

## String and membrane condensation on three-dimensional lattices

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In this paper, we investigate the general properties of lattice spin models that have string and/or membrane condensed ground states. We discuss the properties needed to define a string or membrane operator. We study three three-dimensional spin models which lead to  $Z_2$  gauge theory at low energies. All the three models are exactly soluble and produce topologically ordered ground states. The first model contains both closed-string and closed-membrane condensations. The second model contains closed-string condensation only. The ends of open strings behave like fermionic particles. The third model also has condensations of closed membranes and closed strings. The ends of open strings are bosonic while the edges of open membranes are fermionic. The third model contains a different type of topological order.

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### I. INTRODUCTION

The discovery of fractional quantum Hall (FQH) liquids by Tsui, Stormer, and Gossard<sup>1</sup> in 1982 showed that not all the states of matter are associated with symmetries (or the breaking of symmetries). As an example, FQH liquids cannot be described by the Landau theory<sup>2,3</sup> of broken symmetries and local order parameters.<sup>4,5</sup> In Landau's theory, the internal order is defined by the symmetry of the states. Symmetry is a universal property of the states, i.e., a property shared by all the states in the same phase. The symmetry group (SG) can thus characterize the internal orders of those states. The internal order of FQH liquids is the internal structure of their quantum ground state. The kind of order that explains this internal structure of FQH liquids is a *topological order*.<sup>4,5</sup> Topological order is a special case of more general quantum order.<sup>5,6</sup> Topological or quantum order cannot be characterized by symmetry breaking since all FQH states have the same symmetry. To characterize quantum orders, we need to find universal properties of the wave function. One way to characterize quantum orders is through the projective symmetry group (PSG).<sup>7</sup> The PSG is the group of symmetry of the mean-field ansatz of a mean-field Hamiltonian  $H_{mean}$  that describes a quantum ordered state. Two different physical wave functions obtained from two different mean-field *Ansätze* can have the same symmetry and different PSGs. Thus the PSG gives a more refined characterization of the internal orders than the SG and can describe those internal orders that are not distinguished by the symmetry group.

The topological order, as a special case of quantum order, is a quantum order where all the excitations above ground states have finite energy gaps. FQH liquids present nontrivial topological orders in which the degeneracy of the ground state depends on the topology of the space.<sup>8,9</sup> The ground state of FQH liquids on a Riemann surface of genus  $g$  is  $\tilde{q}^g$ -fold degenerate where  $\tilde{q}$  is the ground-state degeneracy in a torus topology. This degeneracy is robust against any perturbations. For a finite system of size  $L$  the ground-state degeneracy is lifted and the energy splitting is  $\sim e^{-L\xi}$ . This robustness is at the root of the proposal of fault-tolerant quantum computation at the physical level.<sup>10</sup>

A particular class of quantum orders is the one from string condensation.<sup>11</sup> We say we have string condensation in the ground state  $|\xi\rangle$  when (a) a certain closed-string operator  $W(\bar{\gamma})$  satisfies

$$\langle \xi | W(\bar{\gamma}) | \xi \rangle = 1. \quad (1)$$

(b) The closed-string operator cannot be decomposed into smaller pieces  $W(\bar{\gamma}) = W(\gamma_1)W(\gamma_2)\cdots W(\gamma_n)$  where each piece satisfies  $\langle \xi | W(\gamma_\ell) | \xi \rangle = 1$ .

A string net is a branched string. It turns out<sup>11</sup> that quantum ordered states that produce and protect massless gauge bosons and massless fermions are string-net condensed states. Moreover, different string-net condensations are not characterized by symmetries, but by projective symmetry group. In this case, the PSG describes the symmetry in the hopping Hamiltonian for the ends of condensed strings. Then the characterization of different string-net condensations classifies different topological or quantum orders. Systems that feature string-net condensation, if gapped, feature a ground-state degeneracy that depends on the topology of the system and that is robust against arbitrary local perturbations.

Ends of open strings are particlelike objects which can have a nontrivial statistics.<sup>12</sup> When on a two-dimensional system such a quasiparticle winds around another quasiparticle of a different kind, its wave function picks a phase. The particle has undergone an Aharonov-Bohm effect, whose topological nature is described by a Cherns-Simons theory.<sup>13</sup> This phenomenon corresponds to the fact that the end of an open string can be detected by a closed string of another type that encloses its end.

On a three-dimensional (3D) lattice, the end of an open string can be detected by a closed surface. So we can enquire about the meaning of the condensation of closed membranes. Similar to closed-string condensation, closed-membrane condensation is a superposition of closed membranes of arbitrary sizes, shapes, and numbers. Just like closed-string condensation, closed-membrane condensation also implies topological order. Many 3D models have both closed-string

and closed-membrane condensation due to a natural duality between strings and membranes in 3D space. But it may be possible to have 3D models with only string condensation. We will present an example.

We also build a model with a different kind of closed-string and closed-membrane condensation. We argue that this model has a different type of topological order.

## II. $Z_2$ LATTICE GAUGE THEORY: A MODEL WITH STRING AND MEMBRANE CONDENSATION

Let us consider a three-dimensional cubic lattice. Then we can place a spin 1/2 on each link of the lattice. A string operator can be defined by drawing a curve  $\gamma$  connecting the sites of the lattice and acting with a  $\sigma^z$  on all the links belonging to  $\gamma$ :

$$W[\gamma] = \prod_{j \in \gamma} \sigma_j^z. \quad (2)$$

A membrane operator  $M[\Sigma]$  is obtained by drawing a two-dimensional surface  $\Sigma$  in the dual lattice and acting with a spin flip  $\sigma^x$  on all the links orthogonal to  $\Sigma$ :

$$M[\Sigma] = \prod_{j \perp \Sigma} \sigma_j^x. \quad (3)$$

As expected, closed membranes are able to detect the ends of open strings. The open string flips the spin on the membrane where it punctures it because the membrane operator anti-commutes with the string operator when they intersect in only one point,

$$W[\gamma]M[\bar{\Sigma}]|\psi\rangle = -M[\bar{\Sigma}]W[\gamma]|\psi\rangle, \quad (4)$$

where  $\bar{\Sigma}$  is a closed surface in the dual lattice. If the sign of a closed-membrane state is flipped we know that there is the end of an open string inside. If the string punctures the closed membrane in two points, then the end of the string is not inside and the sign of the state will not be flipped because the two operators will in fact commute. A closed membrane can detect the presence of a particle (the end of an open string) inside even if the membrane is actually very far from the particle.

We would like to stress that not all products of operators along a string give us a nontrivial string operator. Similarly, not all products of operators on a membrane give us a nontrivial membrane operator. The product of identity operators along a string or on a membrane is an example of a trivial string operator or membrane operator. However, in our case, the nontrivial algebraic relation (4) between the large string-operator and membrane operator ensures that both the string operator and the membrane operator defined above are nontrivial.

A plaquette operator is the product of  $\sigma^z$  on all the spins belonging to the same plaquette  $p$ . A closed-string operator  $W[\bar{\gamma}]$  can also be expressed as a product of plaquette operators. When we multiply two neighboring plaquette operators, the  $\sigma^z$  acts twice on the shared link, and so the resulting operator is the product of  $\sigma^z$  on the border of the two plaquettes.

