ϕ_3^4 lattice field theory viewed from the high-temperature side

P. Butera* and M. Comi†

Istituto Nazionale di Fisica Nucleare and Dipartimento di Fisica, Università di Milano-Bicocca 3 Piazza della Scienza, 20126 Milano, Italy

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We analyze high-temperature series expansions of the two- and four-point correlation functions in the three-dimensional Euclidean lattice scalar field theory with quartic self-coupling, which have been recently extended through 25th order for the simple-cubic and body-centered-cubic lattices. We conclude that the length of the present series is sufficient for a fairly accurate description of the critical behavior of the model and confirm the validity of universality, scaling, and hyperscaling. In the case of the body-centered-cubic lattice, we determine the value of the quartic self-coupling for which the leading corrections to scaling approximately vanish and correspondingly the universal critical parameters can be determined with high accuracy. In particular, for the susceptibility and the correlation-length exponents we find $\gamma=1.2373(2)$ and $\nu=0.6301(2)$. For the four-point renormalized coupling we find $g=23.56(3)$. In the case of the simple-cubic lattice our results are consistent with earlier estimates.

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I. INTRODUCTION

High-temperature (HT) expansions for scalar spin models with bilinear nearest-neighbor interaction and general single-spin measure, on two- and three-dimensional bipartite lattices, have been extended[1–6] in the last few decades from order K^{10} to K^{25}. Here $K=J/kT$ denotes the usual HT expansion variable, with k the Boltzmann constant, T the temperature, and J some energy scale characterizing the next-neighbor interaction. The main observables expanded include the susceptibility χ_2, the second moment of the correlation-function μ_2, and the zero-momentum four-point function χ_4. The linked-cluster method used in these calculations expresses the series coefficients in a closed form as polynomials in the (normalized) moments of the single-spin measure, thus making the analysis of a broad class of models possible. This fact entails two related benefits. First, it enables one to explore the extent of the Ising universality class to which these models should generally belong. Second, one can take advantage of the nonuniversality of the amplitudes of the corrections to scaling in the critical behavior to obtain very accurate estimates of universal critical parameters, such as exponents and scaling functions, by focusing the numerical analysis on those particular spin models, within the given universality class, which show vanishing (or quite small) leading corrections to scaling (LCS). This prescription, which can be effective only with rather long series, was suggested in Refs. 7 and 8 and extensively tested by a 21-term HT series on the body-centered-cubic lattice, for various single-spin measures.

In the initial studies, however, the available series were simply too short. In Ref. 2 an analysis of the critical properties of the lattice ϕ^4 Euclidean field theory in two-, three- and four-dimensional space, using tenth-order HT series, suggested a complicated and puzzling critical behavior in the 3-D case, which showed failures of universality and hyperscaling and cast doubt on the validity of key mathematical assumptions[9] (see, however, Ref. 10) in the application of the renormalization group to the critical phenomena. This analysis was historically important in raising questions which could be answered only by numerical investigation of specific models and was a strong incentive to further extend the HT series and to repeat and improve simulation studies. The anomalous features observed in Ref. 2 were not confirmed by the successive studies[3,8,11] and were finally ascribed to the shortness of the series analyzed. Therefore over a decade ago the problem was considered as settled.

We have recently extended, through order K^{25}, the HT expansions of the moments of the correlation function and, through order K^{23}, the expansion of the zero-momentum four-point correlation function for a general scalar model defined on three-dimensional bipartite lattices [such as the simple-cubic (sc) and the body-centered-cubic (bcc) lattices]. These expansions have already proved useful for accurately displaying the properties of critical scaling and universality of the spin-S Ising model[5,12] and have also produced high-precision estimates of critical exponents and universal combinations of critical amplitudes, based on the prescriptions of Refs. 7 and 8. Also following the lead of the same authors,[7,8] analogous expansions, independently obtained[6] for the sc lattice case, through order 25 for χ_2 and μ_2 and through order 21 for χ_4, were used to study the lattice ϕ_3^4 model in a neighborhood of a particular value of the self-coupling for which the LCS are negligible. It remains to show that the present HT expansions are also adequate for an extensive and accurate study of the scalar field model for two different lattices and on the whole range of the self-coupling. From our study we conclude that the present HT series are sufficiently long that, by properly resumming them, our analysis can reach an accuracy comparable to or higher than the most extensive Monte Carlo simulations. Moreover, we show that also in the case of the bcc lattice, we can approximately determine a value of the quartic self-coupling such that the LCS vanish, and thereby we can obtain[7,8] very accurate estimates of the universal critical parameters in complete agreement with the other cited recent computations.[5,6]

In the following paragraphs we shall first recall the definitions of the quantities whose HT expansions are studied. We shall then very briefly comment on the results of the series analysis. We shall map out the phase diagram of the model and exhibit its universality properties along the critical line, such as the independence of the critical exponents γ and ν on the self-coupling of the field and on the lattice structure. Then, we shall give evidence that the critical renormalized coupling constant is universal and nonzero, thus verifying the validity of hyperscaling. Finally we shall report our best estimates of the critical exponents γ and ν and of the renormalized four-point coupling from the analysis of the model with minimal LCS on the bcc lattice. The figures summarizing our results are among the main motivations of this study, since some of them qualitatively differ from the analogous ones appearing in Ref. 2.

II. THE MODEL

The lattice ϕ_3^4 model is defined by the Hamiltonian

$$KH\{\phi\} = -K\sum_{\langle i,j\rangle} \phi_i\phi_j + \sum_i [\phi_i^2 + g_0(\phi_i^2-1)^2]. \qquad (1)$$

Here i and j are integer-component (multi)indices denoting the sites of a three-dimensional lattice and the first sum extends to all nearest-neighbor sites.

