N spin-1/2 spins described by H_0.

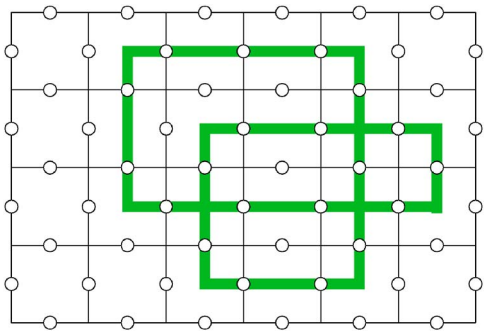

FIG. 3. (Color online) A dual closed string.

The closed-string condensed state in our spin-1/2 model is characterized by a fourfold ground-state degeneracy.

To show the fourfold ground-state degeneracy, we would like to first point out that the $S(C_{\text{closed}})$ are not the only closed-string operators that commute with the Hamiltonian. We can define dual string operators that also commute with the Hamiltonian.[22]

While strings are formed by segments that connect nearest-neighbor vertices, dual strings are formed by segments that connect the centers of nearest-neighbor squares (see Fig. 3). A dual string operator $\tilde{S}(\tilde{C})$ for a dual string $\tilde{C}$ is defined as

$$\tilde{S}(\tilde{C}) = \prod_{i \text{ cross } \tilde{C}} \sigma_i^z. \tag{5}$$

One can check that

$$[\tilde{S}(\tilde{C}_{\text{closed}}), H_0] = 0, \tag{6}$$

which implies that the dual closed strings also condense,

$$\langle \Phi_0 | \tilde{S}(\tilde{C}_{\text{closed}}) | \Phi_0 \rangle = 1. \tag{7}$$

Now we are ready to show that the ground states of H_0 on a torus have at least fourfold degeneracy. Let C_x and C_y ($\tilde{C}_x$ and $\tilde{C}_y$) be the closed (dual) strings that wrap around the torus once in the x and y directions. We find that the four large-string operators $(S(C_x), S(C_y), \tilde{S}(\tilde{C}_x), \tilde{S}(\tilde{C}_y))$ commute with each other except

$$\{S(C_x), \tilde{S}(\tilde{C}_y)\} = 0, \quad \{\tilde{S}(\tilde{C}_x), S(C_y)\} = 0. \tag{8}$$

The above algebra has only one four-dimensional irreducible representation. Since the large-closed-string operators $(S(C_x), S(C_y), \tilde{S}(\tilde{C}_x), \tilde{S}(\tilde{C}_y))$ act within the degenerate ground-states, the ground state degeneracy must be a multiple of 4.

When $U=\infty$, the model becomes the Z_2 gauge theory.[20] One way to see this is to note that the only states with finite energies are closed-string states and closed-string states are gauge-invariant states under local Z_2 gauge transformations. The local Z_2 gauge transformations are generated by the unitary operators $W_{\boldsymbol{I}}$ in Eq. (2):

$$W_{\boldsymbol{I}} |\text{closed strings}\rangle = |\text{closed strings}\rangle \quad \text{for any } \boldsymbol{I}. \tag{9}$$

A generic Z_2 gauge transformation is given by

$$\prod_{\boldsymbol{I}} (W_{\boldsymbol{I}})^{n_{\boldsymbol{I}}} |\text{closed strings}\rangle = |\text{closed strings}\rangle. \tag{10}$$

In the gauge theory language the closed-string operator $S(C_{\text{closed}})$ turns out to be the Wilson-loop operator,[20,24] which is gauge invariant,

$$W_{\boldsymbol{I}} S(C_{\text{closed}}) W_{\boldsymbol{I}}^{\dagger} = S(C_{\text{closed}}).$$

Equation (3) implies that the expectation values of the Wilson loop satisfy the perimeter law

$$\langle \Phi_0 | S(C_{\text{closed}}) | \Phi_0 \rangle \sim e^{-\alpha |C_{\text{closed}}|} \tag{11}$$

with zero coefficient $\alpha=0$. Here $|C_{\text{closed}}|$ is the length of the string C_{closed}. In this case we will call Eq. (3) the zero law.

The perimeter law indicates that the Z_2 gauge theory is in the deconfined phase. The Z_2 deconfined phase has four nearly degenerate ground states on the torus, which is consistent with our previous direct calculation on the spin model. The energy separation between the four nearly degenerate ground states is of order $e^{-L/\xi}$ where L is the linear size of the torus and ξ a finite length scale.

Now let us add a term

$$H_1 = -J_1 \sum_i \sigma_i^z$$

to our spin model Hamiltonian H_0 and assume U is finite. $S(C_x)$ and $S(C_y)$ no longer commute with the modified Hamiltonian H_0+H_1. So it is not clear if H_0+H_1 still has four degenerate ground states on a torus.

To understand the properties of the modified spin system H_0+H_1, we note that H_0+H_1 still does not mix the closed-string and open-string states. So if $U \gg g$, J_1, the low-lying states are still closed-string states. One can check that if we restrict H_0+H_1 to the closed-string subspace, the system is identical to a pure lattice Z_2 gauge theory. A pure Z_2 gauge theory has two phases in $d+1$ dimensions if $d>1$: a deconfined phase where the expectation value of the Wilson loop satisfies the perimeter law, and a confined phase where the expectation value of the Wilson loop satisfies the area law:

$$\langle \Phi_0 | S(C_{\text{closed}}) | \Phi_0 \rangle \sim e^{-\gamma A(C_{\text{closed}})}, \tag{12}$$

where $A(C_{\text{closed}})$ is the area enclosed by the loop C_{closed} and $\gamma>0$. So we expect that our spin model H_0+H_1 also has two phases. If $|J_1| \ll |g|$, the ground state of the spin model is filled with large closed strings which correspond to the Z_2 deconfined phase. If $J_1 \gg |g|$, the ground state of the spin model hardly has any strings (i.e., almost all spins point up) which corresponds to the deconfined phase.

Base on this picture, we guess that when $|J_1| \ll |g|$, the ground states of H_0+H_1 are (nearly) fourfold degenerate, while when $|J_1| \gg |g|$ the ground state is not degenerate. But the result for $|J_1| \ll |g|$ clearly is just a guess. Can we provide a more rigorous proof?

The situation gets even more complicated if we add another term

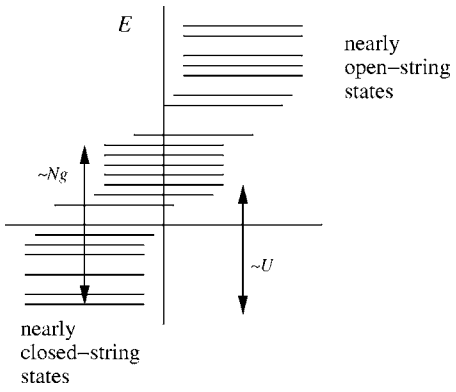

FIG. 4. The energy levels of N spin-1/2 spins described by $H_0+H_1+H_2$, assuming $|J_1|,|J_2| \ll |g| \ll U \ll N|g|$.

$$H_2 = -J_2 \sum_i \sigma_i^x.$$

Such a term can mix the closed-string and open-string states. The low-energy sector of $H_0+H_1+H_2$ is not formed by simple closed-string states, although the mixing with the open-string states may be small for small J_2 (see Fig. 4). It appears that H_2 breaks the Z_2 gauge symmetry since the low-energy states no longer satisfy Eq. (9).

Also when $J_2 \neq 0$ and $U < \infty$, the expectation value of a closed string operators satisfies the perimeter law in both the $|J_1| \ll |g|$ and $J_1 \gg |g|$ limits, which gives no sign of two phases. All of those facts suggest that the fourfold ground-state degeneracy is lifted by a finite J_2.

But this suggestion is incorrect. Reference 4 argues that any local perturbations of H_0 cannot break the Z_2 gauge symmetry. As a result, $H_0+H_1+H_2$ will have four nearly degenerate ground states as long as J_1 and J_2 are not too large. Such a phase contains a nontrivial topological order. However, Ref. 4 only provides a formal argument. A more rigorous understanding is needed.

Certainly, large J_1 and J_2 will polarize the spins and lift the ground-state degeneracy. Such a phase has a trivial topological order. This suggests the phase diagram in Fig. 5. We note that the two phases in Fig. 5 have the same symmetry and are distinguished only by topological orders.

From the phase diagram, we see that when $J_2 \neq 0$ the perimeter and area laws of the closed-string operators (or Wilson loop) cannot determine if the ground state is topologically ordered or not. Thus for a generic bosonic model, the perimeter and area laws of the closed-string operators are not the proper way to test if the ground state has closed-string condensation (or nontrivial topological order).

In Sec. IV, we will show that the topological phase is characterized by dressed closed-string operators which satisfy the zero law. The trivial phase does not contain any such closed-string operator. The closed-string operators in the trivial phase all satisfy the perimeter law. So it is the zero and perimeter laws that distinguish topological and trivial (deconfined and confined) phases, instead of the perimeter and area laws.

For $J_1=J_2=0$, the system has an exact fourfold degeneracy of the ground state on the torus, followed by a gap of order g to the next lowest state. The closed-string operators satisfy the zero law. In Sec. IV we will show that, for small but nonzero J_1 and J_2, a deformed or dressed closed-string operator can still satisfy the zero law. The zero law of the dressed string operator allows us to show the fourfold degeneracy of the ground states in the small-$J_{1,2}$ limit. More precisely, we assume that the gap, from the four lowest states to the rest of the spectrum, remains open, and then we show the fourfold ground-state degeneracy up to an exponentially small splitting. We will also show that, in the small-$J_{1,2}$ limit, the low-energy sector of the model is still formed by Z_2 gauge-invariant states. However, for $J_2 \neq 0$, the gauge invariance is under a deformed Z_2 gauge transformation. So in this sense, none of the small perturbations in the spin model can break the gauge symmetry in the low-energy effective gauge theory.