A star operator is on the other hand the product of  $\sigma^x$  on all the links extruding from a site  $s$ :  $A_s = \prod_{j \in s} \sigma_j^x$ . Similarly then, a closed-membrane operator  $M[\bar{\Sigma}]$  is the product of all the star operators enclosed in such a two-dimensional closed surface  $\bar{\Sigma}$ :

$$M[\bar{\Sigma}] = \prod_{j \perp \bar{\Sigma}} \sigma_j^x = \prod_{s \in V} A_s, \quad (5)$$

where  $V$  is the volume enclosed in  $\bar{\Sigma}$ :  $\bar{\Sigma} = \partial V$ . The star operator is then the operator corresponding to the elementary closed membrane, the cube. If we consider in the dual lattice the six faces orthogonal to the links of a star, we see that they form a cube. Since when multiplied with each other the stars cancel their interiors, they are surface operators and not volume operators. So in general, the product of two membrane operators is still a membrane operator because the interior cancels.

An example of a model featuring both membrane and string condensation is given by the following Hamiltonian:

$$H_{spin} = -g \sum_p \prod_{j \in p} \sigma_j^z - U \sum_s \prod_{j \in s} \sigma_j^x, \quad (6)$$

so that the Hamiltonian is the sum of all the plaquette operators and star operators. Such a Hamiltonian defines a  $Z_2$  lattice gauge theory.<sup>14</sup> The model is exactly soluble since all the plaquette operators and star operators commute with each other.<sup>10</sup>

Having a spin on each link, the dimension of the local Hilbert space is 2. If  $N$  is the number of the sites, on a cubic lattice we have  $3N$  links.<sup>17</sup> The dimension of the global Hilbert space is hence  $2^{3N}$ . How many states can we label with the operators  $\prod_{j \in p} \sigma_j^z$  and  $\prod_{j \in \text{site}} \sigma_j^x$ ? We have  $N$  star operators, and  $3N$  plaquettes in 3D. However, not all these plaquettes are independent. Indeed, in each cube the product of the eight plaquettes is identically 1, as it is immediate to verify. This gives us  $N$  constraints on the plaquettes. The number of independent plaquettes is thus  $3N - N = 2N$ . Together with the star operators, we can then label all the  $2^{3N}$  states. We have a finite degeneracy of the ground state due to topological global constraints on the star and plaquette operators (if we are on a torus for instance).

This model features both closed-string and closed-membrane condensation. Closed-string operators and closed-membrane operators both commute with the Hamiltonian because every closed string (membrane) shares either 0 or 2 links with any star (plaquette):

$$[H_{spin}, W[\bar{\gamma}]] = 0, \quad (7)$$

$$[H_{spin}, M[\bar{\Sigma}]] = 0. \quad (8)$$

Thus the ground state is an eigenstate of  $W[\bar{\gamma}]$  and  $M[\bar{\Sigma}]$  with eigenvalue  $\pm 1$  since  $W[\bar{\gamma}]^2 = M[\bar{\Sigma}]^2 = 1$ . This leads to a condensation of the closed-string and the closed-membrane operators  $\langle W[\bar{\gamma}] \rangle = \langle M[\bar{\Sigma}] \rangle = \pm 1$  regardless the size of the strings and the membranes.

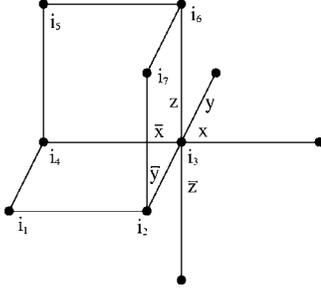


FIG. 1. A three-dimensional model with Majorana fermions on the sites. The six Majorana fermions label the six vectors from a site in the following way:  $x \mapsto +\hat{x}$ ,  $\bar{x} \mapsto -\hat{x}$ ,  $y \mapsto +\hat{y}$ ,  $\bar{y} \mapsto -\hat{y}$ ,  $z \mapsto +\hat{z}$ ,  $\bar{z} \mapsto -\hat{z}$ . The plaquettes in the three planes are shown.  $\hat{F}_{p_{\alpha\beta}}$ ,  $\hat{F}_{p_{yz}}$ ,  $\hat{F}_{p_{zx}}$  are, respectively, the plaquettes  $\hat{F}_{i_1 i_2 i_3 i_4}$ ,  $\hat{F}_{i_2 i_3 i_6 i_7}$ ,  $\hat{F}_{i_4 i_3 i_6 i_5}$ , where  $\mathbf{i}_1 = \mathbf{i}$ ;  $\mathbf{i}_2 = \mathbf{i} + \hat{x}$ ;  $\mathbf{i}_3 = \mathbf{i} + \hat{x} + \hat{y}$ ;  $\mathbf{i}_4 = \mathbf{i} + \hat{y}$ ;  $\mathbf{i}_5 = \mathbf{i} + \hat{y} + \hat{z}$ ;  $\mathbf{i}_6 = \mathbf{i} + \hat{x} + \hat{y} + \hat{z}$ ;  $\mathbf{i}_7 = \mathbf{i} + \hat{x} + \hat{z}$ .

Physically, when acting on the ground state, the open-string operator creates a pair of  $Z_2$  charges at its ends, while the open-membrane operator creates a loop of  $Z_2$  flux at its edge. So it is natural that a 3D  $Z_2$  gauge theory has both the closed-string and closed-membrane condensations.

In order to have string or membrane condensation, it is important that they do not dissolve, i.e., they are not decomposable into smaller objects that also condense. For example, a closed-string operator can be written as a product of two open-string operators  $W[\bar{\gamma}] = W[\gamma_1]W[\gamma_2]$ . We require that the open-string operators do not condense. That is,

$$\frac{\langle W[\bar{\gamma}_1] \rangle \langle W[\bar{\gamma}_2] \rangle}{\langle W[\bar{\gamma}] \rangle} \rightarrow 0 \quad (9)$$

as the size of strings approaches infinity. If closed membranes can be written as the product of smaller objects, and each smaller object still condenses, then there is no need to talk about membrane condensation, and we cannot expect to have topological order. It is the fact that a big closed membrane—which can explore the topology of the lattice—condenses that implies a topological order. For the same reason open membranes must be forbidden in the ground state. We can obtain it by making them pay an energy cost or by means of a constraint, as we will see in Sec. IV.

We would like to remark that the theory (6) can be mapped into a model with Majorana fermions on the links. We put six Majorana fermions at each site and one so-called ghost Majorana fermion at each link. Then we “move” the Majorana fermions from the sites to the midpoints of each link according to the directions shown in Fig. 1. So at this point we have three Majorana fermions at each link, two coming from the sites plus the initial ghost Majorana fermion. The mapping is given by the following representation of the Pauli matrices:

$$\sigma^j = \epsilon_{jk}^i \lambda^i \lambda^k. \quad (10)$$

### III. A MODEL WITH STRING CONDENSATION ONLY

In this section we want to show a model that has closed-string condensation but no closed-membrane condensation. Membranes either pay in energy or, if they commute with the Hamiltonian, dissolve into smaller pieces.

Let us consider the following exactly solvable model on a cubic lattice.<sup>11,12</sup> We introduce six Majorana fermions at each site of the lattice, namely,  $\lambda_i^a$ , where  $a = x, \bar{x}, y, \bar{y}, z, \bar{z}$ . Define the plaquette operator in the plane  $\alpha\beta$ :

$$\hat{F}_{p_{\alpha\beta}} = -\lambda_i^\beta \lambda_i^\alpha \lambda_{i+\hat{\alpha}}^{\bar{\alpha}} \lambda_{i+\hat{\beta}}^{\bar{\beta}} \lambda_{i+\hat{\alpha}+\hat{\beta}}^{\bar{\alpha}} \lambda_{i+\hat{\alpha}+\hat{\beta}}^{\bar{\beta}} \lambda_{i+\hat{\beta}}^\alpha \lambda_{i+\hat{\beta}}^\beta, \quad (11)$$

with  $\alpha, \beta = x, y, z$ . The Hamiltonian is then

$$H_{3D} = g \sum_p (\hat{F}_{p_{xy}} + \hat{F}_{p_{yz}} + \hat{F}_{p_{zx}}), \quad (12)$$

where the sum is taken on all the plaquettes in the  $xy, yz$ , and  $xz$  planes (see Fig. 1).