This Hamiltonian is obtained from the lattice discretization of the Euclidean action of the continuum real field $\psi(x)$ in $2+1$ dimensions:

$$S = \int \left(\frac{1}{2}(\partial_\mu\psi)^2 + \frac{1}{2}m^2\psi^2 + \frac{\lambda}{4!}\psi^4\right)dx \qquad (2)$$

after setting $\psi=\sqrt{2K}\phi$, $m^2=[(1-2g_0)/K]-6$ and $\lambda=6g_0/K^2$.

We will analyze the HT expansion of the moments of the two-point connected correlation function $\langle\phi_i\phi_j\rangle_c$ and of the zero-momentum connected four-point correlation-function $\chi_4(K,g_0)=\Sigma_{i,j,k}\langle\phi_0\phi_i\phi_j\phi_k\rangle_c$.

The susceptibility is defined by the zeroth-order moment of $\langle\phi_0\phi_i\rangle_c$

$$\chi_2(K,g_0) = \sum_i \langle\phi_0\phi_i\rangle_c. \qquad (3)$$

Its behavior, as K tends from below to the critical value $K_c(g_0)$, is expected to be

$$\chi_2(K,g_0) = A(g_0)\tau(g_0)^{-\gamma(g_0)}[1 + a(g_0)\tau^{\theta(g_0)} + \cdots], \qquad (4)$$

where $A(g_0)$ is the critical amplitude of the susceptibility, $\tau(g_0)=1-K/K_c(g_0)$ is called the reduced (inverse) temperature, $\gamma(g_0)$ is the critical exponent of the susceptibility, $a(g_0)$ is the amplitude of the LCS, and $\theta(g_0)$ is the exponent of the LCS.

The square of the correlation length is expressed in terms of the ratio of the second moment of the correlation function $\mu_2(K,g_0)=\Sigma_i i^2\langle\phi_0\phi_i\rangle_c$ and of the susceptibility as

$$\xi^2(K,g_0) = \frac{\mu_2(K,g_0)}{6\chi_2(K,g_0)} \qquad (5)$$

and is expected to show the critical behavior

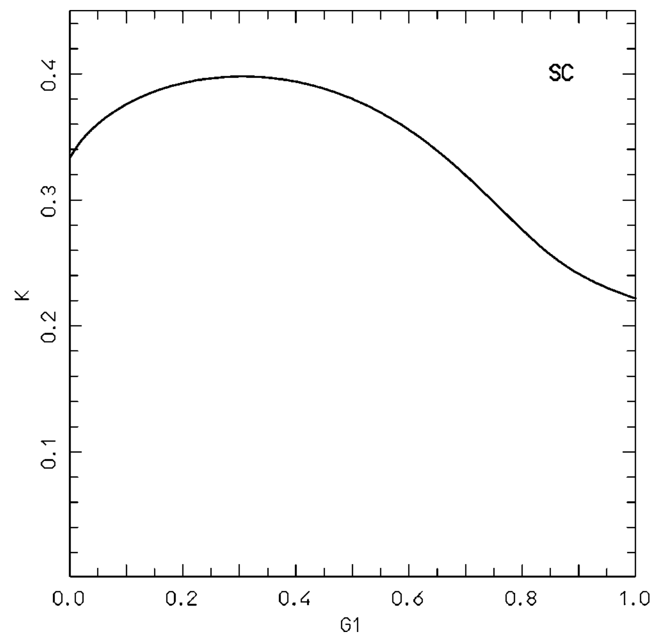

FIG. 1. Phase diagram of ϕ_3^4 on the sc lattice. The critical line $K=K(g_0)$ is plotted vs. $G1=g_0/(g_0+1)$.

$$\xi^2(K,g_0) = B(g_0)\tau(g_0)^{-2\nu(g_0)}[1 + b(g_0)\tau^{\theta(g_0)} + \cdots], \qquad (6)$$

where $\nu(g_0)$ is the critical exponent of the correlation length.

The critical renormalized coupling constant $g(g_0)$ is expressed in terms of $\chi_4(K,g_0)$, $\chi_2(K,g_0)$, and $\xi^2(K,g_0)$ as the value of the quantity

$$g(K,g_0) = \frac{-v\chi_4(K,g_0)}{\xi^{3/2}(K,g_0)\chi_2^2(K,g_0)}, \qquad (7)$$

when K tends from below to $K_c(g_0)$. Here v denotes the volume per lattice site. We have $v=1$ for the sc lattice and $v=4/3\sqrt{3}$ for the bcc lattice. The expected critical behavior of $g(K,g_0)$ is

$$g(K,g_0) = g(g_0)\tau(g_0)^{\gamma(g_0)+3\nu(g_0)-2\Delta_4(g_0)}[1 + c(g_0)\tau^{\theta(g_0)} + \cdots], \qquad (8)$$

where $\Delta_4(g_0)$ is the gap exponent. If $\gamma(g_0)+3\nu(g_0)-2\Delta_4(g_0)=0$, we say that hyperscaling holds and the critical renormalized coupling $g(g_0)$ is the finite positive[13] critical limit of $g(K,g_0)$. Of course, the exponents $\gamma(g_0)$, $\nu(g_0)$, $\Delta_4(g_0)$, and $\theta(g_0)$ and the critical renormalized coupling $g(g_0)$, as determined from the analysis of the HT series, will appear to be independent of g_0, if universality is valid along the critical line.

III. NUMERICAL RESULTS

A. The phase diagrams

In Figs. 1 and 2 we have mapped out the phase diagrams of the ϕ_3^4 model on the sc and the bcc lattices, respectively. There are two phases separated by a line $K=K_c(g_0)$ of second-order critical points. In the disordered phase, below the critical line, the reflection symmetry $\phi\rightarrow-\phi$ is unbroken.