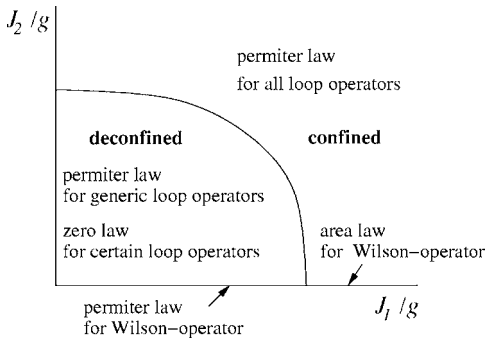

FIG. 5. A likely quantum phase diagram for the spin-1/2 system $H_0+H_1+H_2$. The deconfined phase is characterized by four nearly degenerate ground states on a torus with energy splitting of order $e^{-L/\xi}$ where L is the linear size of the torus and ξ a length scale. The confined phase is characterized by a nondegenerate ground state on the torus. In general, the confined phase and the deconfined phase are distinguished by the zero law and the perimeter law of certain loop operators.

B. A bosonic model with emergent U(1) gauge theory

In our second bosonic model, we consider rotors on the links of a d-dimensional cubic lattice. A rotor can be viewed as a particle moving on a circle. The position of the particle is given by an angle θ, and the angular momentum of the particle by $L^z=-i\partial_\theta$. The Hamiltonian of the rotor model is given by

$$H_{\text{rotor}} = U\sum_{\boldsymbol{I}} Q_{\boldsymbol{I}}^2 - g\sum_{\boldsymbol{p}} (B_{\boldsymbol{p}} + \text{H.c.}) + J_1 \sum_i (L_i^z)^2 + J_2 \sum_i (L_i^+ + L_i^-),$$

$$B_{\boldsymbol{p}} = L_1^+ L_2^- L_3^+ L_4^-, \quad Q_{\boldsymbol{I}} = (-)^{\boldsymbol{I}} \sum_{\text{legs of } \boldsymbol{I}} L_{\boldsymbol{i}}^z, \tag{13}$$

where $\boldsymbol{I}$ labels the vertices, $\boldsymbol{i}$ labels the links, and $\boldsymbol{p}$ labels the squares of the cubic lattice. 1,2,3,4 label the four links that form the edges of the square $\boldsymbol{p}$. $L^+=e^{i\theta}$ is the raising operator of L^z, $L^-=(L^+)^\dagger$, and $(-)^{\boldsymbol{I}}=1$ for the even vertices and -1 for the odd vertices.

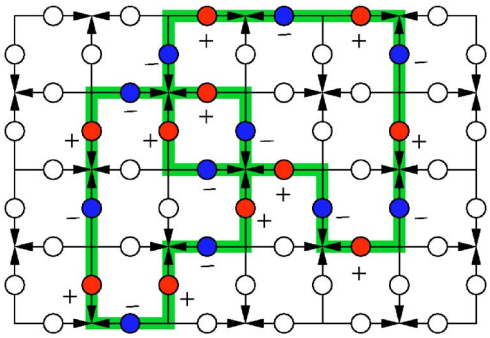

FIG. 6. (Color online) The empty dots represent rotors with $L^z=0$—the no-string states. A closed string is formed by a loop connecting neighboring vertices. A closed-string state is obtained by alternately increasing or decreasing L^z by 1 along the closed string. The filled dots represent rotors with $L^z=\pm 1$. The arrows on the links all point from even vertices to odd vertices.

When $g=J_1=J_2=0$ and $U>0$, the ground states are highly degenerate and form a low-energy subspace. One of the ground states is the state with $L_i^z=0$ for every rotor. Other ground states can be constructed from the first ground state by drawing a loop in the cubic lattice, and then alternately increasing or decreasing L^z by 1 along the loop (see Fig. 6). The sum $\Sigma_{\text{legs of } \boldsymbol{I}} L_i^z$ vanishes on every vertex for such a closed-string state. Such a process can be repeated to construct all of the degenerate ground states. We see that the degenerate ground states are formed by loops, or more precisely string nets, since loops can overlap and form branched strings.

The J_1 term gives the strings in the degenerate ground states a fine energy and represents string tension. The $B_{\boldsymbol{p}}$ operator creates a small loop of closed string or deforms the existing strings. Thus the g term generates string "hopping" or string fluctuations.

The $J_{1,2}$ and g terms lift the degeneracy of the ground states. In the $U \gg J_1 \gg \max(|g|,|J_2|)$ limit, the true ground state correspond to a state with almost no strings (i.e., a state with $L_i^z=0$ on every link). The excitations above such a state have finite energy gaps. In the $U \gg |g| \gg J_{1,2} > 0$ limit, the true ground state is a superposition of many large closed strings.[11,19] Such a state is a string-net condensed state.

When $J_2=0$, the Hamiltonian H_{rotor} does not mix the closed-string and open-string states. When restricted to the closed-string subspace, H_{rotor} is identical to the Hamiltonian of lattice U(1) gauge theory. The closed-string states can be viewed as gauge-invariant states since they are invariant under local U(1) gauge transformations

$$e^{i\phi Q_{\boldsymbol{I}}}|\text{closed string}\rangle = |\text{closed string}\rangle \quad \text{for any } \boldsymbol{I}. \quad (14)$$

A general U(1) gauge transformation is generated by $\exp(i\Sigma_{\boldsymbol{I}}\phi_{\boldsymbol{I}}Q_{\boldsymbol{I}})$.

In the limit $|J_1| \gg |g|$, the lattice U(1) gauge theory is in the strong-coupling limit and is in a confined phase. In the limit $|J_1| \ll |g|$, the lattice U(1) gauge theory is in the weak-coupling limit and has gapless U(1) gauge bosons as its only low-lying excitations if $d>2$. So when $J_2=0$ and when $U \gg |g| \gg |J_1|$, the rotor model H_{rotor} contains emergent gapless U(1) gauge bosons.[11,14–16,19]

When $J_2 \neq 0$, H_{rotor} mixes the closed-string and open-string states. The low-energy states are no longer pure closed-string states and are not invariant under the local U(1) gauge transformation (14). It appears that a nonzero J_2 will break the U(1) gauge symmetry. We may conclude that even a small J_2 will give the U(1) gauge boson a gap and the rotor model H_{rotor} ceases to have emergent U(1) gauge bosons at low energies. In the Sec. IV, we will show that this line of argument is incorrect. For a small J_2 (or any other small perturbation to H_{rotor}), we can define deformed local U(1) gauge transformations so that the low-energy states of H_{rotor} are invariant under the deformed local U(1) gauge transformations. Thus the local U(1) gauge symmetry cannot be broken by any small perturbations if $U \gg |J_{1,2}|, |g|$. Thus far, we will prove these results for U(1) theories; we do not prove, but strongly conjecture, that the stability of the U(1) gauge symmetry protects the gaplessness of the gauge boson. As a result, no translation-invariant perturbation can give the gapless U(1) gauge boson a mass gap. The gaplessness of the emergent U(1) gauge boson is topologically stable.

III. QUASIADIABATIC CONTINUATION

In this section we define the quasiadiabatic continuation. We consider a family of Hamiltonians $\mathcal{H}_s$, depending on a continuous parameter s, where we wish to define a continuation from $s=0$ to 1. We denote eigenstates of $\mathcal{H}_s$ by $\Psi_{a,s}$, where a state $\Psi_{a,0}$ evolves into a state $\Psi_{a,s}$ under an *adiabatic* evolution of $\mathcal{H}_s$. In the event of a level crossing as a function of s, any arbitrary continuation of the states through the level crossing is allowed.

Let us begin with some motivation and discussion. We define the unitary operator $V(s)=\Sigma_a|\Psi_{a,s}\rangle\langle\Psi_{a,0}|$. Then, for any operator O, we could define a corresponding dressed operator by $O_{\text{adiab}}(s)=V(s)OV(s)^\dagger$ so that $O_{\text{adiab}}(s)$ would have exactly the same expectation value in state $\Psi_{a,s}$ as O does in state $\Psi_{a,0}$. Using such a definition of dressed operators, we can show that the dressed string operators and the dressed gauge transformations will have the same properties in the deformed model as the bare string operators and the bare gauge transformations in the exactly soluble model. However, this definition of a dressed operator would not suffice for our purposes at all. In particular, we do not have any reason to believe that the resulting $O_{\text{adiab}}(s)$ would still be a local operator.

Indeed, even if O only involves operators on a few sites, in general, the continued $O_{\text{adiab}}(s)$ will contain operators on every site of the system. However, in this section, we will show that, under certain conditions, we can continue O into an local operator that acts only on a finite number of sites.

To state the result more precisely, let us first assume that $\mathcal{H}_s$ has a gap ΔE separating a low-energy sector and a high-energy sector for all $0<s<1$. For any operator O that acts on a set of sites S_O, we will construct an approximate dressed operator $O(s)$ that only acts on sites within a distance l from the sites S_O. Then, there is a unitary matrix $Q_0(s)$ that acts only within the low-energy sector of $\mathcal{H}_s$, such that

$$\langle \Psi_{\mathrm{low,s}} | Q_0(s)^\dagger O(s) Q_0(s) - O_{\mathrm{adiab}}(s) | \Psi_{\mathrm{low,s}} \rangle < N_{S_{O,l}} e^{-l/\xi} \tag{15}$$

in the large-l limit for a certain fixed length scale ξ, where $N_{S_{O,l}}$ is the number of sites within distance l of a site in S_O. This number grows only as a power of l so is easily overwhelmed by the exponential. The state $|\Psi_{\mathrm{low,s}}\rangle$ is any state in the low-energy sector of $\mathcal{H}_s$.