This model is exactly solvable because all the plaquette operators commute with each other  $[\hat{F}_{\square_i}, \hat{F}_{\square_j}] = 0$ .

Let us introduce now the following complex fermion operators at each site  $\mathbf{i}$ :

$$2\psi_x = \lambda^x + i\lambda^{\bar{x}}, \quad 2\psi_y = \lambda^y + i\lambda^{\bar{y}}, \quad 2\psi_z = \lambda^z + i\lambda^{\bar{z}}. \quad (13)$$

We project down to the physical Hilbert space with an even number of fermions per site  $\mathbf{i}$ :

$$(-)^{\sum_{a=x,y,z} \psi_{a,i}^\dagger \psi_{a,i}} = 1. \quad (14)$$

The projection above makes the theory a gauge theory. The physical states are invariant under local  $Z_2$  transformations generated by

$$\hat{G} = \prod_{\mathbf{i}} G_{\mathbf{i}}^{\sum_{a=x,y,z} \psi_{a,i}^\dagger \psi_{a,i}}, \quad (15)$$

where  $G_{\mathbf{i}}$  is an arbitrary function on the sites  $\mathbf{i}$  with the only two values  $\pm 1$ .

The Hamiltonian (12) acts on spin-3/2 states<sup>11,12</sup> by means of the following mapping:

$$\gamma_{\mathbf{i}}^{ab} = \frac{i}{2} (\lambda_{\mathbf{i}}^a \lambda_{\mathbf{i}}^b - \lambda_{\mathbf{i}}^{\bar{b}} \lambda_{\mathbf{i}}^{\bar{a}}). \quad (16)$$

The operators  $\gamma_{\mathbf{i}}^{ab}$  act on the local four-dimensional physical Hilbert space, that is, after the projection. In terms of the  $\gamma_{\mathbf{i}}^{ab}$  we can write down the Hamiltonian acting on spin-3/2 states:<sup>11</sup>

$$H_{3D} \mapsto H_{3/2} = g \sum_p (\gamma_{\mathbf{i}}^{ab} \gamma_{\mathbf{i}+\hat{a}}^{\bar{b}a} \gamma_{\mathbf{i}+\hat{b}}^{\bar{a}b} \gamma_{\mathbf{i}+\hat{a}+\hat{b}}^{\bar{b}\bar{a}}). \quad (17)$$

The ground state of  $H_{3D}$  has closed-string condensation. This means that we can define closed strings that commute with the Hamiltonian  $H_{3D}$ . What kind of strings can we define in this model (or similar models)? We can define two types of strings running on links: strings that end at the midpoint of the links and strings that end on the sites. The open-string operator of the former type is

$$W[\gamma]_l = -(\lambda_{i_1}^{a_1} \lambda_{i_1}^{b_1})(\lambda_{i_2}^{a_2} \lambda_{i_2}^{b_2}) \cdots (\lambda_{i_n}^{a_n} \lambda_{i_n}^{b_n}), \quad (18)$$

where  $\gamma$  is an open string running on the lattice links. Strings that end on sites would decompose in edges all commuting with the Hamiltonian and hence not giving a closed-string condensation; therefore we will focus only on the strings of type Eq. (18). The pair of indices  $ab$  gives the shape of any element of the string.  $\lambda_i^a \lambda_i^b = \lambda_i^x \lambda_i^{\bar{x}}$  is associated with a horizontal string straddling the site  $\mathbf{i}$ , while  $\lambda_i^{\bar{y}} \lambda_i^y$  is associated with an L-shaped elementary string crossing the site  $\mathbf{i}$ . These strings have end points at the midpoints of links. Moreover, if we close a string around the elementary loop (the square), we obtain the plaquette operator

$$W[\gamma_p] = \hat{F}_p. \quad (19)$$

Large (contractible) loops are products of these elementary loops:  $W[\partial\Sigma] = \prod_{p \in \Sigma} \hat{F}_p$ , where  $\Sigma$  is a surface made of squares of the lattice and  $\partial\Sigma$  its contour. We notice that the product of loops with some overlap gives a loop and not a net because the interior cancels. In fact the interior is identically 1 because the Majorana fermions square the identity. The important fact is that loops—contractible or not—commute with the Hamiltonian, as it is easy to check:

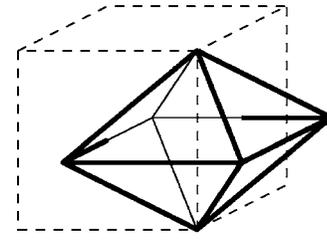
$$[H_{3D}, W[\bar{\gamma}]] = 0. \quad (20)$$

So the ground state of  $H_{3D}$  has closed-string condensation.

We want now to argue about the degeneracy of the ground state if the lattice has a torus topology. The dimension of the total Hilbert space is computed in the following way. We have six Majorana fermions per site and with them we defined three complex fermion operators. Three complex fermion operators generate an eight-dimensional local Hilbert space at each site. So the dimension of the total Hilbert space is  $8^N$ . The projection onto the physical Hilbert space at each site gives us thus a four-dimensional local Hilbert space at each site, because there are four states out of eight with an even number of fermions. Therefore after the projection the Hilbert space is  $4^N = 2^{2N}$  dimensional. How many states can we label with the commuting operators  $\hat{F}_p$ ? We have  $3N$  such operators, but not all of them are independent. We have local constraints and global constraints on them. The local constraint is given by the fact—which it is immediate to prove—that in each cube the product of all the plaquettes is identically equal to 1:

$$\prod_{p \in c} \hat{F}_p = 1, \quad (21)$$

where  $c$  labels the cubes in the lattice. This gives us  $N$  constraints. The global constraints are given by the periodic boundary conditions. These conditions provide three constraints. It turns out that the number of independent plaquette operators  $\hat{F}_p$  is  $3N - N - 3 = 2N - 3$ . So we can label  $2N - 3$  states out of  $3N$  and this means we are left with a  $2^{2N - (2N - 3)} = \text{eightfold}$  degeneracy of the ground state. This degeneracy is due to the closed-loop condensation.



Octahedron

FIG. 2. The dashed lines represent the cube in the cubic lattice. The solid lines represent the octahedron.

Notice that closed strings are not decomposable into smaller objects that still commute with the Hamiltonian. Small elements—called “dimer”—of the type  $\lambda_i^a \lambda_i^b$  (which is an elementary string of the I type) never commute with some of the plaquettes  $\hat{F}_p$ .