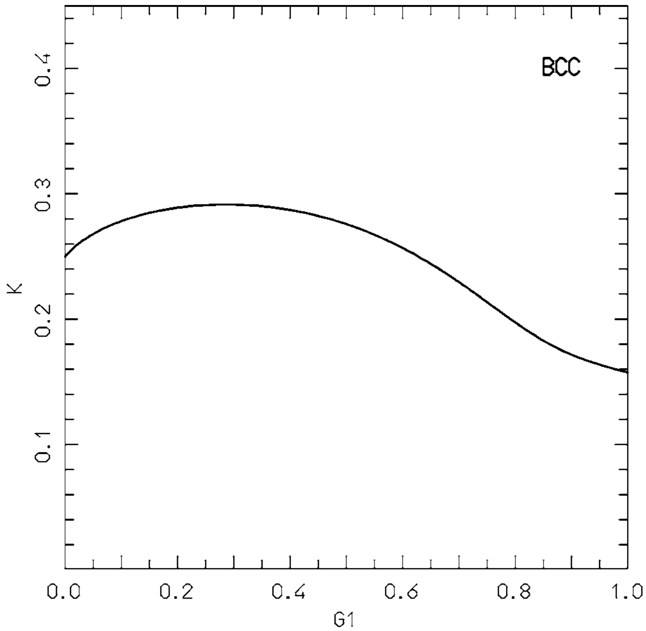

FIG. 2. Phase diagram of ϕ_3^4 on the bcc lattice. The critical line $K=K(g_0)$ is plotted vs. $G1=g_0/(g_0+1)$.

For $g_0=0$ the model reduces to the Gaussian model and we have $K_c(0)=2/q=1/3$ on the sc lattice $[K_c(0)=2/q=1/4$ on the bcc lattice$]$, where q is the lattice coordination number. For $g_0\to\infty$ the model reduces to the usual spin-1/2 Ising model and we have $K_c(\infty)=0.221\,655(2)$ on the sc lattice $[K_c(\infty)=0.157\,372\,5(10)$ on the bcc lattice$]$ as indicated in our general study[5] of the spin-S Ising model. In Table I we have reported numerical estimates of $K_c(g_0)$ for a few values of $0<g_0<\infty$ in the sc and bcc lattice cases. These estimates are obtained by the following simple procedure. Using the susceptibility expansion, for each value of g_0, we form the sequence $\{K_c^{(n)}(g_0)\}$ of Zinn-Justin modified-ratio approximants (MRA)[7,14] of the critical point. We observe that these sequences approach smoothly their expected asymptotic behavior[5]

$$K_c^{(n)}(g_0)=K_c(g_0)[1-C(\gamma,\theta)a(g_0)/n^{1+\theta(g_0)}+o(1/n^{1+\theta(g_0)})], \tag{9}$$

where $a(g_0)$ and $\theta(g_0)$ are, respectively, the amplitude and the exponent of the LCS appearing in Eq. (4) and $C(\gamma,\theta)$ is a positive constant[5] depending on the values of $\gamma(g_0)$ and $\theta(g_0)$. We expect that the exponent $\theta(g_0)$ be universal, namely independent of g_0 and, indeed, to a good approximation, the asymptotic behavior of the MRA sequences is consistent with the value $\theta=0.517(4)$ suggested by a recent simultaneous study[15] of a set of models in the Ising universality class. Since, for all values of g_0, the contributions of the higher-order terms in Eq. (9) appear to be small, particularly so in the bcc lattice case, we shall assume that the MRA sequences $\{K_c^{(n)}(g_0)\}$ can be extrapolated to infinite length n of the series simply by fitting the highest-order (alternate) approximants to the first two terms of the asymptotic expansion (9). The values of $K_c(g_0)$ determined by this prescription are quite stable, both under small variations of θ and of the fitting procedure, such as including in the fit higher-order terms of the expansion (9). Our conclusion is that at least five decimal figures of the result are reliable. These estimates are then refined by comparing the results of the extrapolations with the determination of $K_c(g_0)$ from first-, second- and third-order differential approximants(DA)[14,16] and finally error bars are inferred which are likely to be only upper bounds. As expected, these uncertainties, which reflect the differences between the estimates by extrapolated MRA sequences and by DAs, depend strongly on the size of the corrections to scaling and therefore on the value of g_0, but of course they are generally invisible on the scale of Figs. 1 and 2.

TABLE I. $K_c(g_0)$ for a few values of g_0 in the case of the sc and bcc lattices.

g_0	$K_c^{sc}(g_0)$	$K_c^{bcc}(g_0)$
0.000	0.333 333 3	0.250 000
0.010	0.340 55(1)	0.254 71(1)
0.050	0.359 40(1)	0.267 24(1)
0.100	0.373 40(1)	0.276 55(1)
0.150	0.382 38(1)	0.282 42(1)
0.200	0.388 42(1)	0.286 27(1)
0.250	0.392 50(1)	0.288 76(1)
0.300	0.395 17(1)	0.290 29(1)
0.350	0.396 782(4)	0.291 08(1)
0.400	0.397 585(3)	0.291 31(1)
0.500	0.397 397(3)	0.290 55(1)
0.650	0.394 135(3)	0.287 39(1)
0.760	0.390 343(3)	0.284 131(4)
0.900	0.384 503(3)	0.279 835(3)
1.060	0.377 035(2)	0.273 875(3)
1.100	0.375 096(2)	0.271 844(2)
1.140	0.373 137(3)	0.270 306(2)
1.300	0.365 220(3)	0.264 137(2)
1.600	0.350 584(3)	0.252 883(1)
1.800	0.341 319(3)	0.245 830(1)
1.850	0.339 085(3)	0.244 135 7(5)
2.000	0.332 596(3)	0.239 230(1)
3.000	0.297 990(3)	0.213 328(1)
4.100	0.274 459(3)	0.195 919(1)
5.000	0.262 538(3)	0.187 155(2)
6.400	0.251 303(3)	0.178 926(2)
10.000	0.238 853(4)	0.169 848(2)
15.000	0.232 587(4)	0.165 296(2)
45.000	0.225 112(4)	0.159 875(2)

B. The critical exponents

In Figs. 3 and 4, referring to the sc and the bcc lattices, respectively, we have plotted our estimates of the exponent