If $\mathcal{H}_s$ has no gap between the low-energy and high-energy sectors, we will define a density of states $\rho(E)$ in the high-energy sector. If $\rho(E)$ is bounded by $\rho(E) < E^{\alpha-1}$, then the bound in Eq. (15) is weakened to

$$\langle \Psi_{\mathrm{low,s}} | Q_0(s)^\dagger O(s) Q_0(s) - O_{\mathrm{adiab}}(s) | \Psi_{\mathrm{low,s}} \rangle < N_{S_{O,l}} l^{1-\alpha/2}. \tag{16}$$

For a local operator O, $N_{S_{O,l}}$ grows as l^d, where d is the dimension of the lattice, and so if $d+1-\alpha/2<0$ then the error decays for large l. For a stringlike operator $N_{S_{O,l}}$ is proportional to $l^{d-1}|O|$, where $|O|$ is the length of the strength, for $l<|O|$, and is proportional to l^d for $l>|O|$. (A more rigorous statement of the results will be given later.)

The key in obtaining the above result is to adopt a *different* definition of the dressed operator by $O(s)=\tilde{V}(s)O\tilde{V}(s)^\dagger$, where the unitary operator $\tilde{V}(s)$ is defined following Eq. (17). Physically, the definition of V corresponds to adiabatically changing the Hamiltonian from $\mathcal{H}_0$ to $\mathcal{H}_s$, while the definition (17) of $\tilde{V}(s)$ corresponds to a quasiadiabatic change of the Hamiltonian. So $\tilde{V}(s)$ can be viewed as an approximation of $V(s)$. The operator $\tilde{V}(s)$ will be chosen to achieve the goal of defining dressed operators $O(s)$ that have approximately the same ground-state expectation values in the perturbed Hamiltonian $\mathcal{H}_s$ as the original operators did in the unperturbed Hamiltonian, while preserving the locality of the operators.

The locality of the operators is very important. Suppose, for example, that the ground state of $\mathcal{H}_0$ has long-range correlations. That is, there exist two local operators O_1, O_2 which are correlated even though the two operators are far separated from each other in space. For example, if this is a spin system with long-range spin correlations these may be spin operators acting on two different sites which are far separated from each other. Then, however, the operators $O_1(s)$, $O_2(s)$ will also be correlated in the ground state of $\mathcal{H}_s$, and since the operators remain local under the continuation this implies the existence of long-range correlations in the ground state of $\mathcal{H}_s$.

The major result will be an explicit definition of $\tilde{V}(s)$ in terms of derivatives of the Hamiltonian which accomplishes these goals. In order to show that ground-state expectation values remain approximately unchanged under the continuation, we will make some assumptions on the existence of a gap, though some extensions to gapless systems with sufficiently small low-energy density of states will be discussed. The dressed operators remain local under this evolution: local operators are spread out over a length scale of order the correlation length in a gapped system, while stringlike operators are spread out over a length scale which grows only logarithmically with the string length. Detailed proofs will be given in the Appendix.

The importance of the low-energy density of states in what follows can be understood physically by analogy to another continuation that should be much more familiar, namely, Fermi liquid theory. As discussed by Anderson,[25] the correct way to think of Fermi liquid theory is to think of starting with a noninteracting system and turning the interactions on slowly, but not infinitely slowly; that is, physically exactly the same procedure we imagine here. Anderson's discussion of how fast the interactions need to be turned on is based on considerations of the quasiparticle states, and corresponds very closely to our two criteria: maintaining both the expectation values and the locality of the operators. To show that this is possible, the analysis in the Fermi liquid case relies on the low density of particle-hole excitations near the Fermi surface. Here, we rely on something similar, namely, a low density of states at low energy. Our continuation is very general, and thus valid for a much wider range of systems than Fermi systems, but this generality can in some cases limit what we can prove on specific systems.

A. Definition of system

We consider a family of Hamiltonians $\mathcal{H}_s$ which obey the finite-range conditions[26,27] $\mathcal{H}_s=\Sigma_i \mathcal{H}_s^i$, where letters $i,j,\ldots$ label different lattice sites of the system; each $\mathcal{H}_s^i$ acts only on sites j with $d(i,j)\leq R$ where R is the interaction range and $d(i,j)$ is some metric on the lattice; and $\|\mathcal{H}_s^i\|\leq J$ for some constant J for all i,s. We further assume that $\|\partial_s \mathcal{H}_s^i\|\leq K$ for some constant K for all i,s. It is possible to slightly weaken the finite-range conditions and consider exponentially decaying interactions as well.[26]

To define the concept of "density of states," let us focus on a state $\Psi_{\mathrm{low,s}}$ that we continue. We will consider a $\Psi_{\mathrm{low,s}}$ which is a ground state of $\mathcal{H}_s$, but in general we could continue any eigenstate of $\mathcal{H}_s$. The state $\Psi_{a,s}$ has energy $E_a>0$ compared to $\Psi_{\mathrm{low,s}}$ if a is not in the low-energy sector. We assume that, for $0\leq s\leq 1$, the density of states of the local operator $u_s^i=\partial_s \mathcal{H}_s^i$ is bounded as follows. We assume that there is a $D(E)$ such that $\Sigma_{a\in high,|E_a|\leq E}|\langle \Psi_{a,s}|u_s^i|\Psi_{\mathrm{low,s}}\rangle|^2 \leq D(E)\|u_{si}^i\|$, where $\|\cdots\|$ denotes the operator norm. The sum is restricted to states Ψ_a in the high-energy sector. Note that $D(E)\leq 1$ for all E. We define $\partial_E D(E)=\rho(E)$ to be the density of states at energy E produced by operator u_s^i.

There are a number of systems for which this density of states bound is relevant. For a discrete gauge theory, there will be a set of topological excitations below a gap. These topological excitations form the low energy sector, and thus $D(E)$ vanishes below the gap. For a transverse-field Ising system, $\mathcal{H}^i=J\Sigma_{\langle i,j\rangle}\sigma_i^z\sigma_j^z+B\sigma_i^y$, in the ferromagnetic phase, there will again be two almost degenerate states below a finite gap. At $B=0$, these states correspond to symmetric and antisymmetric combinations of all spins up or all spins down.

The bound $D(E)$ implies a locality bound.[27] Suppose some operator O acts only on some set of sites, S_O. We define the distance between a site j and the operator O to be

equal to $d(j,O)=\min_{i\in S_O}[d(i,j)]$, with the minimum taken over sites $i\in S_O$, and we define the distance between two operators O_1,O_2 to be $d(O_1,O_2)=\min_{i\in S_{O_1},j\in S_{O_2}}[d(i,j)]$. Suppose the system has a unique ground state. Then if the system is gapped, so that $D(E)=0$ for $E\leqslant\Delta E$, the connected expectation value $|\langle Ou_s^i\rangle_s-\langle O\rangle_s\langle u_s^i\rangle_s|$ is exponentially decaying in l, where $\langle\cdot\rangle_s$ denotes the ground-state expectation value with Hamiltonian $\mathcal{H}_s$. On the other hand, if $D(E)\propto E^\alpha$, then $|\langle Ou_s^i\rangle_s-\langle O\rangle_s\langle u_s^i\rangle_s|$ is bounded by some constant times $d(i,O)^{-\alpha/2}$.

B. Definition of quasiadiabatic continuation

We introduce the unitary operator

$$\tilde{V}(s)=\mathcal{S}'\exp\left\{-\int_0^s ds'\int_0^\infty d\tau\, e^{-(\tau/t_q)^2/2}[\tilde{u}_{s'}^+(i\tau)-\text{H.c.}]\right\}, \tag{17}$$

where the symbol $\mathcal{S}'$ denotes that the exponential is S'-ordered, in analogy to the usual time-ordered or path-ordered exponentials. We define $u_s=\partial_s\mathcal{H}_s=\Sigma_i u_s^i$, and define $\tilde{u}_s^+(i\tau)$ following Ref. 27: for any operator A

$$\tilde{A}(t)\equiv A(t)\exp[-(t/t_q)^2/2], \tag{18}$$

$$\tilde{A}^\pm(\pm i\tau)=\frac{1}{2\pi}\int dt\,\tilde{A}(t)\frac{1}{\pm it+\tau}. \tag{19}$$

The time t_q will be chosen later. For comparison with previous work,[27,28] t_q is what was previously called $\sqrt{q}/\Delta E$; in this work, we also consider the possibility of gapless theories where there is no scale ΔE. The time evolution of operators is defined by $A(t)=\exp[i\mathcal{H}_{s'}t]A\exp[-i\mathcal{H}_{s'}t]$. The Hermitian conjugate in Eq. (17) of $\tilde{u}_s^+(i\tau)$ is $\tilde{u}_s^-(-i\tau)$, and $\tilde{V}(s)$ is a unitary operator.

At $t_q=\infty$, the operator $\tilde{V}(s)$ becomes equal to $V(s)=\Sigma_a|\Psi_{a,s}\rangle\langle\Psi_{\text{low},s}|$. To see this, note that at $t_q=\infty$, we have $\tilde{A}^+(i\tau)=A^+(i\tau)$, where $A^+(i\tau)$ is the positive-energy part of A taken at imaginary time $i\tau$. That is, in a basis of eigenstates of $\mathcal{H}_s$ with energies E_a,E_b, we have matrix elements $A^+(i\tau)_{ab}=A_{ab}\Theta(E_a-E_b)\exp[-\tau(E_a-E_b)]$, where $\Theta(x)$ is the step function. Then, $-\int_0^\infty d\tau[u^+(i\tau)-u^-(-i\tau)]=-(E_a-E_b)^{-1}\partial_s\mathcal{H}_s$, which gives the result of linear perturbation theory for the adiabatic evolution of quantum states with a change in the Hamiltonian. We instead keep t_q finite to define a "quasiadiabatic" evolution, which will transform local operators into local operators. To show that keeping t_q finite maintains the locality we will rely on finite-group-velocity results, while we will use the gap to show that we get only a small error in the ground-state expectation values by taking a finite t_q; detailed proofs of this are in the Appendix, while the physical discussion is given in the next section. We will relate the time t_q to the scale l by Eq. (21) below.

