Transition to zero resistance in a two-dimensional electron gas driven with microwaves

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High-mobility two-dimensional electron systems in a perpendicular magnetic field exhibit zero-resistance states (ZRSs) when driven with microwave radiation. We study the nonequilibrium phase transition into the ZRS using phenomenological equations of motion to describe the electron current and density fluctuations in the presence of a magnetic field. We focus on two models to describe the transition into a time-independent steady state. In model I the equations of motion are invariant under a global uniform change in the density. This model is argued to describe physics on small length scales where the density does not vary appreciably from its mean. The ordered state that arises in this case spontaneously breaks rotational invariance in the plane and consists of a uniform current and a transverse Hall field. We discuss some properties of this state, such as stability to fluctuations and the appearance of a Goldstone mode associated with the continuous symmetry breaking. Using dynamical renormalization group techniques, we find that with short-range interactions this model can admit a continuous transition described by mean-field theory, whereas with long-range interactions the transition is driven first order. In model II, we relax the invariance under global density shifts as appropriate for describing the system on longer length scales, and in this case we predict a first-order transition with either short- or long-range interactions. We discuss implications for experiments, including a possible way to detect the Goldstone mode in the ZRS, scaling relations expected to hold in the case of an apparent continuous transition into the ZRS, and a possible signature of a first-order transition in larger samples. Our framework for describing the phase transition into the ZRS also highlights the connection of this problem to the well-studied phenomenon of "bird flocking."

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I. INTRODUCTION

High-mobility two-dimensional electron gases (2DEGs) subject to a perpendicular magnetic field exhibit remarkable physics when driven with microwave radiation. Zudov *et al.*[1] first demonstrated that the longitudinal resistance develops dramatic radiation-induced oscillations at low temperatures ($T \sim 1$ K) and low magnetic fields ($B \lesssim 1$ kG). These oscillations are periodic in $1/B$, with the period set by the ratio of the microwave and cyclotron frequencies. The more spectacular observation, made independently by Mani *et al.*[2] and Zudov *et al.*,[3] is that in even higher-mobility samples the oscillations become sufficiently large that the minima of the resistance oscillations develop into zero-resistance states—the measured resistance vanishes within experimental accuracy over a range of magnetic fields and radiation intensities. Subsequent experiments have confirmed their results[4,6,7] and also observed a similar effect in Corbino samples,[8] where zero-conductance states have been measured. In contrast to the longitudinal resistance, the (transverse) Hall resistance is nearly unaffected by the microwaves,[2,3] although small radiation-induced Hall oscillations have recently been observed.[4,5]

On the theoretical front, several groups have carried out microscopic calculations of the resistance taking into account radiation-induced transitions between Landau levels in the presence of impurities[9–14] or radiation-induced changes in the electron distribution function.[15,16] Both sets of calculations capture the resistance oscillations with the correct period and phase at low radiation intensity. At higher intensities, however, these calculations predict a *negative* resistance in regions of magnetic field where the experiments find a zero-resistance state.

The missing ingredient needed to connect the microscopic theory with the experiments was pointed out by Andreev *et al.*—namely, a state characterized by a negative longitudinal resistance, quite independent of its microscopic origin, is unstable to current fluctuations.[17] They argued that this instability leads to an inhomogeneous state where the system spontaneously develops domains of current with magnitude j_0, where j_0 corresponds to a vanishing longitudinal resistivity, i.e., $\rho_D(j_0^2)=0$. Applying an external current then merely reorganizes the domains in order to accommodate the additional current, leading to zero measured resistance over a range of bias current as observed experimentally. Since the system has a large Hall resistance, this picture indicates that in the absence of a current bias the spontaneous current domains in the zero-resistance state should reveal themselves through spontaneous Hall voltages transverse to the domains. Willett *et al.*[7] have indeed measured spontaneous voltages between internal and external contacts with no applied current, which lends support to this idea.