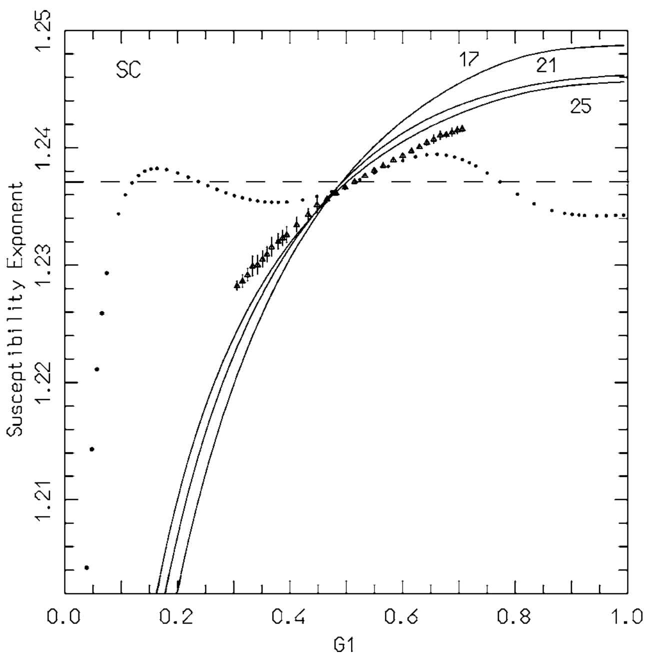

FIG. 3. The susceptibility exponent $\gamma(g_0)$ as a function of $G1=g_0/(g_0+1)$ for the sc lattice. The continuous curves represent the estimates obtained by the MRAs of orders 17, 21, and 25, respectively. The dashed curve indicates the central value of our estimate (Ref. 5) of γ for the Ising universality class. The dotted curve shows the results of an extrapolation of the MRA sequences by a fit to the asymptotic expansion (10) supplemented by a single additional higher order correction $O(1/n^4)$. The triangles show estimates of $\gamma(g_0)$ by unbiased first-order DAs.

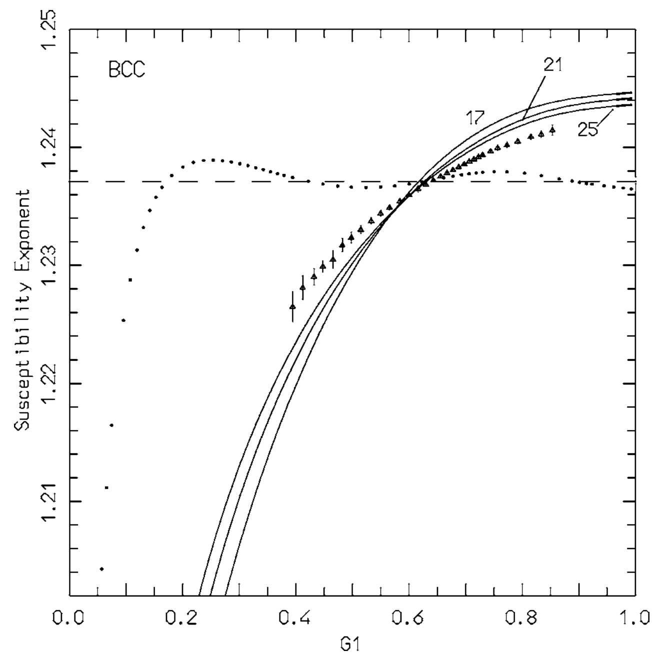

FIG. 4. The same as in Fig. 3, but for the bcc lattice.

of the susceptibility $\gamma(g_0)$ vs. $G1=g_0/(g_0+1)$. Also in this case, using the susceptibility HT expansion, for each value of g_0, we can form the sequences $\{\gamma^{(n)}(g_0)\}$ of MRAs of the exponent $\gamma(g_0)$ and compare them to their expected asymptotic expansion[5]

$$\gamma^{(n)}(g_0) = \gamma(g_0) - D(\gamma,\theta)a(g_0)/n^{\theta(g_0)} + o(1/n^{\theta(g_0)}). \quad (10)$$

Here $D(\gamma,\theta)$ is a positive constant[5] depending on $\gamma(g_0)$ and $\theta(g_0)$ and $a(g_0)$ is the amplitude of the LCS in Eq. (4). We can observe that also these MRA sequences are smooth, but a simple minded extrapolation based on a fit of the sequences $\{\gamma^{(n)}(g_0)\}$ to the first two terms of the asymptotic expansion (10) cannot lead to very accurate estimates of $\gamma(g_0)$, because, even at the present order of expansion, the contribution of higher-order corrections is not sufficiently small. In Figs. 3 and 4, we have represented by solid lines the estimates $\gamma(g_0)$ obtained from the MRAs $\gamma^{(17)}(g_0)$, $\gamma^{(21)}(g_0)$, and $\gamma^{(25)}(g_0)$, which use the susceptibility series only up to the orders 17, 21, and 25, respectively. These estimates show a rapid crossover from the Gaussian value $\gamma(0)=1$ to a behavior which, for $g_0>0$, tends to become independent of g_0 and to approach the dashed line in the figure indicating the central value $\gamma=1.2371$ of our estimate in Ref. 5 for the Ising universality class. If we include higher correction terms in the asymptotic expansion (10), somewhat more accurate estimates of $\gamma(g_0)$ can be obtained. The simplest possibility is to introduce a single effective higher correction $O(1/n^s)$ with g_0-independent exponent s. Choosing the value $s=4$, we obtain the estimates represented by the dotted curve. The accuracy of the approximation can be further improved at the expense of the simplicity of the fitting procedure, but here it is sufficient to indicate only the qualitative trend. In the same figures, we have also reported estimates of $\gamma(g_0)$ obtained by simple first-order unbiased DAs using all available series co-