C. Results

For any operator O, we define $O(s)=\tilde{V}(s)O\tilde{V}(s)^\dagger$, where $O(s)$ has been "smeared out" over a scale l given by Eq. (21) below. For a gapped theory, we only need to take a length l of order the correlation length ξ to get a small error in Eq. (15). To understand how this works, define $Q(s)=\tilde{V}(s)V(s)^\dagger$. Then, for a state $|\Psi_{\text{low},s}\rangle$ in a low-energy subspace, $\langle\Psi_{\text{low},s}|O(s)|\Psi_{\text{low},s}\rangle=\langle\Psi_{\text{low},0}|Q(s)OQ^\dagger(s)|\Psi_{\text{low},0}\rangle$. If we can show that $Q(s)$ has nearly vanishing matrix elements between the low-energy and high-energy states, we can bound the difference in expectation values in Eq. (15). The gap between the states in the low-energy subspace and the high-energy subspace enables us to show this; for a given gap ΔE, if we perform the quasiadiabatic continuation sufficiently slowly, we can show that (loosely speaking) the matrix elements of $Q(s)$ between the low-energy and high-energy states almost vanishes and $Q(s)$ almost acts within the low-energy subspace as a unitary operator.

However, there is one complication: in a macroscopic system, $Q(s)$ produces excitations throughout the sample. Thus, for infinitesimal s, $Q(s)=1+s\Sigma_i e^i+\cdots$, where the anti-Hermitian operator e^i creates local excitations near site i. The probability that $Q(s)$ produces some excitation acting on the ground state diverges with the system size. Strictly speaking, the matrix elements of $Q(s)$ between the low-energy and high-energy states are not small. However, the terms e^i in Q with i sufficiently far from O can be commuted through O and do not affect the final expectation value. We will show that the error from the terms e^i with i near O is exponentially small in l divided by the correlation length. The proof in the Appendix follows this argument, using a triangle inequality to make the bound precise.

If $D(E)$ is not gapped, but has a power-law behavior, $D(E)\propto E^\alpha$ for small α, then in a d-dimensional system we can still find a local $O(s)$ if $\alpha/2>d+1$ according to Eq. (16). This requirement on the exponent can be physically understood as follows: if we keep t_q finite so that $O(s)$ is spread out over a length scale l under the quasiadiabatic evolution, then we have to worry about any error in $Q(s)$ on the length scale l. The correlation function of u_s with O decays as $l^{-\alpha/2}$, and this also is how the error terms decay, while the space-time volume at scale l is of order l^{d+1}. Thus, this requirement can be understood in terms of the relevance or irrelevance of u_s at large scales in the given correlation function.

This physical description is based on two technical results. First, we claim that $O(s)$ is *local* up to a scale l. Specifically, we claim that for any operator O_j which acts only on site j,

$$\|[O_j,O(s)]\|\leqslant X_1\|O_j\|\|O\|\times\max(\exp[-d(j,O)/\xi_C],\exp\{-[d(j,O)/l_q]^2/2\}), \tag{20}$$

for some constant X_1 of order unity and constant l_q of order t_q/c_1, where c_1 is a characteristic inverse velocity of the system. The length ξ_C is a microscopic length of order the interaction range of the Hamiltonian and is defined later. Equation (20) implies that, while $O(s)$ does involve sites more than a distance l_q from S_O, the commutator becomes exponentially small once $d(j,O)$ becomes larger than l_q. To relate l_q and l, we ask for the commutator to be smaller than the error in Eq. (16) or Eq. (15) once $d(j,O)$ becomes larger than

l. For a gapless theory, we only need to take $d(j,O)$ logarithmically larger than l_q before the exponential decay becomes much smaller than $l^{d+1-\alpha/2}$; in this case, l is of order l_q. For a gapped theory, we need that $\exp[-(l/l_q)^2] \sim \exp[-l/\xi]$, so that

$$l \sim l_q^2/\xi \sim t_q^2 \Delta E/c_1. \tag{21}$$

The correlation length ξ is at most of order $1/(c_1\Delta E)$.

If one prefers to have an operator which involves only sites within a distance l of S_O, and *exactly* commutes with sites more than a distance l from S_O, one may define an operator $O_{trunc}(s)$ such that

$$\|O_{trunc}(s) - O(s)\| \leq X_2 \sum_{j,d(i,j)\geq l} \max(\exp[-d(j,O)/\xi_C], \exp\{-[d(j,O)/l_q]^2/2\}), \tag{22}$$

for some constant X_2 of order unity. In order for the error in Eq. (22) to be of order the error in Eq. (16) or Eq. (15), we need to pick l as above: $l \sim l_q^2/\xi$ in the gapped case, up to logarithmic corrections.

Second, we claim that $O(s)$ has almost the same expectation value as $O_{\text{adiab}}(s)$, up to the unitary matrix Q_0. Specifically, we show that for $0 \leq s \leq 1$

$$|\langle \Psi_{\text{low,s}}|Q_0(s)^\dagger O(s) Q_0(s) - O_{\text{adiab}}|\Psi_{\text{low,s}}\rangle| \leq 2\|O\|[c_2(s) + c_3(s)], \tag{23}$$

where $c_2(s)$ and $c_3(s)$ are given by Eqs. (A9) and (A2) in the Appendix. One may verify from the calculation in the Appendix that the error term $c_2(s)+c_3(s)$ gives the error described above in Eqs. (15) and (16). Note that if $O = O_1 O_2 \cdots O_n$, then $O(s) = O_1(s) O_2(s) \cdots O_n(s)$, so the expectation values of products of operators are also approximately preserved under this quasiadiabatic evolution. Thus, $|\langle \Psi_{\text{low,s}}|Q_0(s)^\dagger O_1(s) O_2(s) \cdots Q_0(s) - O_{1,\text{adiab}} O_{2,\text{adiab}} \cdots |\Psi_{\text{low,s}}\rangle| \leq 2\|O_1\|\|O_2\| \cdots [c_2(s)+c_3(s)]$, where here the $c_2(s), c_3(s)$ are the error terms appropriate for the product operator $O_1 O_2 \cdots$.

IV. EXAMPLES

We illustrate the quasiadiabatic continuation by a series of examples. We start with local operators, considering a system with Z_2 (Ising) symmetry, a fractional quantum Hall system, and then a system with U(1) symmetry. We then repeat the process in the case of nonlocal, string operators.

A. Local Ising model

The first and simplest example is a quantum Ising ferromagnet in a transverse field. Let the Hamiltonian be $\mathcal{H} = J\Sigma_{\langle i,j\rangle} \sigma_i^z \sigma_j^z + B\Sigma_i \sigma_i^y$ where each site has a spin 1/2 and the ferromagnetic interaction J couples nearest-neighbor spins on the lattice.

For $B=0$, the system has two exact ground states, one state which we denote $\Psi_\uparrow$ with all spins up and one state $\Psi_\downarrow$ with all spins down. The gap to the lowest excited state above these two states is $2Jq$ where q is the coordination number of the lattice; this state has one flipped spin. We define symmetric and antisymmetric combinations $\Psi_{S,A} = (1/\sqrt{2})(\Psi_\uparrow \pm \Psi_\downarrow)$. The states $\Psi_{S,A}$ are eigenvectors of the operator $\Pi_i \sigma_i^y$, with eigenvalues ± 1. This operator, which flips the spin on every site, commutes with the Hamiltonian for all B.

We now consider the quasiadiabatic continuation with $\mathcal{H}_s = J\Sigma_{\langle i,j\rangle} \sigma_i^z \sigma_j^z + sB\Sigma_i \sigma_i^y$. At $s=0$, this has the exact ground states $\Psi_{S,A}$. For sB small enough, the two lowest eigenstates of $\mathcal{H}_s$ are adiabatic continuations of $\Psi_{S,A}$. Hence, the matrix element of the operator $\partial_s \mathcal{H}_s = B\Sigma_i \sigma_i^y$ between these two states vanishes for all B, since $\partial_s \mathcal{H}_s$ commutes with $\Pi_i \sigma_i^y$ and these two states are eigenstates of $\Pi_i \sigma_i^y$ with different eigenvalues.

Above these two states, there is a gap to the rest of the spectrum. It is known that for small enough B, the gap will remain open for all s with $0 \leq s \leq 1$. Thus, we can perform the continuation. The vanishing of the matrix elements of $\partial_s \mathcal{H}_s$ between the two low-energy states implies that $Q_0(s)$ is equal to the identity matrix. We consider the continuation of the operator σ_i^z. The ground state for $B=0$ may be taken to be Ψ_S. For $B=0$, we have $\langle \Psi_S|\sigma_{i_1}^z \sigma_{i_2}^z \cdots \sigma_{i_n}^z|\Psi_S\rangle = 1$ for n even and -1 for n odd. The quasiadiabatic continuation $\sigma_i^z \to \sigma_i^z(s)$ spreads out the σ_i^z operators over a distance l. Certainly l is less than the linear size of the system. Then $\langle \Psi_{S,s}|\sigma_{i_1}^z(s) \sigma_{i_2}^z(s) \cdots \sigma_{i_n}^z(s)|\Psi_{S,s}\rangle$, again equal to 1 or -1, up to some error, depending on whether n is even or odd. Here, $\Psi_{S,s} = V(s)\Psi_S$. The error is exponentially small in l/ξ. We conjecture that the operators $\sigma_{i_1}^z(s)$ correspond to *block spin* operators: they are equal to plus or minus 1 depending on whether the average spin over a correlation volume is positive or negative.