The experiments together with the microscopic calculations and phenomenological arguments provide strong evidence for the existence of a *nonequilibrium* phase transition from a normal state with nonzero resistance to a zero-resistance state whose detailed properties remain largely unexplored. In this paper we attempt to gain an understanding of the nature of this transition, and to learn about the properties of the zero-resistance state, going beyond the important initial step taken by Andreev *et al.*[17] Indeed, the work in Ref. 17 provides only a description of the *local* physics, while a theory capable of treating fluctuations, noise, and inhomogeneities is needed for a more comprehensive understanding of the long-distance physics seen experimentally. We introduce a symmetry-based hydrodynamic approach that allows one to incorporate such effects in a systematic way. As is well known, this technique exposes universal, instrinsic physics, and has proven extremely successful for a broad range of problems including liquid crystals,[18] superfluids, magnets,[19] and nonequilibrium flocks of birds.[20,21] We believe that the approach developed here can similarly be utilized to obtain a detailed understanding of the nonequilibrium zero-resistance transition as well as the nature of the ordered state.

As a first step, guided by symmetries and conservation laws we construct and analyze equations of motion for the current and density fluctuations, including the effects of white noise. Even in this relatively simple setting (i.e., without disorder), the theory is rather rich, and can describe a number of different ordered states. We focus on the transition to the ordered state that appears to be relevant experimentally, namely, a steady state with current domains. Under a physically reasonable assumption, we demonstrate that the transition to such a state can be described in terms of *only* the density modes, without explicit reference to current fluctuations. Applying renormalization group methods to several pertinent models with increasing generality, we show that generically these modes lead to a *first-order* transition. Interestingly, as we show, the zero-resistance transition is closely related to the nonequilibrium "flocking" transition,[20,21] and the theories describing these very different systems become identical in the limit of zero magnetic field and short-range electron-electron interactions. Recent simulations of the "flocking" model also suggest a first-order transition.[22]

A. Strategy

We begin with the observation that while the microscopic mechanism for how radiation induces the transition to a zero-resistance state is a matter of some debate, this knowledge is not crucial for studying universal properties close to the phase transition. Indeed, in order to study the long-wavelength, low-frequency dynamics near the transition it is sufficient to identify the appropriate hydrodynamic variables and construct the most general local equations of motion for them consistent with symmetries and conservation laws. The magnetic field, temperature, microwave radiation, and quantum effects will determine the various parameters of this theory; these may be calculated in principle from a microscopic approach, but we do not attempt to do this here. Our idea will be to view the equations of motion as a nonequilibrium analog of Landau-Ginzburg-Wilson theory. We will use them to study universal physics near the phase transition, going beyond mean-field theory by including nonlinearities and fluctuations within a renormalization group framework.

In the vicinity of the transition into the zero-resistance state in the 2DEG, the relevant hydrodynamic degrees of freedom are the current density $\mathbf{j}(\mathbf{r},t)$ and the charge density $n(\mathbf{r},t)$, which are constrained by a continuity equation

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = 0 \tag{1}$$

that enforces local charge conservation.

The dynamics of the current density $\mathbf{j}(\mathbf{r},t)$ is governed by a nonequilibrium equation of motion (akin to the Navier-Stokes equation) for a 2D charged fluid in a perpendicular magnetic field. Because of the nonequilibrium nature of the system (microwave-driven 2D electron liquid) the equation for $\mathbf{j}$ includes nonconservative forces, i.e., those not derivable from a free-energy functional. Hence, the generic symmetry-allowed form of the equation for the current density is only restricted by the translational and rotational invariances in the plane. Keeping the leading order (at long length and time scales) terms in powers of the charge and current densities and their gradients leads to

$$\begin{aligned}\omega_0^{-1}\partial_t^2\mathbf{j} + \partial_t\mathbf{j} = &-r\mathbf{j} - u|\mathbf{j}|^2\mathbf{j} + \eta_1\nabla^2\mathbf{j} + \eta_2\nabla(\nabla\cdot\mathbf{j}) - \eta_3\nabla^4\mathbf{j}\\ &- \eta_4\nabla^3(\nabla\cdot\mathbf{j}) + \tilde{\omega}_c\hat{\mathbf{z}}\times\mathbf{j} - \mu\nabla\Phi - \nu_1(\mathbf{j}\cdot\nabla)\mathbf{j}\\ &- \nu_2\nabla\mathbf{j}^2 - \nu_3(\nabla\cdot\mathbf{j})\mathbf{j} + \gamma_1\Phi\nabla\Phi + \gamma_2\Phi\mathbf{j}\\ &+ \gamma_3\Phi\hat{\mathbf{z}}\times\mathbf{j} + \boldsymbol{\zeta} + \cdots .\end{aligned} \tag{2}$$