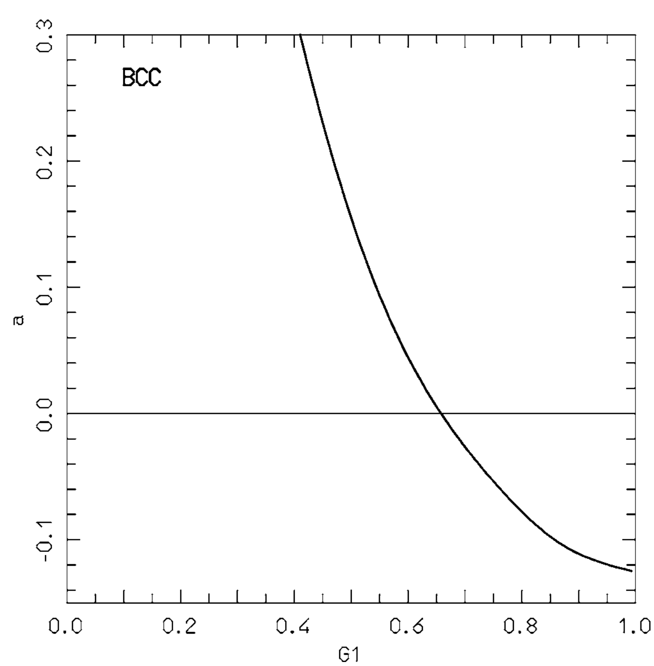

FIG. 5. The amplitude $a(g_0)$ of the leading correction to scaling in (4) vs. $G1=g_0/(g_0+1)$, as obtained from a fit to the asymptotic expansion (9) supplemented by a single higher-order correction $O(1/n^{6+\theta})$ in the bcc lattice case.

TABLE II. The HT expansions of the susceptibility χ_2, the second moment of the correlation function μ_2, and the zero-momentum four-point function χ_4 at $g_0=1.85$ on the bcc lattice.

Order	χ_2	μ_2	χ_4
0	0.607 989 217 187 788 034 965	0.000 000 000 000 000 000 000	−0.530 149 722 377 688 222 012
1	2.957 207 105 732 954 317 729	2.957 207 105 732 954 317 729	−10.314 410 070 423 482 079 15
2	13.094 299 007 411 010 484 93	28.767 200 532 427 891 489 65	−116.861 966 354 011 135 934 3
3	57.793 235 691 488 110 468 67	197.714 417 550 647 733 790 2	−1037.709 279 795 438 675 000
4	248.217 922 950 611 561 565 1	1185.408 945 561 549 742 349	−7930.113 134 914 392 838 073
5	1064.858 590 911 481 926 754	6551.599 918 767 001 108 148	−54 882.228 300 207 536 464 24
6	4512.534 104 619 384 245 591	34 421.021 315 133 593 395 22	−353 349.351 394 343 652 961 6
7	19 111.259 511 559 953 208 31	174 321.717 203 564 550 926 9	−2 155 038.392 864 457 644 076
8	80 383.219 481 734 788 004 67	859 681.096 255 125 245 363 7	−12 593 879.170 232 762 669 10
9	337 958.277 647 198 316 812 7	4 151 521.352 493 417 270 411	−71 122 511.167 490 721 434 98
10	1 414 677.216 907 545 784 892	19 721 977.308 240 405 664 87	−390 463 929.572 442 747 952 6
11	5 919 934.511 772 996 630 729	92 423 661.550 596 365 661 34	−2 093 673 545.070 737 911 244
12	24 698 142.582 260 792 997 01	428 322 758.435 417 967 997 2	−11 002 848 880.383 865 245 75
13	103 016 201.643 607 322 343 6	1 966 128 942.161 860 887 810	−56 833 474 652.325 228 433 81
14	428 735 652.964 013 824 808 4	8 952 466 557.867 528 601 072	−289 181 057 575.143 246 457 5
15	1 783 970 290.009 394 230 477	40 476 319 807.178 661 887 75	−1 452 144 744 869.861 166 239
16	7 410 672 453.259 276 817 146	181 885 433 500.913 180 229 6	−7 207 354 502 435.721 114 810
17	30 779 127 150.295 103 731 62	812 879 825 813.359 874 423 9	−35 401 801 340 419.050 978 54
18	127 668 343 287.704 088 852 2	3 615 487 616 835.276 408 009	−172 273 869 711 328.586 515 2
19	529 480 497 190.951 194 230 4	16 011 252 152 185.131 508 69	−831 302 587 175 201.555 232 9
20	2 193 593 476 094.224 975 766	70 632 026 759 664.657 402 19	−3 980 914 502 340 770.007 282
21	9 086 796 325 630.050 694 943	310 488 939 709 344.326 094 0	−18 931 663 273 718 966.908 98
22	37 608 562 966 496.064 399 66	1 360 524 482 339 925.373 685	−89 460 744 874 166 538.460 43
23	155 638 849 919 474.903 498 3	5 944 225 695 917 982.872 040	−420 283 001 120 399 961.613 9
24	643 624 490 532 272.253 215 9	25 901 506 033 628 787.278 95	
25	2 661 386 657 660 504.869 714	112 585 550 276 346 019.871 7	

efficients. In order to keep the figures readable, these data are restricted to a smaller range of values of g_0 of particular interest. Completely analogous results are obtained starting with the HT series for the correlation-length squared ξ^2 to determine the critical exponent ν but, for brevity, they will not be reported here.