What we want to argue now is that the model Eq. (12) is a model with string condensation only. The membrane operator that we want to construct should trap the ends of strings which now exist on the centers of the links. Because of this, it is not natural to use the faces of the cubic lattice or the dual lattice to form the membrane. The dual lattice of the links is formed by octahedrons (see Fig. 2). We can put many octahedrons together to form a volume. The surface of such a volume is a natural choice of membrane which traps the centers of the links. The elementary (smallest) membrane corresponds to the faces of a single octahedron (see also Ref. 15). What is the operator for the elementary membrane? Notice that each elementary membrane contains a single link, say  $(\mathbf{i}, \mathbf{i} + \hat{x})$ . So a natural choice of the elementary membrane operator is  $\lambda_i^x \lambda_{i+\hat{x}}^{\bar{x}}$ . A generic membrane operator is the product of the elementary membrane operators for the enclosed octahedrons. On each interior lattice site  $\mathbf{j}$ , enclosed by the membrane, we have a product

$$\prod_{\alpha \in \mathbf{j}} \lambda_j^\alpha, \quad (22)$$

where  $\alpha = x, \bar{x}, y, \bar{y}, z, \bar{z}$ . In the projected physical Hilbert space,  $\prod_{\alpha \in \mathbf{j}} \lambda_j^\alpha = 1$ .<sup>11</sup> So the product of the elementary membrane operators is a membrane operator that only acts on the sites on the membrane. The membrane operator also satisfies an important condition in that it commutes with all the closed-string operators. The membrane operator anticommutes with the open-string operators of one end of the open string enclosed by the membrane. So the membrane operator can detect the presence of the trapped ends of strings. However, the membrane operator, in general, changes the fermion number by an odd number (on a site on the membrane). So the membrane operator defined above, although having many of the right properties, does not act within the physical Hilbert space.

Let us compare the model  $H_{3D}$  and the model Eq. (6). Notice that the plaquette terms in  $H_{3D}$  map well onto the spin model Eq. (6). We can associate  $\sigma^z$  with  $i\lambda_i^{\bar{a}} \lambda_i^a$  and see that  $\hat{F}_p \mapsto \prod_{j \in p} \sigma_j^z$ . So the string operator in the model  $H_{3D}$  can be mapped into the string operator in the model Eq. (6). What

does not map well is the star term for the reasons stated above. In order to build a good star operator so that the star maps onto the term  $-U \sum_s \prod_{j \in s} \sigma_j^x$ , we need to put an additional ghost Majorana fermion on each link and then realize the mapping Eq. (10). Because of this, we obtain the membrane operator in the model  $H_{3D}$  from the membrane operator in the model Eq. (6).

After many trials, we fail to obtain a membrane operator with the right properties. This leads us to believe that the model  $H_{3D}$  (12) has no membrane condensation.

#### IV. EXACTLY SOLVABLE MODEL WITH MEMBRANE CONDENSATION

The two models discussed above were constructed to have string condensation. The first model also has a membrane condensation. In this section we will directly construct a model that has a membrane condensation. The model is the following. We start with a cubic lattice with  $N$  sites and put four Majorana fermions at each link; thus we have  $12N$  Majorana fermions in total. We label the Majorana fermions according to the directions orthogonal to the link. For example, on a link  $\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{x}} \rangle$  we have the following Majorana fermions:

$$\lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{x}} \rangle}^y, \quad \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{x}} \rangle}^{\bar{y}}, \quad \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{x}} \rangle}^z, \quad \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{x}} \rangle}^{\bar{z}}. \quad (23)$$

We can define the complex fermion operators at each link  $\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle$ :

$$2\psi_{b, \langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle} = \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^b + i\lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^{\bar{b}}, \quad (24)$$

where  $a, b = x, y, z$  and  $a \neq b$ . Of course on a link  $\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{x}} \rangle$  we can only define  $\psi_y$  and  $\psi_z$  and so on. So at each link we have defined two complex fermion operators and thus at each link there is a four-dimensional local Hilbert space  $\mathcal{H}$ . Since the number of links on a 3D cubic lattice with  $N$  sites is  $3N$ , the total Hilbert space  $\mathcal{H}^{\otimes 3N}$  has dimension  $4^{3N} = 2^{6N}$ .

On each link we define the Link operator in this way:

$$\hat{L}_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle} = \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^b \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^{\bar{b}} \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^c \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^{\bar{c}}, \quad (25)$$

where  $a \neq b \neq c$  and they take values in the set  $\{x, y, z\}$ . In a cubic lattice each link is shared by four crossing faces and every face has as contour four links. For each link, we can uniquely associate one Majorana fermion to each of the four faces that share that link. Since each face is bordered by four links, each face receives a total of four Majorana fermions from the links that border it. This assignment is univocal. Each Majorana fermion is assigned to one and only one face. The corresponding face operator is defined as

$$\hat{F}_s = \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^b \lambda_{\langle \mathbf{i} + \hat{\mathbf{a}}, \mathbf{i} + \hat{\mathbf{a}} + \hat{\mathbf{b}} \rangle}^{\bar{a}} \lambda_{\langle \mathbf{i} + \hat{\mathbf{a}} + \hat{\mathbf{b}}, \mathbf{i} + \hat{\mathbf{b}} \rangle}^{\bar{b}} \lambda_{\langle \mathbf{i} + \hat{\mathbf{b}}, \mathbf{i} \rangle}^a, \quad (26)$$

where the face  $s$  is on the plane  $\{ab\}$  and  $a \neq b$  and again they take the values  $\{x, y, z\}$ . Notice that the face operator corresponds to a link operator in the dual lattice. Notice also that the operator on the link  $\langle l \rangle$  anticommutes with any of the four adjacent face operators because they have a Majorana fermion in common:

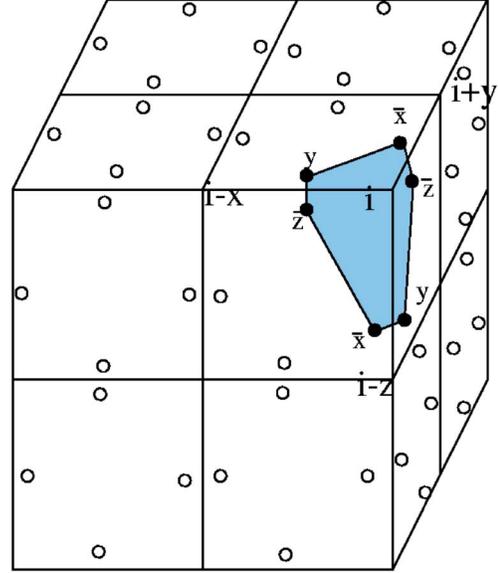


FIG. 3. (Color online) The cube operator and the corner loop operator with the nomenclature of the Majorana fermions forming a corner loop.

$$\{\hat{L}_{\langle l \rangle}, \hat{F}_{s \pm}\} = 0. \quad (27)$$

After the discussion of the previous sections, we know that we can define a good elementary closed-membrane operator by taking the product of the six face operators on a cube. This is equivalent to a star operator on the dual lattice. The cube operator is then

$$\hat{S}_c = \prod_{s \in c} \hat{F}_s. \quad (28)$$

The cube operator shares two Majorana fermions with any adjacent link  $\langle l \rangle$  so they commute:

$$[\hat{L}_{\langle l \rangle}, \hat{S}_c] = 0. \quad (29)$$

Bigger membrane operators are the product of all the faces that make the surface on which the operator is defined:

$$\hat{M}_\Sigma = \prod_{s \in \Sigma} \hat{F}_s. \quad (30)$$

Open membranes do not commute with the link operator, because they share one Majorana fermion on the border of the membrane. But a closed-membrane operator, being the product of cubes, does commute with the link operator.