If we instead consider the Hamiltonian $\mathcal{H}_s = J\Sigma_{\langle i,j\rangle} \sigma_i^z \sigma_j^z + sB\Sigma_i \sigma_i^z$, the operator $\partial_s \mathcal{H}_s$ will have nonvanishing matrix elements between the states $\Psi_{S,A}$, and thus we must start with states $\Psi_\uparrow, \Psi_\downarrow$ to perform the continuation if we want to have $Q_0(s)$ be equal to the identity. In this case, $\sigma_i^z(s) = \sigma_i^z$ and $\langle \Psi_{\uparrow,s}|\sigma_{i_1}^z(s) \sigma_{i_2}^z(s) \cdots \sigma_{i_n}^z(s)|\Psi_{\uparrow,s}\rangle = (+1)^n$, where we have assumed that the sign of B is such that $\Psi_{\uparrow,s}$ is the ground state, rather than $\Psi_{\downarrow,s}$. For the opposite sign of B, the correlation function is instead $(-1)^n$.

The two cases, depending on the different ways to add the magnetic field transverse or parallel to the z axis, lead to different ground-state correlation functions. In the second case, the correlation function of the continued operator is not a very interesting result; the unit operator would have the same correlation function. However, in the first case, the ability to find a continuation of the operator with the given correlation functions is a much more interesting result. There are long-range correlations in the operator, since $\langle \Psi_{S,s}|\sigma_{i_1}^z(s) \sigma_{i_2}^z(s)|\Psi_{S,s}\rangle - \langle \Psi_{S,s}|\sigma_{i_1}^z(s)|\Psi_{S,s}\rangle \langle \Psi_{S,s}|\sigma_{i_2}^z(s)|\Psi_{S,s}\rangle$ is nonvanishing even when sites i_1, i_2 are far from each other, and in particular even when the distance between i_1, i_2 is much larger than l. This implies[27] the presence of another state close in energy to the ground state when the magnetic field is added in the transverse direction.

However, we would like to show the ground-state degeneracy in another way, based directly on continuing the states $\Psi_{S,A}$. This way will be very valuable for more complicated

systems such as the quantum Hall system. We will consider a more general class of Hamiltonians $\mathcal{H}_s$: we consider arbitrary Hamiltonians which are sums of local terms, and which commute with $\Pi_i \sigma_i^y$. We show that, under the assumption that the gap between the two lowest states and the rest of the spectrum remains open, the energy difference between the continuation of the two lowest states is exponentially small. As long as that gap remains open, it is possible to adiabatically continue these two states, giving states $\Psi_{S,s}=V(s)\Psi_S$ and $\Psi_{A,s}=V(s)\Psi_A$ as eigenstates of $\mathcal{H}_s$. Here, we rely on the fact that $\mathcal{H}_s$ commutes with $\Pi_i \sigma_i^y$ and thus has vanishing matrix elements between the states.

To compute the difference in energies for given s, we compute $\langle\Psi_{A,s}|\mathcal{H}_s|\Psi_{A,s}\rangle - \langle\Psi_{S,s}|\mathcal{H}_s|\Psi_{S,s}\rangle = \langle\Psi_A|V(s)^\dagger \mathcal{H}_s V(s)|\Psi_A\rangle - \langle\Psi_S|V(s)^\dagger \mathcal{H}_s V(s)|\Psi_S\rangle$. Thus, $V(s)^\dagger \mathcal{H}_s V(s)$ defines the continuation of the operator $\mathcal{H}_s$ from s back to 0 and $\tilde{V}(s)^\dagger \mathcal{H}_s \tilde{V}(s)$ defines a quasiadiabatic continuation from s back to 0: $\langle\Psi_{S,s}|\mathcal{H}_s|\Psi_{S,s}\rangle$ is approximately equal to $\langle\Psi_S|\tilde{V}(s)^\dagger \mathcal{H}_s \tilde{V}(s)|\Psi_S\rangle$. The error in this continuation is exponentially small in l/ξ. The operator $\mathcal{H}_s$ is a sum of local operators, while the operator $\tilde{V}(s)^\dagger \mathcal{H}_s \tilde{V}(s)$ is a sum of terms spread out over length scale l. The only operators O such that $\langle\Psi_A|O|\Psi_A\rangle - \langle\Psi_S|O|\Psi_S\rangle \neq 0$ are operators which flip every spin in the system and thus have nonvanishing matrix elements between $\Psi_\uparrow$ and $\Psi_\downarrow$. Thus all *local* operators have vanishing matrix elements between the two states $\Psi_\uparrow$ and $\Psi_\downarrow$. In particular, if the length scale l is smaller than the system size L, then $\langle\Psi_A|\tilde{V}(s)^\dagger \mathcal{H}_s \tilde{V}(s)|\Psi_A\rangle = \langle\Psi_S|\tilde{V}(s)^\dagger \mathcal{H}_s \tilde{V}(s)|\Psi_S\rangle$. Thus we can pick l just smaller then the system size to show that $\langle\Psi_A|V(s)^\dagger \mathcal{H}_s V(s)|\Psi_A\rangle - \langle\Psi_S|V(s)^\dagger \mathcal{H}_s V(s)|\Psi_S\rangle$ is of order $\|\mathcal{H}_s\|\exp(-L/\xi) \sim L^d \exp(-L/\xi)$. Here, the bound for this system is not a very tight bound: one expects the level splitting to be exponentially small in $(L/\xi)^d$ instead.

The key steps in this argument were that (1) matrix elements of operators which commute with $\Pi_i \sigma_i^y$ vanish between Ψ_S and Ψ_A; and (2) all local operators have the same expectation values in state Ψ_A as in state Ψ_S.

B. Quantum Hall effect

We now turn to the case of the fractional quantum Hall effect. Consider a system with no disorder on a torus at filling factor p/q, with p and q coprime. The magnetic translation group implies at least a q-fold degeneracy of the ground state.[18] Assume that in the absence of disorder there is a q-fold degenerate ground state, with a gap to the rest of the spectrum. Now, consider adding disorder to the system, defining $\mathcal{H}_s = \mathcal{H}_0 + s\int dx\, dy\, U(x,y)\Psi^\dagger(x,y)\Psi(x,y)$, where $U(x,y)$ is a disorder potential and $\mathcal{H}_0$ is the clean Hamiltonian. Wen and Niu argued[18] that to first order in $sU(x,y)$ the splitting between the q-fold degenerate states was exponentially small. We will use the continuation to show that the splitting is exponentially small for nonvanishing disorder strength under the sole assumptions that the gap to the rest of the spectrum remains open and that at $s=0$ all local operators have the same expectation value up to exponentially small terms in the q lowest states and that at $s=0$ all local operators have exponentially small matrix elements between the q lowest states. We note that since these last statements involve only $s=0$ they can be checked in specific model systems without disorder, and in fact were checked in Ref. 18 when they showed the exponentially small splitting of the q lowest states in linear perturbation theory.

Suppose, then that for $0 \leqslant s \leqslant 1$ there are q states, $\Psi_{n,s}$ for $n=0,\ldots,q-1$, and a gap to the rest of the spectrum. We wish to show that these states are close in energy. Thus, we compute $\langle\Psi_{m,s}|\mathcal{H}_s|\Psi_{m,s}\rangle - \langle\Psi_{n,s}|\mathcal{H}_s|\Psi_{n,s}\rangle$, for some $m,n=0,\ldots,q-1$. As above, $V(s)^\dagger \mathcal{H}_s V(s)$ defines the continuation of the operator $\mathcal{H}_s$ from s back to 0, and $\tilde{V}(s)^\dagger \mathcal{H}_s \tilde{V}(s)$ defines a quasiadiabatic continuation from s back to 0.

However, unlike the case of the quantum Ising model, we do not have any symmetries to make matrix elements of $\mathcal{H}_s$ and $\partial_s \mathcal{H}_s$ vanish between the low-lying states. Thus, we do not have any control on the matrix $Q_0(s)$. Then, $\langle\Psi_{m,s}|\mathcal{H}_s|\Psi_{m,s}\rangle$ is equal to $\langle\Psi_{m,0}|Q_0(s)^\dagger \tilde{V}(s)^\dagger \mathcal{H}_s \tilde{V}(s) Q_0(s)|\Psi_{m,0}\rangle$, up to an error of order $\|\mathcal{H}_s\|\exp(-l/\xi)$. Since we are continuing from nonzero s to $s=0$, now the matrix Q_0 acts within the low-energy sector of $\mathcal{H}_0$.

However, this matrix $Q_0(s)$ causes no problem. As before, for $l<L$, $\tilde{V}(s)^\dagger \mathcal{H}_s \tilde{V}(s)$ is a local operator, and then under the assumptions above, the expectation value $\langle\Psi_{m,0}|Q_0(s)^\dagger \tilde{V}(s)^\dagger \mathcal{H}_s \tilde{V}(s) Q_0(s)|\Psi_{m,0}\rangle$ is independent of $Q_0(s)$, up to exponentially small corrections, thus giving the desired result.

C. Local rotor model

The next example is a system with a continuous symmetry. We take the Hamiltonian $\mathcal{H}_s = k(s)^{-1}\Sigma_i \Pi_i^2 + k(s)\Sigma_{\langle i,j\rangle} \bar{z}_i z_j$, where z_i is a continuous complex field with $|z_i|=1$, and Π_i is a momentum with $[z_i, \Pi_i] = i z_i$. The parameter $k(s)$ is an s-dependent stiffness of the field z.