C. Leading corrections to scaling and accurate estimates

In Fig. 5 we have plotted vs. g_0, the amplitude $a(g_0)$ of the LCS in (4), as obtained by fitting the MRA sequence $\{K_c^{(n)}(g_0)\}$ to the asymptotic expansion (9) in the bcc lattice case. Of course, the determination of $a(g_0)$ is much more sensitive than $K_c(g_0)$ to the inclusion of higher-order corrections in the fit to the expansion (9). The most convincing results are obtained when a single additional correction $O(1/n^{6+\theta})$ is included. This three-parameter fit changes the results of the two-parameter fit for $K_c(g_0)$ only within the estimated uncertainties and does not alter the qualitative structure of $a(g_0)$, but only brings its evaluation into closer agreement with the corresponding estimate of $a(g_0)$ from a similarly improved fit to Eq. (10) and with the values of $a(\infty)$ obtained in earlier works.[3,5] The uncertainty of the final results should not exceed 5%–10%. In the case of the sc lattice, the behavior of $a(g_0)$ is similar, but $a(g_0)$ cannot be determined with comparable accuracy by this straightforward method because it is even more sensitive to the presence of higher-order corrections in (9). The important fact is, however, that, in both the sc and the bcc lattice cases, when g_0 increases from 0 to ∞, the amplitude $a(g_0)$ of the LCS in Eq. (4) varies from positive to negative values. In the sc lattice case $a(g_0)$ vanishes at $\hat{g}_0^{sc} \approx 1.10(2)$. We recall that the above cited numerical analysis of the ϕ_3^4 HT series on the sc lattice[6] was indeed performed precisely at $g_0=1.10$. In the bcc lattice case we find that the zero of $a(g_0)$ occurs at $\hat{g}_0^{bcc} \approx 1.85(5)$. In Ref. 6 $\hat{g}_0^{sc}$ had been determined by an appropriate Monte Carlo simulation. In our approach no preliminary Monte Carlo simulation is needed to determine $\hat{g}_0^{bcc}$, since, knowing the structure of the function $a(g_0)$, our study of the model on the whole range of values of g_0 directly gives a sufficiently accurate indication. For g_0 in a neighborhood of the zero of the LCS amplitude, we can observe that the various approximations of $\gamma(g_0)$, shown in Figs. 3 and 4, tend to coincide and come nearest to our earlier estimate[5] for the spin S Ising model: $\gamma=1.2371(1)$ shown in the figure by a dashed line. Equation (10) and the measured behavior of $a(g_0)$ simply explain why, for $g_0 < \hat{g}_0^{sc}$ (or $g_0 < \hat{g}_0^{bcc}$ in the bcc lattice case),

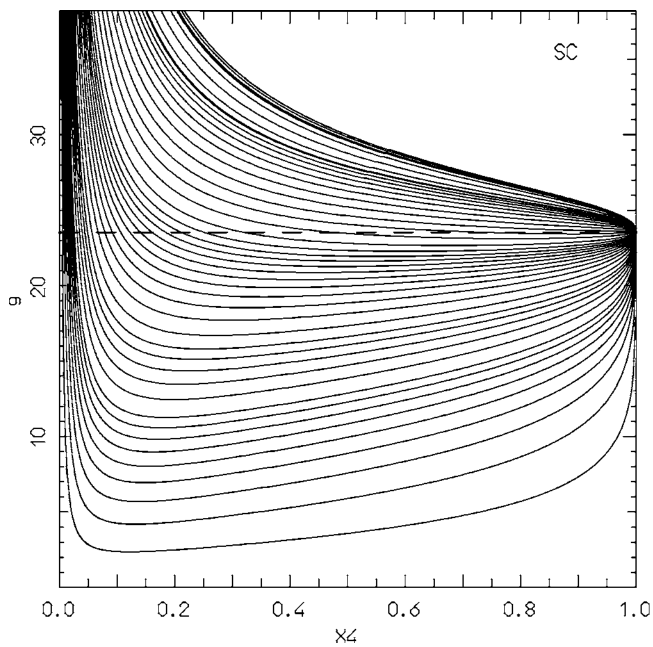

FIG. 6. The renormalized coupling $g(\xi^2, g_0)$ as a function of $X4=\xi^2/(\xi^2+4)$ for several fixed values of the ϕ^4 self-coupling g_0 on the sc lattice. The values of g_0 increase from the lowest curve up. The dashed line represents the central value of our estimate $g=23.52(5)$, obtained in Ref. 5, of the renormalized coupling in the Ising universality class.

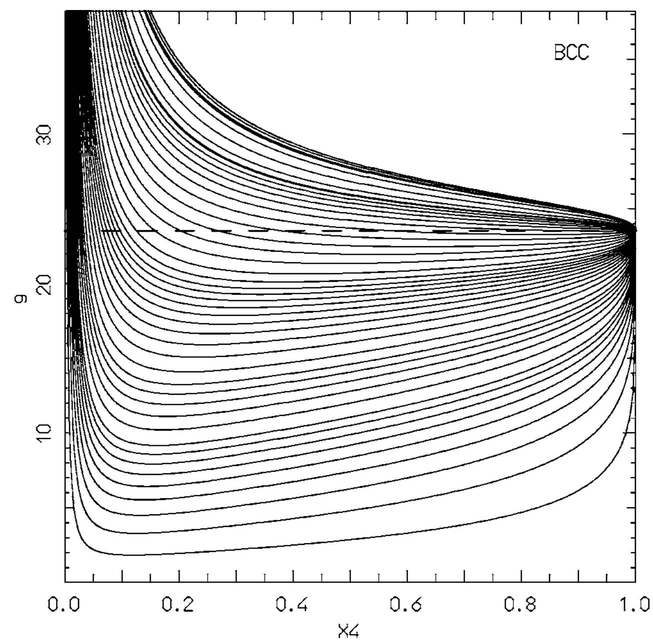

FIG. 7. The same as in Fig. 6, but for the bcc lattice.

the MRAs of Figs. 3 and 4 approach their constant limiting value from below and otherwise from above. Of course, the same mode of approach to the limit can be observed for the estimates obtained from DAs, when varying the number of HT series coefficients used in the approximants; however, for clarity we have reported in the figures only the highest-order results.