The last operator we want to define is the corner loop operator. It is the operator that collects in a loop the six Majorana fermions that are associated with each corner of a cube. For instance, the corner made by, say, the links  $\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{x}} \rangle, \langle \mathbf{i} + \hat{\mathbf{x}}, \mathbf{i} + \hat{\mathbf{x}} + \hat{\mathbf{y}} \rangle, \langle \mathbf{i} + \hat{\mathbf{x}}, \mathbf{i} + \hat{\mathbf{x}} - \hat{\mathbf{z}} \rangle$  is labeled by the vector  $\mathbf{j} = (\bar{x}, y, \bar{z})$  and the associated corner loop operator  $\hat{C}_i^{\bar{x}y\bar{z}}$  is

$$\hat{C}_i^{\bar{x}y\bar{z}} \equiv \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{x}} \rangle}^{\bar{z}} \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{x}} \rangle}^y \lambda_{\langle \mathbf{i} + \hat{\mathbf{x}}, \mathbf{i} + \hat{\mathbf{x}} + \hat{\mathbf{y}} \rangle}^{\bar{x}} \lambda_{\langle \mathbf{i} + \hat{\mathbf{x}}, \mathbf{i} + \hat{\mathbf{x}} + \hat{\mathbf{y}} \rangle}^{\bar{z}} \times \lambda_{\langle \mathbf{i} + \hat{\mathbf{x}}, \mathbf{i} + \hat{\mathbf{x}} - \hat{\mathbf{z}} \rangle}^y \lambda_{\langle \mathbf{i} + \hat{\mathbf{x}}, \mathbf{i} + \hat{\mathbf{x}} - \hat{\mathbf{z}} \rangle}^{\bar{x}} \quad (31)$$

(see Fig. 3). At each site  $\mathbf{i}$  there are eight corners that are

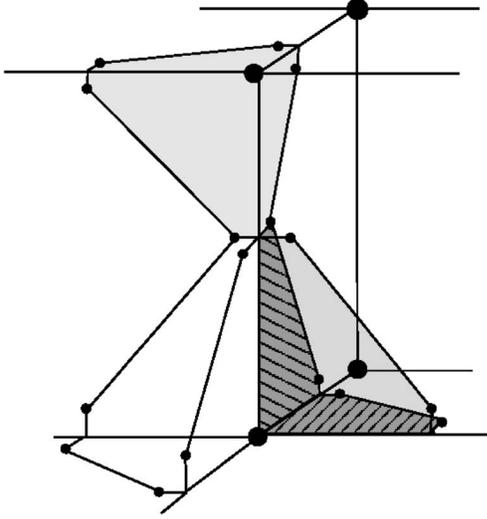


FIG. 4. A triple of noncommuting corner loops. The corner operator above does not commute with the two below. Each corner term affects three cubes.

labeled by  $\mathbf{j}=(a,b,c)$  with  $a=x,\bar{x},b=y,\bar{y},c=z,\bar{z}$ .

So the expression for the generic corner loop operator on the site  $\mathbf{i}$  is

$$\hat{C}_{\mathbf{i}}^{(a,b,c)} = \lambda_{(\mathbf{i},\mathbf{i}+\mathbf{a})}^b \lambda_{(\mathbf{i},\mathbf{i}+\mathbf{a})}^c \lambda_{(\mathbf{i},\mathbf{i}+\mathbf{b})}^a \lambda_{(\mathbf{i},\mathbf{i}+\mathbf{b})}^c \times \lambda_{(\mathbf{i},\mathbf{i}+\mathbf{c})}^a \lambda_{(\mathbf{i},\mathbf{i}+\mathbf{c})}^b, \quad (32)$$

where  $a,b,c=x,\bar{x},y,\bar{y},z,\bar{z}$ . The corner loop operators commute with both the link and cube operators but, interestingly enough, not all the corner loop operators commute with the others. As can be seen in Fig. 4, a corner loop shares only one Majorana fermion with two other corner loops belonging, respectively, to two other adjacent cubes. These corner loop operators do not commute with each other when they are based on adjacent sites and the indices in the direction of the connecting link are conjugate and have only one index in common. Consider the case of corner operators at the sites  $\mathbf{i}$  and  $\mathbf{i}+\bar{\mathbf{z}}$ ; then for example we have

$$\{\hat{C}_{\mathbf{i}}^{xy\bar{z}}, \hat{C}_{\mathbf{i}+\bar{\mathbf{z}}}^{x\bar{y}z}\}_+ = 0, \quad (33)$$

$$[\hat{C}_{\mathbf{i}}^{xy\bar{z}}, \hat{C}_{\mathbf{i}+\bar{\mathbf{z}}}^{x\bar{y}z}] = 0. \quad (34)$$

Notice that we have the following constraint for the product of the eight corners coming from a site  $\mathbf{i}$ :  $\prod_{\mathbf{j}=1}^8 \hat{C}_{\mathbf{i}}^{\mathbf{j}}=1$ . This means that on a cubic lattice with  $N$  sites we have  $N$  constraints on the corner operators. Notice also that the cube is not the product of its eight corners. Indeed, also the product of eight corners in a cube is identically 1 because in the product each Majorana fermion appears twice and they square the identity. We thus have also the following constraint at each cube:  $\prod_{\mathbf{j}=1}^8 \hat{C}^{\mathbf{j}}=1$ . Here  $\mathbf{j}$  of course runs instead on the eight corners belonging to the same cube. The cube is almost the square root of  $\prod_{\mathbf{j}=1}^8 \hat{C}^{\mathbf{j}}$ . Actually, the cube operator

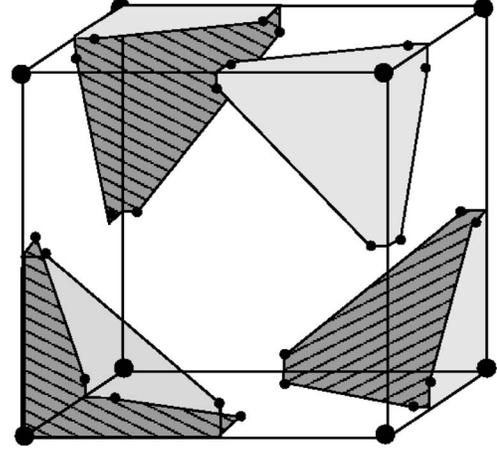


FIG. 5. The cube operator and four corner loop operators. The cube is made of the product of only four such loops. These four operators all commute with each other because they have no Majorana fermion in common. The four corner loops shown in the figure are chosen on the odd sites.

is made of the product of four of these corner loop operators. We have two different ways of choosing the four corners that make the cube out of the eight possible corners on the cube, namely, the corners on the even or odd sites, as is shown in Fig. 5. So we can write

$$\sum_c \hat{S}_c = \sum_c \prod_{\mathbf{j} \in \mathbf{i}(c)_{\text{even}}} \hat{C}^{\mathbf{j}} = \sum_c \prod_{\mathbf{j} \in \mathbf{i}(c)_{\text{odd}}} \hat{C}^{\mathbf{j}}, \quad (35)$$

where  $\mathbf{j} \in \mathbf{i}(c)_{\text{even}}$  means that  $\mathbf{j}$  runs on the four corners at even sites in each cube  $c$ , and similarly for the odd term.

In this model we want closed-membrane condensation. As we have seen, it is necessary that (i) the cube operator commutes with the Hamiltonian; (ii) all the smaller pieces that compose a closed membrane must be forbidden or pay some energy cost, so that the closed membranes will not dissolve. In order to make it impossible for the membrane to decompose into open parts, we can use the link term in the Hamiltonian. The border of an open-membrane operator does not commute with it because they share one Majorana fermion. The link term has also the effect of making it impossible for the strings to run on the links. So if this model has closed-string condensation, it will be of a different type.