We pick a quasiadiabatic continuation using $k(s) = k_0(k_1/k_0)^s$, so $k(0)=k_0$ and $k(1)=k_1$; thus $\partial_s \mathcal{H}_s = \ln(k_1/k_0) \times [-k(s)^{-1}\Sigma_i \Pi_i^2 + k(s)\Sigma_{\langle i,j\rangle} \bar{z}_i z_j]$. We choose the initial k_0 to be large compared to the final k_1. We assume that we k_1 is sufficiently large that the system is still in a phase with gapless excitations and algebraic correlations. In this phase, we can compute the density of states by writing $z=\exp(i\phi)$ for some ϕ with a Gaussian action for ϕ. At low energy, the system acquires relativistic invariance, and thus in d dimensions the density of single particle states at energy E is of order E^{d-1}. The matrix element of ϕ between the ground state and such a single-particle state is of order $E^{-1/2}$, and thus the integrated density of states created by ϕ is $D(E) \propto E^{d-1}$. The integrated density of states below energy E created by $\partial_s \mathcal{H}_s$ instead is $KD(E) \sim \ln(k_0/k_1) E^{2d+2}$. Then, $\alpha/2 = d+1$ and we are in a marginal case: the error in the continuation is of order $\log(L/l)$.

D. Emergent discrete gauge theories

We now turn from these theories with local operators to emergent gauge theories. We consider the Hamiltonian of the

emergent Z_2 gauge theory in the introduction.

For $J_1=J_2=0$, we have 4 exactly degenerate ground states on a torus, and a gap to the rest of the spectrum. For nonzero J_1, J_2, the deformed Hamiltonian is still local $\mathcal{H}_s=\Sigma_{\mathbf{i}}\mathcal{H}_s^{\mathbf{i}}$, where $\mathcal{H}_s^{\mathbf{i}}$ is a local operator defined near the site $\mathbf{i}$. We can use the same reasoning used in fractional quantum Hall states to show that $\mathcal{H}_s$ still has four exactly degenerate ground states on a torus. However, here we will use a slightly different approach. Continuing $\mathcal{H}_s$ from s back to 0, we find that

$$\langle\Psi_{m,s}|\mathcal{H}_s^{\boldsymbol{i}}|\Psi_{n,s}\rangle=\langle\Psi_{m,0}|V^\dagger(s)\mathcal{H}_s^{\boldsymbol{i}}V(s)|\Psi_{n,0}\rangle,$$

where $|\Psi_{n,s}\rangle$, $n=1,\ldots,4$, are the four low-lying states of $\mathcal{H}_s$. Due to the energy gap separating the four low-lying states from the rest of the states, the operator $V^\dagger(s)\mathcal{H}_s^{\mathbf{i}}V(s)$ is almost a local operator. More precisely, up to an error of order $e^{-l/\xi}$ and a unitary rotation $Q_0(s)$ between the four low-lying states $|\Psi_{n,0}\rangle$, $V^\dagger(s)\mathcal{H}_s^{\mathbf{i}}V(s)$ can be replaced by a truncated operator $H^{\boldsymbol{i}}$ which only acts on sites within a distance l from the site $\boldsymbol{i}$. Thus we have

$$\langle\Psi_{m,s}|\mathcal{H}_s^{\boldsymbol{i}}|\Psi_{n,s}\rangle=\langle\Psi_{m,0}|Q_0(s)^\dagger H^{\boldsymbol{i}}Q_0(s)|\Psi_{n,0}\rangle+O(e^{-l/\xi}).$$

Since $Q_0(s)|\Psi_{n,0}\rangle$ are the ground states of the exactly soluble model (1), they from a irreducible representation of the algebra of the large-closed-string operators (8). We can choose the length l over which the operators are smeared to be one-quarter of the linear size L of the system. In this case $H^{\boldsymbol{i}}$ will be local enough that we can choose the positions large-closed-string operators to avoid any overlap between $H^{\boldsymbol{i}}$ and the closed-string operators. So $H^{\boldsymbol{i}}$ commutes with the closed-string operators. As a result, $H^{\boldsymbol{i}}$ must be proportional to the identity operator within the irreducible representation. This way, we have shown that $\langle\Psi_{m,s}|\mathcal{H}_s^{\mathbf{i}}|\Psi_{n,s}\rangle\propto\delta_{mn}$ up to an error $e^{-L/4\xi}$, which implies that $\langle\Psi_{m,s}|\mathcal{H}_s|\Psi_{n,s}\rangle\propto\delta_{mn}$ up to an error $L^2e^{-L/4\xi}$. So the energy splitting between the four low-lying states is less than $L^2e^{-L/4\xi}$ for the deformed Hamiltonian $\mathcal{H}_s$.

We can also continue any closed-string or dual closed-string operators $S(C_{\text{closed}})$ and obtain

$$\langle\Psi_{m,0}|S(C_{\text{closed}})|\Psi_{n,0}\rangle=\langle\Psi_{m,s}|V(s)S(C_{\text{closed}})V^\dagger(s)|\Psi_{n,s}\rangle.$$

Then there exist dressed closed-string operators $S^{\text{dre}}(C_{\text{closed}})$ that have a width l such that

$$\begin{aligned}&\langle\Psi_{m,s}|Q_0(s)^\dagger S^{\text{dre}}(C_{\text{closed}})Q_0(s)|\Psi_{n,s}\rangle\\&\quad=\langle\Psi_{m,s}|V(s)S(C_{\text{closed}})V^\dagger(s)|\Psi_{n,s}\rangle+O(e^{-l/\xi}).\end{aligned}$$

Here $Q_0(s)$ is a unitary rotation between the four low-lying states $|\Psi_{n,s}\rangle$ and $Q_0(s)$ is independent of the closed-string operators. This implies that the dressed string and dual string operators $S^{\text{dre}}(C_{\text{closed}})$ have the same algebra among the low-energy states of the perturbed Hamiltonian as the original operators $S(C_{\text{closed}})$ do for the original Hamiltonian [up to an error $\exp(-l/\xi)$].

The above also implies that the expectation of the dressed closed-string operators in the ground state of the perturbed Hamiltonian satisfies the zero law (3), up to an error of order $|S|\exp(-l/\xi)$, where $|S|$ is the string length. We see that the error is exponentially small for long strings since l can be chosen to be a fraction of $|S|$.

Our final task is to show the invariance of certain states under a deformed gauge transformation for $J_2\neq 0$. First consider $J_1, J_2=0$. The Hamiltonian has the four ground states which are invariant under the local gauge transformation $W_{\boldsymbol{I}}$ given by Eq. (2). We can create gauge-invariant excited states by acting on the four ground states by open dual string operators. Following Eq. (5), we define an open dual string operator as $\tilde{S}(\tilde{C})=\Pi_{\boldsymbol{i}\ \text{cross}\ \tilde{C}}\sigma_{\boldsymbol{i}}^z$, where now the dual string $\tilde{C}$ is open, with two end points. The states created by acting with $\tilde{S}(\tilde{C})$ are still invariant under the local Z_2 gauge transformations, since $[\tilde{S}(\tilde{C}),W_{\boldsymbol{I}}]=0$. They are excited states which introduce Z_2 gauge flux at the end points of the open string. By acting on the ground state with the open dual strings, we can create all gauge-invariant states, and for $U\gg g$, these are the lowest-energy excited states. Continuing to nonzero J_1 at $J_2=0$ still leaves these states invariant under the local Z_2 gauge transformation. Continuing to nonzero J_2 breaks the local Z_2 gauge invariance. However, these states are invariant under the deformed local Z_2 gauge transformation $W_{\boldsymbol{I}}(s)$. To see this, we use the continuation to show that $\langle\Psi_{\text{low,s}}|W_{\boldsymbol{I}}(s)|\Psi_{\text{low,s}}\rangle=1$, up to exponentially small error in l/ξ. Since $W_{\boldsymbol{I}}(s)$ is unitary, this implies the gauge invariance of the ground state under the deformed gauge transformation up to exponentially small error. The gauge invariance, up to the same small error, of the states created by acting on the ground state with operators $\tilde{S}(\tilde{C},s)=\tilde{V}(s)\tilde{S}(\tilde{C},s)\tilde{V}(s)^\dagger$ then follows from the exact commutator $[\tilde{S}(\tilde{C},s),W_{\boldsymbol{I}}(s)]=0$. The size of the deformed gauge group is, however, much smaller than the original: if the linear system size is L, there are $(L/l)^d$ different local gauge transformations rather than L^d as is the case at $J_2=0$.

E. Emergent continuous gauge theories

Our final problem is the theory with the U(1) gauge symmetry, Eq. (13). The absence of a gap makes it much more difficult to obtain results on this system. We will obtain only one result, the existence of a deformed U(1) gauge invariance of the system.

We consider some gauge transformation acting on a site, $W_{\boldsymbol{I},\phi}=e^{i\phi Q_{\boldsymbol{I}}}$. When $J_2=0$, the ground state has an exact gauge invariance: $\langle\Psi_0|W_{\boldsymbol{I},\phi}|\Psi_0\rangle=1$. If we continue to nonzero J_2, the ground state breaks the gauge invariance. However, we claim that the expectation value of the continued operator $\langle\Psi_{0,s}|W_{\boldsymbol{I},\phi}(s)|\Psi_{0,s}\rangle$ is still close to unity. Unlike the Z_2 case, this does *not* follow simply from the results derived previously as there is no gap, and the low-energy density of states is too high to use the results for gapless systems. However, it is still possible to show this result. We only very briefly sketch the argument, leaving a more detailed presentation for future work. The idea is as follows: at $J_2=0$, the system has gauge-noninvariant states at an energy of order U above the ground state. Under the continuation, at finite t_q, the operator $\tilde{V}(s)$ will take the ground state at $s=0$ into some state which

is a mix of ground and excited states. That is, the expectation value $\langle\Psi_{0,s}|W_{\boldsymbol{I},\phi}(s)|\Psi_{0,s}\rangle=\langle\Psi_0|Q(s)^\dagger W_{\boldsymbol{I},\phi}Q(s)|\Psi_0\rangle$, where the unitary matrix $Q(s)=\tilde{V}(s)^\dagger V(s)$ mixes the ground state with the excited states. However, since all the low-lying excited states at $J_2=0$ are gauge invariant, it is no longer necessary to show that $Q(s)^\dagger W_{\boldsymbol{I},\phi}Q(s)$ is close to $W_{\boldsymbol{I},\phi}$; it suffices to show that we do not mix in the states which are not gauge invariant under $W_{\boldsymbol{I},\phi}$ and all such states are at energy of order U above the ground state. At infinitesimal s, the mixing into such states is exponentially small, since the energy U acts like a gap; however, one has some mixing into the low-lying gauge-invariant states. As s increases, one mixes into progressively higher-energy states, until eventually one begins to excite the gauge-noninvariant states. However, if t_q is sufficiently big (roughly of order U^{-1}) the mixing into the gauge-noninvariant states can be bounded at $s=1$, thus obtaining the desired result.