As we will discuss in more detail below, terms appearing on the right-hand side of the above equation are forces that determine the local acceleration $[\partial_t\mathbf{j}(\mathbf{r},t)]$ of the electron fluid, each having a simple physical interpretation. The r and u terms are the linear and nonlinear longitudinal resistivities (frictional drag forces on the electron fluid). The η_i terms describe viscous forces associated with a nonuniform flow and the $\tilde{\omega}_c$ term is the Lorentz force on the charged moving electron fluid. The ν_i terms are convectivelike nonlinearities, where the absence of Galilean invariance permits more general types of convective terms ν_2 and ν_3 in addition to the conventional ν_1 term, with generic (symmetry-unrestricted) values of these couplings.

Here, the potential $\Phi \equiv \Phi[n]$ is determined by the density via

$$\Phi = \int_{\mathbf{r}'} V(\mathbf{r}-\mathbf{r}')n(\mathbf{r}'). \tag{3}$$

For long-range interactions, $V(\mathbf{r}-\mathbf{r}') \sim 1/|\mathbf{r}-\mathbf{r}'|$ is the Coulomb potential. For a screened interaction, we can set $V(\mathbf{r}-\mathbf{r}') \approx \delta(\mathbf{r}-\mathbf{r}')$, so that $\Phi \approx n$. With this, the μ and γ_1 terms incorporate Fick's law (diffusion down a local chemical potential gradient), with the latter accounting for a density-dependent diffusion coefficient. Similarly, γ_2 and γ_3 account for the lowest-order density dependence of the linear resistivity and the Lorentz force.

In addition, we have included in Eq. (2) a zero-mean white noise force $\boldsymbol{\zeta}$ with a correlator

$$\langle \zeta^{\alpha}(\mathbf{r},t)\zeta^{\beta}(\mathbf{r}',t')\rangle = 2g\,\delta^{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{4}$$

Apart from thermal noise, this incorporates the effect of microscopic fluctuations that arise from the coarse graining implicit in our formulation. Since we are dealing with a system far from equilibrium, the strength g of the noise is not fixed by the fluctuation-dissipation relation, but is an independent quantity.

Focusing on the terms $\partial_t \mathbf{j} = -r\mathbf{j} + \boldsymbol{\zeta}$ in Eq. (2), it is clear that (i) for large positive values of r, the zero-current state is stable and current fluctuations decay exponentially, while (ii) for large negative values of r, current fluctuations grow exponentially and the zero-current state is unstable. Thus, as r changes from positive to negative (in the experiments tunable by a microwave power and/or frequency), Eqs. (1)–(4) describe the phase transition from a conventional resistive state for $r>0$ to a nonequilibrium steady state with spontaneous currents for $r<0$.

As we will argue in Sec. II, this set of equations can potentially describe various types of current and density ordering, including circulating current states and domain patterns of current and density. In this paper our main focus will be on the nature of the transition into a time-independent steady state with possible density and current domains since, given the observations of Willett *et al.*, this appears to be relevant to the 2DEG experiments. We defer to future work questions regarding the detailed nature of the ordered state in this case, as well as a study of the phase transition into the circulating state.

A quite different theoretical motivation for studying this problem arises from the observation that the current and density evolution equations studied here reduce, for $B=0$ and short-range interactions, to the continuum equations used to investigate the problem of "flocking."[20,21] In that case, the system has been shown to develop an expectation value for the particle current, thus spontaneously breaking the continuous rotational symmetry even in two spatial dimensions. This is particularly striking since the Mermin-Wagner theorem[23] forbids such symmetry breaking in $d=2$ for classical equilibrium systems. This "violation" was identified as arising from nonlinear convective terms which are only allowed in nonequilibrium systems, and turn out to be relevant for this problem in dimensions $d<4$. Much is known about the universal dynamics in the flocking state in $d=2$, but the nature of the phase transition into this state has not been addressed analytically. The question we study is equivalent to asking: What is the fate of the flocking transition and the flocking state in two dimensions in the presence of a magnetic field that breaks time-reversal symmetry? As we show, one can make more progress in this modified problem. This "flocking" point of view is also useful for carrying out numerical simulations, since many simple particle models for flocking have been studied in the absence of a magnetic field and can be adapted to our problem, although we do not pursue this here.