We also need to put in the Hamiltonian the corner operators. They are needed to make the model not infinite degenerate. Consider the following set of commuting corner operators:

$$S_{\text{even}} = \{C_{\mathbf{i}}^{xyz}, C_{\mathbf{i}}^{x\bar{y}z}, C_{\mathbf{i}}^{xy\bar{z}}, C_{\mathbf{i}}^{x\bar{y}\bar{z}}\}, \quad (36)$$

$$S_{\text{odd}} = \{C_{\mathbf{i}}^{x\bar{y}z}, C_{\mathbf{i}}^{x\bar{y}\bar{z}}, C_{\mathbf{i}}^{xy\bar{z}}, C_{\mathbf{i}}^{xy\bar{z}}\}. \quad (37)$$

This choice is equivalent to taking all the corner operators in the even cubes and none in the odd cubes, as is shown in Fig. 6. All the operators in  $S_{\text{even}} \cup S_{\text{odd}}$  commute with each other, as it is straightforward to check. We are now ready to write down the Hamiltonian for this model:

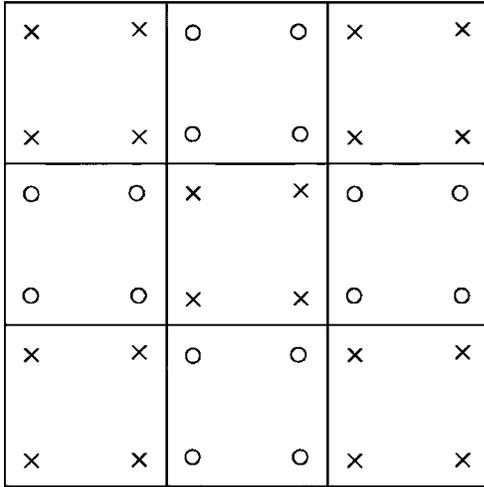


FIG. 6. A section of the cubic lattice is shown. The picture shows the set of commuting corner operators selected for the Hamiltonian. At even sites, we choose the upper corners (marked with a  $\circ$  in the  $xy$  and  $\bar{x}\bar{y}$  quadrants and the lower corners (marked with an  $\times$ ) in the other two quadrants. We make the complementary choice on odd sites. This is equivalent to alternately taking all the lower corners or all the upper corners in a square. This means that in even cubes all the corners are taken, but in the odd ones none.

$$H_f = -g \sum_c \hat{S}_c - h \sum_{\langle l \rangle} \hat{L}_{\langle l \rangle} - U \sum_{\text{even } c} \hat{C}_c^\dagger. \quad (38)$$

Since all the terms commute with each other, the model is exactly solvable. The ground-state manifold is

$$\mathcal{L} = \{ |\psi\rangle \in \mathcal{H}^{\otimes 3N} | \hat{S}_c |\psi\rangle = \hat{L}_{\langle l \rangle} |\psi\rangle = \hat{C}_c^\dagger |\psi\rangle = |\psi\rangle \}. \quad (39)$$

What is the ground state of this model like? All the states obtained acting on a ground state with operators commuting with the Hamiltonian are still states in the ground state. The ground state of the Hamiltonian (38) cannot contain open membranes, because their border does not commute with the link term. If the membrane is closed, it is the product of cube operators and it commutes with the Hamiltonian, so closed-membrane states are allowed in the ground state.

It is of crucial importance to consider nontopologically trivial closed membranes. They are noncontractible closed membranes and are not the product of cubes. A noncontractible closed-membrane operator is the product of all the faces on a plane  $\alpha$ :

$$\hat{M}_\alpha^{\text{big}} = \prod_{s \in \alpha} \hat{F}_s, \quad (40)$$

where  $\alpha = xy, yz, xz$ . This big noncontractible membrane still commutes with the Hamiltonian because it commutes with the cubes, the corners, and has no border, so it commutes with the links as well. The noncontractible membrane operator is of capital importance for the topological structure of the ground-state manifold, as we will see soon.

Also strings on the links are forbidden for the same reason. Small loops corresponding to the corner loops in the

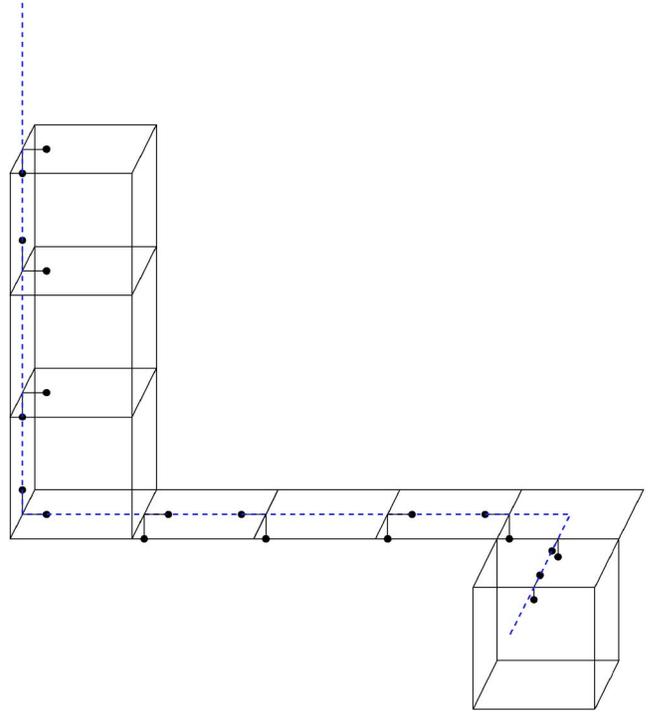


FIG. 7. (Color online) The string featured in the model.

Hamiltonian are allowed, but they cannot join to form bigger loops because they are disconnected. A binary term of the type  $\lambda_{(i,i+a)}^b \lambda_{(i,i+a)}^c$  always commutes with the links but not with the cube operators. We call it a “hinge term.” It does not commute with corner operators situated at an adjacent cube. This term corresponds to the edge of a cube. We can join the hinge terms in a string orthogonal to the links. Thus the following string operator can be defined. We first draw a string that cuts the faces in two and is orthogonal to the links. The elementary string  $\gamma_a^b$  is orthogonal to the links in the  $a$  direction and runs in the  $b$  direction. To this elementary string we associate the operator

$$w[\gamma_a^b] = \lambda_{(i,i+a)}^c \lambda_{(i,i+a)}^{\bar{b}} \lambda_{(i-b,i-b+a)}^b \lambda_{(i-b,i-b+a)}^c, \quad (41)$$

where  $a, b, c$  are three orthogonal directions. So each elementary string operator defines a particular cube  $c$ . A big string operator  $\hat{W}[\Gamma]$  is the product of many elementary strings and is obtained making the product of all the  $w(c)$  on the even cubes crossed by the string (see Fig. 7):

$$\hat{W}[\Gamma] = \prod_{\text{even } c \in \Gamma} w(c). \quad (42)$$

Because we take the hinge operators only on the even cubes, these strings always commute with the corners that we have put in the Hamiltonian. The ends of an open string do not commute with the cubes so these strings commute with the Hamiltonian only when they close. Therefore this model has a different type of closed-string condensation. Notice that

strings and membranes anticommute if they intersect in a single point and thus an open-string operator anticommutes with a closed-membrane operator if its end is trapped inside the membrane.