We can now take advantage of our estimate of $\hat{g}_0^{bcc}$ to compute some universal critical parameters of the ϕ_3^4 model on the bcc lattice with approximately vanishing LCS. The HT series coefficients of χ_2, μ_2, and χ_4 on the bcc lattice, for $g_0=\hat{g}_0^{bcc}$, are reported in Table II. (The corresponding expansions in the sc lattice case, for $g_0=\hat{g}_0^{sc}$, can be found in Ref. 6.) Very accurate estimates are thus obtained for the critical exponents and the critical renormalized coupling. In particular, using first- and second-order DAs [either unbiased or biased with the critical value $K_c(\hat{g}_0^{bcc})=0.244\,135\,7(5)$], we find $\gamma=1.2373(2)$, $\nu=0.6301(2)$, and $g=23.56(3)$ in good agreement with our previous estimates[5] and with the results of Ref. 6. Our estimated errors account also for the $\approx 3\%$ uncertainty in the estimate of $\hat{g}_0^{bcc}$.

Since our HT series for $g(K, g_0)$ is two terms longer, we have also repeated the analysis of Ref. 6 with our sc lattice series computed at $\hat{g}_0^{sc}=1.10$. The final estimates, $\gamma=1.2372(2)$, $\nu=0.6301(2)$, and $g=23.56(4)$, evaluated at $K_c(\hat{g}_0^{sc})=0.375\,097(1)$, are consistent both with our results for the bcc lattice and with those of Ref. 6.

D. The renormalized coupling

In Figs. 6 and 7, which refer to the sc and the bcc lattices, respectively, we have plotted the renormalized four-point coupling constant $g(\xi^2, g_0)$ vs. $X4=\xi^2/(\xi^2+4)$ for several fixed values of g_0. For fixed g_0 and $\xi^2 \to \infty$, all curves appear to tend to a g_0- and lattice-independent limiting value $g(\infty, g_0)$ which is consistent with our estimate in Ref. 5 of the critical renormalized coupling in the Ising universality class $g=23.52(5)$ indicated in the figure by the dashed line. This

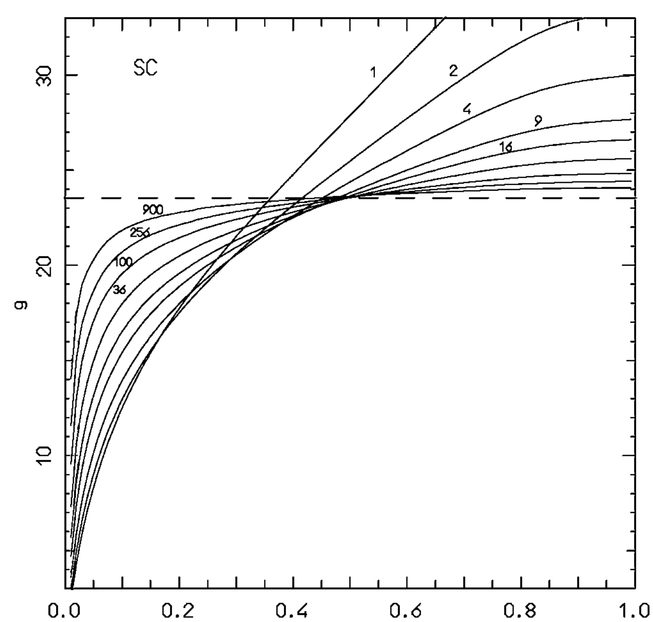

FIG. 8. The renormalized coupling $g(\xi^2, g_0)$ as a function of $G1=g_0/(g_0+1)$ for several values of the correlation length square ξ^2 on the sc lattice. The values of ξ^2 are indicated beside the corresponding curves. The dashed line represents the central value of our estimate $g=23.52(5)$, obtained in Ref. 5, of the renormalized coupling in the Ising universality class.

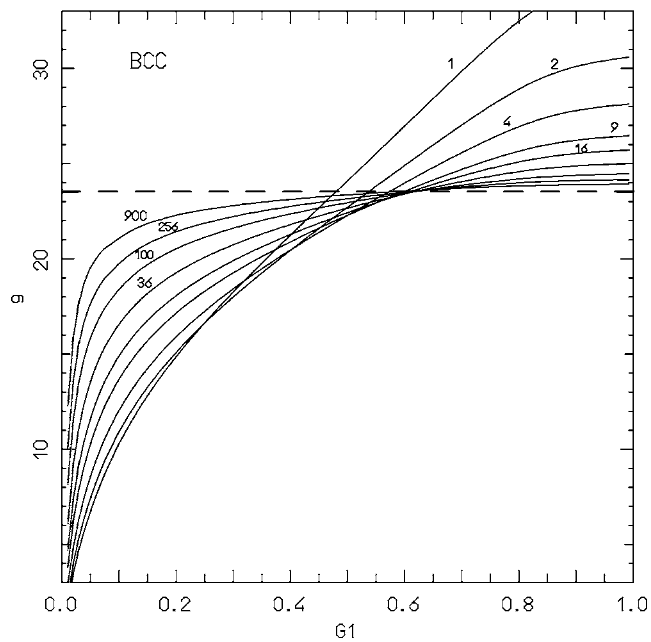

FIG. 9. The same as in Fig. 8, but for the bcc lattice.