B. Summary of the paper

We begin in Sec. II by showing how some terms in Eq. (2) can be related to the full nonlinear resistivity. We do this by formally expanding the relation

$$\mathbf{E}(\mathbf{k},\omega) = \rho_D(\mathbf{j},\Phi)\mathbf{j} + \rho_H(\mathbf{j},\Phi)\mathbf{j}\times\hat{z} \tag{5}$$

at low frequency and wave vector, and for small current and potential fluctuations. Here ρ_D and ρ_H represent the diagonal and Hall resistivities, and the electric field $\mathbf{E}$ is determined via the electrostatic potential, i.e., $\mathbf{E}=-\boldsymbol{\nabla}\Phi$. Upon Fourier transforming back to real space, one can arrive at an equation with a form similar to Eq. (2). This proves to be a useful exercise since we can then relate different possible forms of the frequency- and wave-vector-dependent resistivity in the presence of microwaves to the model parameters appearing in Eq. (2) and therefore to the kinds of ordered states that might emerge from our description. More importantly, this helps us to identify the correct set of critical modes near the phase transition into these putative ordered states. Specifically, we show that if the resistivity is an increasing function of frequency at low frequency, so that the zero-resistance state is achieved when the dc resistance first goes negative, then a time-independent steady state with inhomogeneous density will result. The only critical mode near the transition into this state involves density fluctuations accompanied by current fluctuations that balance the Lorentz force. Since the current and density are tied to one another in this mode, one can reexpress current fluctuations in terms of the density. Inserting the resulting expression into the continuity equation results in an equation of motion involving only the density at the critical point.

This equation of motion for the density at the critical point depends on terms involving the absolute magnitude of the density, as well as terms that depend only on density gradients. We warm up in Sec. III by analyzing a model which neglects terms that depend on the absolute magnitude of the density, and is instead invariant under shifting the density by a constant. This model is expected to describe physics on short length scales where the density does not vary appreciably from its mean so that such terms can be safely ignored. In the "ordered phase" of this model the system develops a uniform current with a transverse density gradient that balances the Lorentz force. We show that this state is stable to small fluctuations, and discuss the "Goldstone mode" associated with the spontaneously broken rotational symmetry. The ordered state described by this model is argued to be relevant for the experiments at short length scales $L<L_{c1}$, where L_{c1} is estimated to be roughly 1 mm, comparable to sample sizes used in the experiments. We then turn to the critical properties of this model, considering both short- and long-range interactions. With short-range interactions, we show that the upper critical dimension is $d_{\mathrm{UC}}=2$. In this case, we use dynamical renormalization group calculations to demonstrate that the Gaussian fixed point has a finite-volume basin of attraction; hence, a finite fraction of initial nonlinear couplings all flow to zero upon renormalization. In such cases, the transition is *continuous* and governed to a good approximation by mean-field theory. Various scal-

ing relations should hold near the transition in this regime. For instance, at fixed magnetic field strength and in the absence of an applied voltage, below the transition, the spontaneous current j_0 should scale with the microwave power P as

$$j_0(P) \sim |P - P_c|^{\beta}, \tag{6}$$

where $\beta > 0$ and P_c is the critical microwave power at which the longitudinal resistance first vanishes. Approaching the transition from the resistive side, with $P < P_c$, we expect a universal scaling relation to hold between the imposed current j and the induced longitudinal electric field,

$$j(P,E) \sim (P_c - P)^{\beta} f(E^{1/\delta}(P_c - P)^{-\beta}), \tag{7}$$

where $f(x)$ is a scaling function with the properties that $f(x) \sim x^{\delta}$ as $x \to 0$, and $f(x) \sim x$ as $x \to \infty$. The behavior as $x \to 0$ recovers linear response behavior, $j \propto E$, in the resistive phase, with the resistivity $\sim (P_c - P)^{\beta(1-\delta)}$. The behavior for $x \to \infty$ leads to a universal longitudinal nonlinear *IV* characteristic

$$j(P_c,E) \sim E^{1/\delta} \tag{8}$$

at the transition, $P = P_c$, with mean-field value of $\delta = 3$ for a current-biased experiment. We then show that long-range interactions appear to drive the transition first order. The experimental signatures of the Goldstone mode in the "ordered phase" and the mean-field transition with short-range interactions are qualitatively discussed in Sec. V.