Now what about the closed-membrane condensation? We established that they commute with the Hamiltonian. Now we have to prove that they do not dissolve into smaller pieces. Even cubes are elementary closed membranes that actually dissolve in the corners. But we are interested in closed membranes of arbitrary size. Can they dissolve? A bigger closed membrane can at most “lose” its corners if they belong to even cubes. Sometimes the corners get smoothed as can be seen in Fig. 3. So closed membranes do not dissolve and we have a particular type of closed-membrane condensation.

Does this model have topological order? The answer is yes. It has a finite ground-state degeneracy that is stable against perturbation. The degeneracy depends on the existence of a nontrivial algebra of noncontractible membranes and strings. Consider the noncontractible membrane operators (40) and now consider the following noncontractible string operator orthogonal to the plane  $xy$ :

$$\hat{W} = (\lambda_{(i,i+x)}^y \lambda_{(i,i+x)}^{\bar{z}}) (\lambda_{(i-z,i-y+x)}^z \lambda_{(i-z,i-y+x)}^y) \cdots \quad (43)$$

This noncontractible string commutes with the Hamiltonian but flips all the noncontractible membranes in the plane  $xy$ . These two operators realize the four-dimensional algebra  $\sigma^x, \sigma^z$  on the ground-state manifold  $\mathcal{L}$ . We have three such algebras so the dimension of the total algebra is 64. This nontrivial algebra acts on the vector space  $\mathcal{L}$  which is therefore eight dimensional. The model has topological order.

We can constrain this model onto a physical Hilbert space of even number of fermions on each link by putting a constraint on the links as follows. The constraint is that of an even number of physical fermions on each link, so for example, on each link  $\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{z}} \rangle$  we require the constraint

$$(-) \psi_x^\dagger \psi_x + \psi_y^\dagger \psi_y = 1 \quad (44)$$

of an even number of fermions, and analogous constraints on the links along the other two axes. Again we can define projection operators to project down to the physical Hilbert space. The local projection operators are obviously

$$\mathcal{P}_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle} = \frac{1 + (-)^{N_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}}}{2}, \quad (45)$$

where

$$N_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle} \equiv \psi_{b, \langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^\dagger \psi_{b, \langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle} + \psi_{c, \langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^\dagger \psi_{c, \langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle} \quad (46)$$

and  $a \neq b \neq c$  take a value in  $x, y, z$ . The global projection operator is

$$\mathcal{P} = \prod_{\text{links}} \mathcal{P}_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}. \quad (47)$$

After the projection, the physical Hilbert space is  $2^{3N}$  dimensional. The projection makes the link term trivial:  $\hat{L}_{\langle l \rangle} = 1$ .

This is a full local bosonic model because the total Hilbert space is a product of finite-dimensional local Hilbert spaces and the Hamiltonian is the sum of local bosonic operators. They are bosonic in the sense that they all commute with each other when they are far apart.

After the projection, the model becomes a system of spins  $1/2$  on the links and we can map the Hamiltonian (38) onto a Hamiltonian with  $\sigma$  operators acting on the links. All we have to do is to map the corner operators correctly. The correct mapping is

$$C_{\mathbf{i}}^{xyz} = \sigma_{\langle x \rangle_{\mathbf{i}}}^x \sigma_{\langle y \rangle_{\mathbf{i}}}^x \sigma_{\langle z \rangle_{\mathbf{i}}}^x, \quad (48)$$

$$C_{\mathbf{i}}^{xy\bar{z}} = \sigma_{\langle x \rangle_{\mathbf{i}}}^x \sigma_{\langle \bar{y} \rangle_{\mathbf{i}}}^x \sigma_{\langle z \rangle_{\mathbf{i}}}^x, \quad (49)$$

$$C_{\mathbf{i}}^{x\bar{y}z} = \sigma_{\langle x \rangle_{\mathbf{i}}}^z \sigma_{\langle \bar{y} \rangle_{\mathbf{i}}}^z \sigma_{\langle z \rangle_{\mathbf{i}}}^z, \quad (50)$$

$$C_{\mathbf{i}}^{x\bar{y}\bar{z}} = \sigma_{\langle x \rangle_{\mathbf{i}}}^z \sigma_{\langle \bar{y} \rangle_{\mathbf{i}}}^z \sigma_{\langle \bar{z} \rangle_{\mathbf{i}}}^z, \quad (51)$$

$$C_{\mathbf{i}}^{\bar{x}yz} = \sigma_{\langle \bar{x} \rangle_{\mathbf{i}}}^x \sigma_{\langle y \rangle_{\mathbf{i}}}^x \sigma_{\langle z \rangle_{\mathbf{i}}}^x, \quad (52)$$

$$C_{\mathbf{i}}^{\bar{x}\bar{y}z} = \sigma_{\langle \bar{x} \rangle_{\mathbf{i}}}^x \sigma_{\langle \bar{y} \rangle_{\mathbf{i}}}^x \sigma_{\langle z \rangle_{\mathbf{i}}}^x, \quad (53)$$

$$C_{\mathbf{i}}^{\bar{x}y\bar{z}} = \sigma_{\langle \bar{x} \rangle_{\mathbf{i}}}^z \sigma_{\langle \bar{y} \rangle_{\mathbf{i}}}^z \sigma_{\langle z \rangle_{\mathbf{i}}}^z, \quad (54)$$

$$C_{\mathbf{i}}^{\bar{x}\bar{y}\bar{z}} = \sigma_{\langle \bar{x} \rangle_{\mathbf{i}}}^z \sigma_{\langle \bar{y} \rangle_{\mathbf{i}}}^z \sigma_{\langle \bar{z} \rangle_{\mathbf{i}}}^z \quad (55)$$

for the (commuting) corner operators in  $\mathcal{G}$ , that is, at even sites  $\mathbf{i}$ . On the odd sites  $\mathbf{i} + \mathbf{a}$  we have to assign the  $\sigma$  operators in a complementary way, that is, sending  $\sigma^x \mapsto \sigma^z$  and vice versa in order to have the right commutation-anticommutation properties.

In order to write the Hamiltonian (38) in terms of the new variables, we have to find the expression for the cube operator. It turns out that

$$\hat{S}_c = \hat{C}_{\mathbf{i}}^{xyz} \hat{C}_{\mathbf{i}+\mathbf{x}+\mathbf{y}}^{\bar{x}\bar{y}\bar{z}} \hat{C}_{\mathbf{i}+\mathbf{x}+\mathbf{z}}^{\bar{x}\bar{y}z} \hat{C}_{\mathbf{i}+\mathbf{y}+\mathbf{z}}^{\bar{x}y\bar{z}} \quad (56)$$

so we see that neighboring cubes have complementary expressions in terms of  $\sigma^x, \sigma^z$ .