Now consider the excited states. At $J_2=0$, the gauge-invariant excited states of the system can be obtained by acting on the ground state with operators of the form $O=\Pi_i e^{i\phi_i L_i^z}$, where the phase ϕ_i is some arbitrary function of the leg. These operators commute with $W_{\boldsymbol{I},\phi}$ and therefore at nonzero J_2 the continued operators commute with the continued gauge transformation: $[O(s),W_{\boldsymbol{I},\phi}(s)]=0$. Therefore, there are a class of excited states for nonzero J_2, namely, the excited states created by acting on the ground state with the operators $O(s)$, which are also gauge invariant under the deformed gauge transformation, up to the same error.

As in the Z_2 case, the size of the deformed gauge group is much smaller than the original gauge group. The original gauge group had L^d different generators, while the deformed group has only $(L/l)^d$ such generators. A more careful analysis should be able to then use this deformed gauge invariance to show that the gapless photon is protected. This is also a job for the future.

The compactness of the gauge group was important here. Consider a noncompact U(1) theory, with Lagrangian $(1/2)(\partial_\mu A_\nu-\partial_\nu A_\mu)^2+(\lambda/2)(\partial_\mu A^\mu)^2+MA_\mu A^\mu$. The term in λ is a gauge-fixing term and M is a term that breaks the gauge symmetry. With this quadratic action, it is easy to verify that a nonzero M opens a gap. Why does the continuation not work here? The reason is that this theory has no gap to the gauge-noninvariant states, unlike the compact cases before, despite the gauge-fixing term. The gauge-fixing term does not open a gap to the gauge-noninvariant states in this case; instead it adds *gapless* longitudinal and scalar photons to the theory. To say it differently, in a compact theory, the charge is quantized and the open-string states have a minimum possible energy because they necessarily terminate in an end with a charge that is a multiple of the charge quantum. In a noncompact theory, the charge may be arbitrarily small. So the energy cost is also arbitrarily small.

V. DISCUSSION

The standard wisdom is that as long as gaps remain open, a system does not have a quantum phase transition and thus the long-distance structure of correlation functions remains the same. We have shown a precise form of this statement. We have found that, by appropriately dressing operators, the long-distance structure can in fact be preserved to a much greater degree than one might have expected. In particular, we have shown the presence of the zero law (3) for gauge theories in the deconfined phase, and we propose this as a test of confinement. Further, we have considered the stability of topological order under perturbations of the Hamiltonian, and shown that the order is robust unless the gap to the rest of the spectrum (local excitations) closes.

Topologically ordered states are described by emergent gauge theories at low-energies. The topological order is closely tied to the emergent gauge invariance of the low-energy gauge theories. From the point of view of the low-energy gauge theory, our result shows that the emergent low-energy gauge invariance is topological. It cannot be broken by any local perturbations in the parent bosonic model. We hope this result will shed light on the true meaning of gauge invariance and gauge theory.

The continuation is also useful for systems without emergent local gauge structure, and may have a wider applicability. For example, an outstanding question is to prove that in some neighborhood of the Affleck-Kennedy-Lieb-Tasala AKLT point,[29,30] a spin-1 chain remains gapped.[31] The continuation might be useful in doing this, or at least in showing, under the assumption of the existence of a gap, the persistence of string order[32] throughout the Haldane phase.

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APPENDIX: PROOF OF LOCALITY AND APPROXIMATION RESULTS

1. Locality result

To show Eq. (20), we use the finite-group-velocity result, proven in Refs. 26 and 28. This result uses the finite-range conditions on the Hamiltonian above to bound the commutator $\|[A(t),B(0)]\|$, where $A(t)=\exp(i\mathcal{H}_s t)A\exp(-i\mathcal{H}_s t)$. One can show that this commutator is exponentially small for times t less than $c_1 l$ where l is the distance between A and B and c_1 is some characteristic inverse velocity which depends on J, R, and the lattice structure. The specific bound is that $\|[A(t),B(0)\|\leqslant\|A\|\|B\|\Sigma_j g(t,d(A,j))$, where the sum ranges over sites j which appear in operator B and where the function g has the property that for $|t|\leqslant c_1 l$, $g(c_1 l,l)$ is exponentially decaying in l for large l with decay length ξ_C for some constant ξ_C which is of order R. Recall that $d(A,j)$ is the minimum over sites i acted on by A of the distance $d(i,j)$.

Before giving the proof of Eq. (20), we give a physical description. The finite group velocity result has a very simple interpretation. Consider a local operator A. Under time evolution, we get an operator $A(t)$ which spreads out over space as time passes. The finite-group-velocity result implies that for finite t the operator A is still local up to some length t/c_1

(here c_1^{-1} is some characteristic velocity of the system) in the following sense: the commutator of $A(t)$ with any operator B is exponentially small if B is at least distance t/c_1 from $A(0)$. This applies in particular for $A=u_s$. Thus, in the definition (17) of U, the operators $u_{s'}^+(i\tau)$ are local in the same sense: Eq. (19) gives $\tilde{u}_{s'}^+(i\tau)$ as an integral over t of $u_{s'}(t)$ and for $t \gg t_q$ the integral is cut off exponentially, while for $t \sim t_q$ the $u_{s'}(t)$ are local up to length of order t_q/c_1. Then, we define $O(s)$ by the unitary transformation $\tilde{V}(s)$. We view this unitary transformation as defining a fictitious time evolution with time parameter s and Hamiltonian given by the exponent of Eq. (17). We have just established that this exponent is local and we can then apply the finite-group-velocity result to this evolution to show that $O(s)$ is also local. The rest of this subsection consists of a few precise error bounds following these statements.

We compute the commutator $[\tilde{u}_s^{i+}(i\tau), O_j]$ where O_j is some operator which acts only on site j, and where $\tilde{u}_s^{i+}(i\tau)$ is defined following Eq. (19) taking $A=u_s^i$. We separate the integral over times t into times with $t<c_1 l$, where $l=d(i,j)-R$, and times with $|t|>c_1 l$. For the first set of times, we have $(2\pi)^{-1}\int_{|t|\leq c_1 l} dt(it+\tau)\|[u_s(t), O_j]\| \leq (2\pi)^{-1}\|u_s^i\|\|O\|\exp(-l/\xi_C)$. For the second set of times, we have $(2\pi)^{-1}\int dt(it+\tau)\|[u_s(t), O_j]\| \leq \pi^{-1}\|u_s^i\|\|O\|(\sqrt{2\pi}t_q/c_1 l)\exp[-(c_1 l/t_q)^2/2]$. Thus, for large l, we find $\|[\tilde{u}_s^{i+}(i\tau), O_j]\|$ is exponentially decaying in $d(i,j)$ with decay length ξ_C. Note that $\exp(-l/\xi_C) \approx \exp[-(c_1 l/t_q)^2/2]$ for $l \approx 2(t_q/c_1)^2/\xi_C$.

Now, here is the trick. We regard Eq. (17) as defining the "time" evolution of states, where the parameter s' is an effective time parameter and $D=i\int_0^\infty d\tau \exp[-(\tau/t_q)^2/2] \times[\tilde{d}_{s'}^+(i\tau)-\text{H.c.}]$ is some effective s'-dependent "Hamiltonian," so that $\tilde{V}(s)=\mathcal{S}' \exp[-i\int_0^s ds' D]$. We write $D=\Sigma_i D^i$, where $D^i=\int_0^\infty d\tau \exp[-(\tau/t_q)^2/2][\tilde{u}_{s'}^{i+}(i\tau)-\text{H.c.}]$. Then, $\|[D^i, O_j]\| \leq \|O_j\| F(d(i,j))$, where the function $F(l)$ is exponentially decaying as $\exp[-l/\xi_C]$ for large l and decaying as $(\sqrt{2\pi}t_q/c_1 l)\exp[-(c_1 l/t_q)^2/2]$ for small l.

This exponential decay is in fact good enough to prove the finite-group-velocity result[26] using D as an effective Hamiltonian; it is not necessary that D^i act only on sites within some finite range, but an exponential decay also suffices. Following Ref. 28, define G_i by the differential equations, for $s>0$, $\partial_s G_i(s)=\Sigma_j G_j(s) F(d(i,j))$ with initial conditions $G_i(0)=2$ if $i \in S_O$ and $G_i(0)=0$ otherwise. Then, one can show that $\|[O(s), O_j]\| \leq \|O\|\|O_j\| G_j(s)$. Solving the equations for G_j, one arrives at the bound (20).

One can define the operator O_{trunc} by setting $\tilde{V}_{\text{trunc}}(s)=\mathcal{S}' \exp[-i\int_0^s ds' \text{Tr}_{j,d(O,j)\geq l}(D)]$, where the trace is the trace of operator D over sites j with $d(O,j) \geq l$. Then, set $O_{\text{trunc}}(s)=\tilde{V}_{\text{trunc}}(s) O \tilde{V}_{\text{trunc}}(s)^\dagger$, getting Eq. (22).