In Sec. IV we analyze the phase transition in the more general model, where terms that depend on the absolute magnitude of the density are taken into account. These terms, which become important on length scales $L > L_{c2}$, are argued to drive the transition first order with either short- or long-range interactions based on renormalization group calculations. We derive an expression for L_{c2} that depends on the density- and wave-vector-dependent resistivity, and suggest that microscopic calculations may be used to estimate this length. Experimental consequences of the first-order phase transition for sample sizes larger than L_{c2} are briefly noted in Sec. V.

II. DERIVING AND SIMPLIFYING THE EQUATIONS OF MOTION

A. "Microscopic derivation" of equations of motion

Before we turn to the analysis of the phases and transitions described by the set of Eqs. (1)–(4), let us consider a derivation of some terms in the equation of motion for $\mathbf{j}$ in Eq. (2).

We begin with the linear response relation

$$\mathbf{E}^{\alpha}(\mathbf{k},\omega) = \rho_D^{\alpha\beta}(\mathbf{k},\omega)\mathbf{j}^{\beta}(\mathbf{k},\omega) + \rho_H(\mathbf{k},\omega)\epsilon^{\alpha\beta}\mathbf{j}^{\beta}(\mathbf{k},\omega) \tag{9}$$

where ρ_D and ρ_H represent the diagonal and Hall resistivities, the electric field $\mathbf{E}$ is determined from the electrostatic potential via $\mathbf{E} = -\nabla\Phi$, and $\epsilon^{\alpha\beta}$ is the antisymmetric tensor.

We know from experiments that $\rho_H(0,0) \approx B/(ne)$ even in the presence of microwave radiation. We are interested in the case where the dissipative part of the microscopic diagonal resistivity becomes negative. With increasing microwave intensity, this would first happen at some particular wave vector and frequency $(\mathbf{K},\Omega)$. Two specific cases for the behavior of $\rho_D(\mathbf{k},\omega)$ are illustrated in Fig. 1.

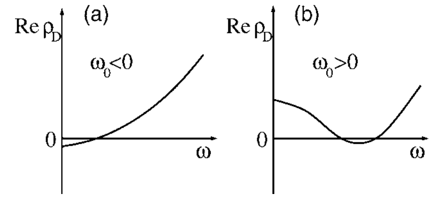

FIG. 1. Schematic behavior of the real part of ρ_D when (a) $\omega_0 < 0$ and (b) $\omega_0 > 0$.

If $\mathbf{K},\Omega$ are small, we can access the resistivity minimum shown in Fig. 1 by expanding $\rho_D(\mathbf{k},\omega) = \rho_1(\mathbf{k},\omega) + i\rho_2(\mathbf{k},\omega)$ in a Taylor series as

$$\rho_D^{\alpha\beta} = \delta^{\alpha\beta}\left[\rho_1(0,0) + i\omega\left(\frac{\partial\rho_2}{\partial\omega}\right) + \frac{\omega^2}{2}\left(\frac{\partial^2\rho_1}{\partial\omega^2}\right) + \bar{\eta}_1\mathbf{k}^2\right] + \bar{\eta}_2\mathbf{k}^{\alpha}\mathbf{k}^{\beta} + \cdots, \tag{10}$$

where the frequency derivatives and coefficients $\bar{\eta}_{1,2}$ are evaluated at $(\mathbf{k}=0, \omega=0)$. Using this expansion inside Eq. (9), and assuming that the Hall resistivity is independent of wave vector and frequency in the regime of interest, we find the following relations between the coefficients in Eq. (2) and the microscopic linear response resistivity:

$$r = \rho_1(0,0)/G, \tag{11}$$

$$\omega_0^{-1} = \frac{1}{2G}\frac{\partial^2\rho_1}{\partial\omega^2}, \tag{12}$$

$$\mu = 1/G, \tag{13}$$

$$\eta_{1,2} = \bar{\eta}_{1,2}/G, \tag{14}$$

$$\tilde{\omega}_c = \rho_H(0,0)/G, \tag{15}$$

where $G \equiv -(\partial\rho_2/\partial\omega)$.