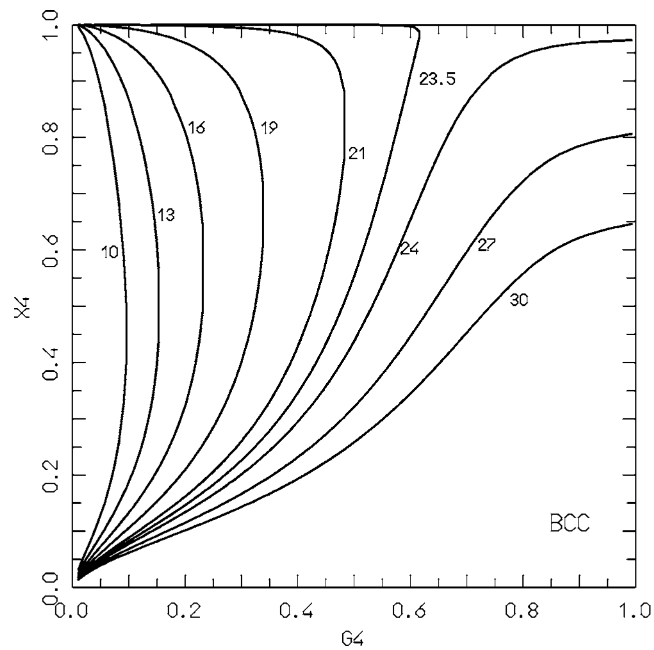

FIG. 11. The same as in Fig. 10, but for the bcc lattice.

fact suggests that the critical renormalized coupling is universal with respect to the self-coupling g_0 and to the structure of the lattice and moreover that it is nonzero (so that hyperscaling is valid). The curves in Figs. 6 and 7 are obtained in a most straightforward way from the highest-order nondefective "simplified differential approximants" (SDA)[17] biased with the values of θ and $K_c(g_0)$. Completely equivalent results are obtained also, slightly more laboriously, using ordinary biased DAs instead of SDAs. The estimated width of the error bars is comparable to the thickness of the lines. Also for $g(\xi^2,g_0)$, we can observe that, as g_0 varies from 0 to ∞, the amplitude $c(g_0)$ of the LCS in (8), which is negative for small g_0, changes sign at $\hat{g}_0$, as it should, because the ratio $c(g_0)/a(g_0)$ is expected to be universal. Correspondingly, as shown clearly by these figures, the curves tend to their common limiting value from below if $g_0<\hat{g}_0$ and otherwise from above.

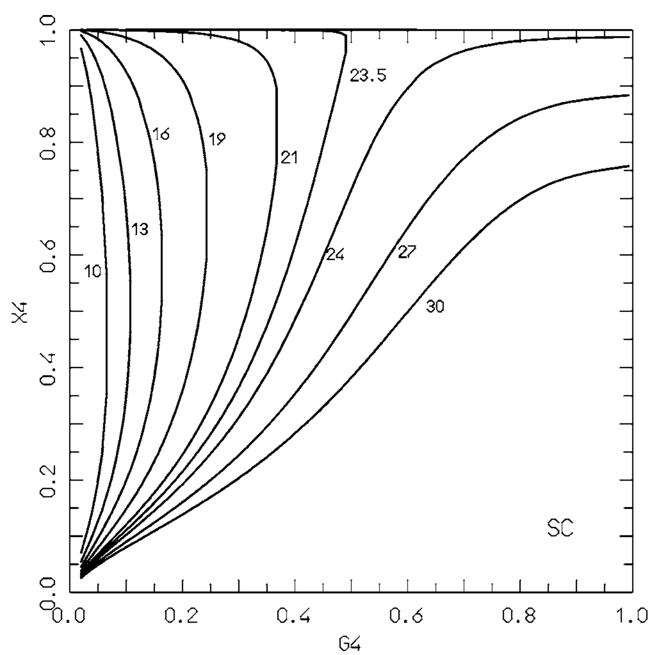

FIG. 10. Contour plot of the renormalized coupling $g=g(\xi^2,g_0)$ in the G_4, X_4 plane for the sc lattice. Here $G_4=g_0/(g_0+4)$ and $X_4=\xi^2/(\xi^2+4)$. The values of $g(\xi^2,g_0)$ are shown beside the corresponding level curves.

Figures 8 and 9, for the sc and the bcc lattices, respectively, offer a different view of the same data. Here the renormalized coupling $g(\xi^2,g_0)$ is plotted vs. $G1=g_0/(g_0+1)$ for several fixed values of ξ^2. All curves are monotonically increasing in g_0 and show a rapid crossover from the Gaussian value $g(\xi^2,0)=0$ to a shape increasingly flatter as ξ^2 becomes large. For $g_0<\hat{g}_0$, as $\xi^2\to\infty$, the sequence of curves tends from below to the constant $g(\infty,g_0)$; for $g_0>\hat{g}_0$ the same value is approached from above. We have already stressed that this pattern of behavior simply reflects the fact that the amplitude $c(g_0)$ of the LCS in $g(K,g_0)$ is negative for $g_0<\hat{g}_0$ and positive for $g_0>\hat{g}_0$. Here we should also recall that in a similar plot reported in Ref. 2 (only the case of the bcc lattice case is discussed in that study) the curves show quite a different structure for $\xi^2>64$. In particular, the curves $g(\xi^2,g_0)$ in Ref. 2 are not monotonic in g_0 and their limiting behavior as $\xi^2\to\infty$ does not appear to be universal and nonzero. We can see no reason for these anomalous features other than the insufficient length of the ten-term HT series used in the analysis of Ref. 2.

One more view of the same data is presented in Figs. 10 and 11, for the sc and the bcc lattices, respectively, showing contour plots of $g(\xi^2,g_0)$ in the $\hat{\xi}^2=\xi^2/(\xi^2+1)$, $\hat{g}_0=g_0/(g_0+1)$ plane. Also these figures are qualitatively different from the corresponding ones of Ref. 2 which show a spurious saddle point structure and suggest a failure of universality. Nothing like that can be inferred from our updated figures.

E. Conclusions

In conclusion, we have analyzed HT expansions for a one-parameter family of continuous spin models which interpolate between the Gaussian and the spin-1/2 Ising models. In the case of the bcc lattice, we have taken advantage of the parameter dependence of the amplitudes of the LCS to improve the accuracy in the determination of some universal critical quantities. Moreover, we have shown that, in the light of our extended series, all puzzling and unexpected features which emerged from the old HT analysis of Ref. 2 can only be ascribed to numerical inaccuracies deriving from the use of too short series.

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*Electronic address: butera@mib.infn.it
†Deceased

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