In terms of the  $\sigma$  operators, the one-half-spin model thus becomes

$$\begin{aligned} H_{1/2} &= -g \sum_c \hat{S}_c - U \sum_{\text{even } c} \hat{C}_{\mathbf{i}}^j \\ &= -g \sum_{\mathbf{i}} \sigma_{\langle x \rangle_{\mathbf{i}}}^x \sigma_{\langle y \rangle_{\mathbf{i}}}^x \sigma_{\langle z \rangle_{\mathbf{i}}}^x \sigma_{\langle \bar{x} \rangle_{\mathbf{i}+\mathbf{x}+\mathbf{y}}}^z \sigma_{\langle \bar{y} \rangle_{\mathbf{i}+\mathbf{x}+\mathbf{y}}}^z \sigma_{\langle z \rangle_{\mathbf{i}+\mathbf{x}+\mathbf{y}}}^x \\ &\quad \times \sigma_{\langle \bar{x} \rangle_{\mathbf{i}+\mathbf{x}+\mathbf{z}}}^z \sigma_{\langle y \rangle_{\mathbf{i}+\mathbf{x}+\mathbf{z}}}^x \sigma_{\langle \bar{z} \rangle_{\mathbf{i}+\mathbf{x}+\mathbf{z}}}^z \sigma_{\langle x \rangle_{\mathbf{i}+\mathbf{y}+\mathbf{z}}}^x \sigma_{\langle \bar{y} \rangle_{\mathbf{i}+\mathbf{y}+\mathbf{z}}}^z \sigma_{\langle \bar{z} \rangle_{\mathbf{i}+\mathbf{y}+\mathbf{z}}}^x \\ &\quad - U \sum_{\text{even cubes}} [\sigma_{\langle \bar{x} \rangle}^z \sigma_{\langle y \rangle}^x \sigma_{\langle z \rangle}^z + \sigma_{\langle \bar{x} \rangle}^x \sigma_{\langle y \rangle}^z \sigma_{\langle z \rangle}^x \\ &\quad + \sigma_{\langle x \rangle}^z \sigma_{\langle \bar{y} \rangle}^z \sigma_{\langle z \rangle}^z + \sigma_{\langle x \rangle}^x \sigma_{\langle \bar{y} \rangle}^x \sigma_{\langle \bar{z} \rangle}^x]. \end{aligned} \quad (57)$$

We can give an explicit expression for the ground states. Let  $|0\rangle = |0_1 \cdots 0_N\rangle$  be the totally polarized state with all the spins down. Then if we consider the state

$$|\xi_{000}\rangle = \prod_c \frac{1 + \hat{S}_c}{2} \prod_s \frac{1 + \hat{C}_s}{2} |0\rangle, \quad (58)$$

this is obviously a ground state; indeed it is immediate to see that for any  $\hat{S}_c, \hat{C}_i^j$

$$\hat{S}_c |\xi_{000}\rangle = \hat{C}_i^j |\xi_{000}\rangle = |\xi_{000}\rangle. \quad (59)$$

The cube operators generate the group of the (contractible) closed-membrane operators,

$$\mathcal{M} = \langle \hat{S}_c \rangle; \quad (60)$$

the corners do not make a group because the product of corners is not a corner. Since the cubes generate a group, the ground state  $|\xi_{000}\rangle$  contains the sum of all possible contractible closed-membrane states, immersed in a broth of corners:

$$|\xi_{000}\rangle = \prod_s \frac{1 + \hat{C}_s}{2} \sum_{\hat{M}_\sigma \in \mathcal{M}} \hat{M}_\sigma |0\rangle, \quad (61)$$

where  $\hat{M}_\sigma$  are membrane operators defined on the contractible surfaces  $\sigma$ . The other sectors of the ground state can be reached by means of the noncontractible membranes  $\hat{M}_\alpha$ . The ground-state manifold can thus be written as

$$\mathcal{L} = \text{span}\{|\xi_{ijk}\rangle\}, \quad (62)$$

where  $i, j, k = 0, 1$  and

$$|\xi_{ijk}\rangle = \hat{M}_{xy}^i \hat{M}_{yz}^j \hat{M}_{xz}^k \prod_s \frac{1 + \hat{C}_s}{2} \sum_{\hat{M}_\sigma \in \mathcal{M}} \hat{M}_\sigma |0\rangle. \quad (63)$$

## V. EDGES OF OPEN MEMBRANES

An open string in the model (12) has two ends that are particlelike excitations. Open-string states are excitations because they do not commute with some of the plaquettes. Open strings have no tension, which means that their ends are free to hop in the lattice without paying additional energy. A longer string does not cost more energy than a shorter one. These elementary excitations are shown to be fermions by means of their hopping algebra.<sup>11</sup>

In the model of Hamiltonian (57), open membranes are forbidden because their border violates the constraint on the links. So open-membrane states are out of the physical Hilbert space. To make open membranes possible, we have to contour them with a ring of Majorana fermions. Then the following membrane operator defined on the open surface  $\Sigma$  makes states that are within the physical Hilbert space:

$$\hat{M}_\Sigma = \prod_{s \in \Sigma} \hat{F}_s \prod_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle \in \partial \Sigma} \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^a \quad (64)$$

where  $b \neq a$ . Notice that  $b$  can be chosen among the other three different Majorana operators that exist on the link

$\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle$ , whereas the Majorana fermion  $\lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^a$  belongs to the face operator  $\hat{F}_s$ .

Now the open-membrane states pay an energy cost because the membrane operator does not commute with the cube operators. In the model Eq. (57) the elementary excitations are open strings or open membranes. An open membrane does not commute with the link term because of its edges. So bigger membranes cost more energy than smaller ones: the membranes have a tension. The elementary excitation is therefore a single edge that exists on the link of the lattice.

The contour  $\prod_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle \in \partial \Sigma} \lambda_{\langle \mathbf{i}, \mathbf{i} + \hat{\mathbf{a}} \rangle}^b$  is a closed- (fermionic) string operator. This fact makes the edge of the membrane have a fermionic property. To give a precise definition of a fermionic edge, let us consider a system whose linear size in the  $z$  direction is given by  $L_z$ . We also assume periodic boundary conditions in the  $z$  direction and  $L_z$  to be odd. Now consider an open membrane that wraps around the  $z$  direction once. The open membrane has the topology of a cylinder with two circles as its two edges. Clearly both circles contain odd numbers of links since  $L_z$  is odd. When the cylinder is very long, it looks like a string with two ends. Using the statistical algebra of Ref. 12, we find that the ends can be viewed as fermions in the  $x$ - $y$  plane. In contrast, the edges of the membrane in the first model (6) do not have such a fermionic property. It is in this sense that we call the edge of the membrane in the present model fermionic.

## VI. CONCLUSIONS

In this paper we discussed three bosonic models on three-dimensional lattices with nontrivial topological orders. All models contain  $Z_2$  string condensations, and hence are described by  $Z_2$  gauge theories at low energies. Despite this, the three models contain three different  $Z_2$  topological orders. The first model (6) contains both string and membrane condensations. The ends of condensed strings (the  $Z_2$  charges) and edges of condensed membranes (the  $Z_2$  vortex loops) are bosonic. The first model gives rise to the standard  $Z_2$  gauge theory at low energies. The second model (12) appears only to have a string condensation. The ends of strings are fermions. The third model (57) also contains both string and membrane condensations. But this time, the ends of the strings are bosonic and the edges of the membranes are fermionic. The third model gives rise to a different topological order.

String-net and membrane condensed states are not just an abstract construction, they can be physically implemented. One way to realize the string-net and membrane condensed states is through ultracold atoms confined in an optical lattice. The optic lattice has the very useful property that the confining potential of the optic lattice may depend on the atomic spins. By tuning the laser frequency, we can tune such a spin-dependent confining potential. This allows us to tune the spin-spin interaction between atoms confined in the

optical lattice. This may allow us to realize some of the Hamiltonians studied in this paper.

Frustrated spin systems appear quite commonly in nature. So another possibility is to find the string-net or membrane condensed states in those frustrated spin systems. The third type of promising system is Josephson junction arrays.<sup>16</sup> The charge on a superconducting island or the flux through a superconducting ring may play the role of the spin degree of freedom. The Josephson junction array is quite tunable. Recent progress in quantum computing has found ways to reduce the decoherence. So building a Josephson junction array to

realized the constructed Hamiltonians may also be possible.

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