We can similarly match some of the nonlinear terms in Eq. (2) as follows. Let us take $\mathbf{k}=\mathbf{0}, \omega=0$ and consider the nonlinear resistivity which depends in general on the local potential and the current magnitude, namely,

$$\rho_D^{\alpha\beta}(\mathbf{j}^2,\Phi) = \delta^{\alpha\beta}[\rho_1(0,0) + \bar{u}\mathbf{j}^2 - \bar{\gamma}_2\Phi + \cdots], \tag{16}$$

$$\rho_H(\mathbf{j}^2,\Phi) = \rho_H(0,0) + \bar{u}_H\mathbf{j}^2 - \bar{\gamma}_3\Phi + \cdots. \tag{17}$$

Using this expansion and comparing with the nonlinear terms in Eq. (2), we find

$$u = \bar{u}/G, \tag{18}$$

$$\gamma_{2,3} = \bar{\gamma}_{2,3}/G. \tag{19}$$

As we shall see below, the type of ordering expected to emerge from our description depends on the frequency and wave-vector dependence of the resistivity—measuring these in the disordered phase close to the transition would offer clues to the nature of the zero-resistance state.

B. Identifying critical modes and simplifying the equations of motion

On general grounds, one would expect that the type of order that develops near the transition should depend on where the minimum of ρ_1 occurs in (k,ω) space. The resistance will in general depend on both k and ω, and close to the transition will be negative in only a small region of frequencies and wave vectors about the minimum. Modes away from the minimum remain stable. Two possibilities for where this minimum occurs as a function of frequency are sketched in Fig. 1. If the minimum occurs at zero frequency as in Fig. 1(a), then zero-frequency modes that become critical at the transition should give rise to a time-independent ordered state (e.g., static domains of current). If on the other hand the minimum occurs at a nonzero frequency as in Fig. 1(b), then finite-frequency modes should give rise to a state ordered at finite frequency (e.g., circulating currents). In either case, the wave vector at which the minimum occurs would determine the wave vector at which the system orders. Thus, the signs of η_1 and ω_0, which determine whether the minimum of ρ_1 occurs at zero (or nonzero) wave vector and frequency, should play an important role in the ordering.

To make this more concrete, let us consider the mode structure in the disordered state, where $\mathbf{j}$ and n represent fluctuations about a stable zero-current state. We start with the case $\eta_1>0$ and focus on wave vectors $k\rightarrow 0$ since the resistivity is minimized when $k=0$. The modes obtained from the linearized equations of motion are given by

$$\omega_{\pm} = -i\left(r + \frac{r^2-\omega_c^2}{\omega_0}\right) \pm \omega_c\left(1+\frac{2r}{\omega_0}\right) + O(k^2V(k),\omega_0^{-2}), \tag{20}$$

$$\omega_D = \frac{-ir}{r^2+\omega_c^2}\mu V(k)k^2 + O(k^4V^2(k)), \tag{21}$$

where $V(k)$ is the Fourier transform of the interaction potential $V(r)$. Equation (20) is written out only to order ω_0^{-1} for simplicity. We only want to consider here the effect of adding a small frequency dependence to the resistivity, so the exact expression is not important. The modes in Eq. (20) correspond to current fluctuations that circulate due to the magnetic field as they dissipate. The associated density fluctuations for these modes vanish in the $k\rightarrow 0$ limit. Equation (21) represents a diffusive mode involving both current and density fluctuations that survive in the $k\rightarrow 0$ limit. These current fluctuations are undeflected by the magnetic field because the Lorentz force is balanced by an electric field set up by density fluctuations.

In order for the zero-current state to be stable, the imaginary part of these frequencies must be negative so that fluctuations are damped exponentially in time. For the diffusive mode, stability requires $r\geqslant 0$. The circulating current modes are stable when $r\gtrsim \omega_c^2/\omega_0$, assuming $\omega_0^2\gg\omega_c^2$ for simplicity. Violation of either inequality renders the zero-current state of the system unstable to current fluctuations. Since $r\sim\rho_1$, this instability occurs approximately where the longitudinal resistivity changes sign, consistent with the findings of Andreev *et al.*[17]