2. Approximation result

The idea behind the proof of the approximation result is that, for any site i, the difference between $\tilde{u}_s^{i+}(i\tau)$ and $u_s^{i+}(i\tau)$ can be made small by taking large enough t_q. Here, u_s^{i+} is the positive energy part of u_s and $\tilde{u}_s^{i+}$ is defined following Eq. (19) with $A=u_s^i$. The difference between these two is closely related to e^i, as given below. Equation (17) involves summing over all sites i, but sites i which are sufficiently far from O will turn out to have little effect on defining $O(s)$. Thus, the task in this subsection is to figure out how much difference there is between $\tilde{u}_s^{i+}(i\tau)$ and $u_s^{i+}(i\tau)$, and then sum that error over sites near enough to O, giving the difference between the quasiadiabatic continuation and the adiabatic continuation. Since the adiabatic continuation preserves expectation values, this will give an estimate in the error in the expectation values. We now do this carefully.

The proof of Eq. (23) involves defining an additional operator and using triangle inequalities. We define

$$\tilde{V}_l(s)=\mathcal{S}' \exp\left(-\int_0^s ds' \int_0^\infty d\tau \sum_i \exp[-(\tau/t_{q_i})^2/2] \times[\tilde{u}_{s'}^{i+}(i\tau)-\text{H.c.}]\right), \tag{A1}$$

where now t_{q_i} may depend on i and we define $\tilde{u}_s^{i+}(i\tau)$ by $\tilde{u}_s^i(t)=u_s^i(t)\exp[-(t/t_{q_i})^2/2]$, again using the t_{q_i} which depend on i.

We then pick $t_{q_i}=t_q$ for $d(i,O) \leq 2l$, and $t_{q_i}=t_q+c_1[d(i,O)-2l]$ otherwise. Define $O_l(s)=\tilde{V}_l(s) O \tilde{V}_l(s)^\dagger$. Thus, $\tilde{V}_l(s)$ has a t_{q_i} which increases the further one gets from operator O. While the operator $\tilde{V}(s)$ would create local excitations everywhere acting on a ground state, $\tilde{V}_l(s)$ only creates local excitations near O.

Using a triangle inequality,

$$\begin{aligned}&|\langle\Psi_{\text{low,s}}|Q_0(s)O(s)Q_0(s)^\dagger - O_{\text{adiab}}(s)|\Psi_{\text{low,s}}\rangle| \\ &\quad \leq |\langle\Psi_{\text{low,s}}|Q_0(s)O(s)Q_0(s)^\dagger - Q_0(s)O_l(s)Q_0(s)^\dagger|\Psi_{\text{low,s}}\rangle| \\ &\quad + |\langle\Psi_{\text{low,s}}|Q_0(s)O_l(s)Q_0(s)^\dagger - O_{\text{adiab}}(s)|\Psi_{\text{low,s}}\rangle|.\end{aligned}$$

The difference between $\tilde{V}_l(s)$ and $\tilde{V}(s)$ is the excitations far from O. That is, if for small s $\tilde{V}(s)V(s)^\dagger=1+s\Sigma_i e^i+\cdots$ and $\tilde{V}_l(s)V(s)^\dagger=1+s\Sigma_i e_l^i+\cdots$, then e_l^i and e^i differ only for i far from O. However, for i far from O, the e^i commute through O (as discussed physically before) and so it is possible to bound the difference $|\langle\Psi_{\text{low,s}}|Q_0(s)O(s)Q_0(s)^\dagger - Q_0(s)O_l(s)Q_0(s)^\dagger|\Psi_{\text{low,s}}\rangle|$. Precisely, to bound the first difference, we note that the difference between the definition of $O_l(s)$ and $O(s)$ has to do with terms u_s^i with sites i which are at least a distance $2l$ from O. Using the locality bound, one can bound the commutator of $O(s)$ and $O_l(s)$ with $\tilde{u}_s^i$ for these sites. This gives $\|O(s)-O_l(s)\| \leq c_3(s)$, where

$$c_3(s)=X_3 \sum_{j,d(i,j)\geq 2l} \max(\exp[-d(j,O)/\xi_C], \exp[-(d(j,O)/l_{q_j})^2/2]), \tag{A2}$$

for some constant X_3 and where l_{q_j} is of order $c_1 t_{q_j}$.

We now bound the difference $|\langle\Psi_{\text{low,s}}|Q_0(s)O_l(s)Q_0(s)^\dagger - O_{\text{adiab}}(s)|\Psi_{\text{low,s}}\rangle|$. To do this, it suffices to bound $|\langle\Psi_{\text{low,s}}\, Q_0(s)\tilde{V}_l(s) - \langle\Psi_{\text{low,s}}\, V(s)|$. This is equal to $|\langle\Psi_{\text{low,0}}\; V(s)^\dagger Q_0(s)\tilde{V}_l(s) - \langle\Psi_{\text{low,0}}|$. The operator $\tilde{V}_l(s)$ is equal to $\mathcal{S}'\exp\{-\int_0^s ds'[\partial_{s'}\tilde{V}_l(s')]\tilde{V}_l(s')^\dagger\}$. This equals $V(s)\mathcal{S}'\exp\{-\int_0^s ds' V(s')^\dagger\{[\partial_{s'}\tilde{V}_l(s')]\tilde{V}_l(s')^\dagger - [\partial_{s'}V(s')]V(s')^\dagger\}V(s)\}$. Thus,

$$V(s)^\dagger Q_0(s)\tilde{V}_l(s) = [V(s)^\dagger Q_0(s)V(s)] \times\mathcal{S}'\exp\left\{-\int_0^s ds' V(s')^\dagger e_l(s')V(s')\right\}, \tag{A3}$$

where

$$e_l(s') = [\partial_{s'}\tilde{V}_l(s')]\tilde{V}_l(s')^\dagger V(s') - [\partial_{s'}V(s')]V(s')^\dagger. \tag{A4}$$

We have grouped the operators $V(s)^\dagger Q_0(s)V(s)$ together in Eq. (A3) for a reason: the matrix $Q_0(s)$ is an operator between the low-energy states of $\mathcal{H}_s$ so therefore the operator $V(s)^\dagger Q_0(s)V(s)$ is an operator between the low-energy states of $\mathcal{H}_0$. Now we turn to the exponential $\mathcal{S}'\exp\{-\int_0^s ds' V(s')^\dagger e_l(s')V(s')\}$. We want to show that this operator is also equal, up to some bounded error, to an operator between the low-energy states of $\mathcal{H}_0$.

Define P_{high} to project onto the high-energy states of $\mathcal{H}_0$. Then, we can pick $Q_0(s)$ such that

$$|\langle\Psi_{\text{low,0}}V\;(s)^\dagger Q_0(s)\tilde{V}_l(s) - \langle\Psi_{\text{low,0}}| \leq \int_0^s ds'|\langle\Psi_{\text{low,0}}V\;(s')^\dagger e_l^i(s')V(s')P_{\text{high}}\;|, \tag{A5}$$

We will bound the integral of Eq. (A5) below; combining this bound with Eq. (A2) will give Eq. (23).

Using linear perturbation theory,

$$[\partial_{s'}V(s')]V(s')^\dagger = -\int_0^\infty d\tau[u_s^+(i\tau) - \text{H.c.}] + P, \tag{A6}$$

where P only has nonvanishing matrix elements between states of the same energy: $P_{ab}=0$ if $E_a \neq E_b$.

The error $e_l(s) = -P + \Sigma_i e_l^i(s)$ where

$$e_l^i(s) = -\int_0^\infty d\tau\{\exp[-(\tau/t_{q_i})^2/2]\tilde{u}_s^{i+}(i\tau) - u_s^{i+}(i\tau) - \text{H.c.}\}. \tag{A7}$$

For $s=0$, $e_l^i(s) = e_l^i$ defined above.

We now bound the projection into the high-energy sector $|\langle\Psi_{\text{low,s}'}\, e_l^i(s')P_{\text{high}}|$. We can show by performing some elementary integrations[27] that, for any eigenstate $\Psi_{a,s}$ with energy $E_a > \tau/t_{q_i}^2$,

$$|\langle\Psi_{\text{low,s}}|\exp[-(\tau/t_{q_i})^2/2]\tilde{u}_s^{i+}(i\tau) - u_s^{i+}(i\tau)|\Psi_{\text{a,s}}\rangle| \leq \exp[-(\tau/t_{q_i})^2/2]\exp[-(t_{q_i}E_a)^2/2]|(u_s^i)_{0a}|,$$

where $|(u_s^i)_{0a}|$ is the absolute value of the matrix element of u_s^i between states $\Psi_{\text{low,s}}$ and $\Psi_{a,s}$. Similarly, for $E_a < \tau/t_{q_i}^2$, $|\langle\Psi_{\text{low,s}}|\exp[-(\tau/t_{q_i})^2/2]\tilde{u}_s^{i+}(i\tau) - u_s^{i+}(i\tau)|\Psi_{a,s}\rangle| \leq \exp[-\tau E_a]|(u_s^i)_{0a}|$. Integrating over τ and summing over states Ψ_a outside the sector of ground states, using the bound on the density of states, we have

$$|\langle\Psi_{\text{low,s}}e_l^i(s)P_{\text{high}}\;|^2 \leq \int dE\;\rho(E)\{(K/E)2\exp[-(t_{q_i}E)^2] + Kt_{q_i}\exp[-(t_{q_i}E)^2/2]\;\sqrt{2\pi})\}^2. \tag{A8}$$

Summing over sites i and using Eq. (A5) we obtain the bound $|\langle\Psi_{\text{low,0}}\;V(s)^\dagger Q_0(s)\tilde{V}_l(s) - \langle\Psi_{\text{low,0}}| \leq c_2(s)$ where we define

$$c_2(s) = s\sum_i\left(\int dE\;\rho(E)\{(K/E)2\exp[-(t_{q_i}E)^2] + Kt_{q_i}\exp[-(t_{q_i}E)^2/2]\sqrt{2\pi}\}^2\right)^{1/2}. \tag{A9}$$

This completes the calculation.

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