As the longitudinal resistance tends to zero and the ordered state is approached, the circulating current modes become critical before the diffusive mode if $\omega_0>0$. (Note that since these modes propagate at a finite frequency, this is consistent with the above discussion on the frequency dependence of the longitudinal resistivity.) Once the circulating current modes become unstable, the system should undergo a transition into an ordered state where circulating currents spontaneously develop but the density remains uniform. If $\omega_0<0$, however, the diffusive mode becomes critical while the circulating current modes remain damped. In this case one would expect the system to undergo a transition into a phase with nonuniform density and spontaneous currents ordered at zero wave vector. A distinguishing characteristic of the latter phase would be the development of voltages resulting from the nonuniform density. Since spontaneous voltages in the absence of a net current have indeed been observed in the ordered state, the case $\omega_0<0$ seems to be the experimentally relevant one. We consequently focus on the transition into the density-ordered state and leave an analysis of the circulating-current state to future studies.

These same ideas can be applied to the case $\eta_1<0$, where the resistivity is minimized at finite wave vector. Assuming $\eta_3>0$, one is then interested in wave vectors with magnitude close to $k_0=(|\eta_1|/2\eta_3)^{1/2}$, corresponding to the resistivity minimum. Since we can no longer perturb in k, we cannot in general write down simple expressions for the modes in the disordered state. We will therefore focus on the point where the resistance at zero frequency and $k=k_0$ drops to zero since this simplifies the mode structure. (This happens when $r=\eta_1^2/4\eta_3$.) A critical diffusive mode then emerges whose frequency is given to lowest order by

$$\omega_D = \frac{-i\mu k_0^2V(k_0)}{\mu k_0^2V(k_0)+\tilde{\omega}_c^2}2|\eta_1|\delta k^2, \tag{22}$$

where $\delta k=|\mathbf{k}|-k_0$. The circulating current modes to lowest order are

$$\omega_{\pm} = -ik_0^2\tilde{\eta}/2 \pm \sqrt{-k_0^4\tilde{\eta}^2/4+\mu_0^2V(k_0)+\tilde{\omega}_c^2}, \tag{23}$$

where $\tilde{\eta}=\eta_2+\eta_4k_0^2$. We have set $\omega_0^{-1}=0$ here since the modes already do not become critical simultaneously. In the limit where the $\tilde{\omega}_c$ term is dominant in Eq. (23), the square root is positive. We will assume that $\tilde{\eta}>0$ so that these modes remain damped when the diffusive mode becomes critical since this appears to be the experimentally relevant situation. As the resistance decreases further, one expects the diffusive mode to give rise to a time-independent state with nonuniform density ordered at wave vector k_0.

It follows from the preceding discussion that only the diffusive mode should be important for describing the transition into a nonuniform density phase ordered at either zero or finite wave vector. Since the circulating-current modes have a finite damping rate when the diffusive mode becomes critical, they can be neglected provided we focus on frequencies smaller than their decay rate. This provides a large simplification in that it allows us to eliminate the currents altogether and obtain a theory in terms of the density alone. Physically, this is possible because at long time scales the current and density fluctuations are dominated by a diffusive mode characterized by a gradient of the density fluctuations that just balances the Lorentz force associated with the current fluctuations. One would thus expect to be able to write the "fast" current in terms of the "slow" density. This can be done by dropping the time derivatives on the left-hand side of Eq. (2) compared to $\tilde{\omega}_c$ and then solving order by order for the current as a function of the density. Inserting the resulting expression into the continuity equation yields a decoupled equation of motion for the density alone. The transition within this simplified description of the system will be analyzed in Secs. III and IV for the case of zero-wave-vector ordering; finite-wave-vector ordering is briefly mentioned in Sec. V but will not be studied in detail here.

III. TRANSITION TO DENSITY-ORDERED STATE AT ZERO WAVE VECTOR WITH $\Phi \to \Phi + \text{CONST}$ SYMMETRY

When $\eta_1 > 0$ so that the resistivity is minimized at zero wave vector, we saw in the previous section that the zero-current state of the system becomes unstable when $r < 0$. Identifying the precise ordered state that develops in this regime is complicated by the presence of nonlinear terms in Eq. (2) involving the magnitude of the potential Φ. If one ignores such terms by manually imposing the symmetry $\Phi \to \Phi + \text{const}$, then a simple ordered state emerges, namely, a state with a uniform current and a transverse electric field that balances the Lorentz force. We will begin this section by discussing some mean-field properties of this ordered state and then analyze the transition to this state using dynamical renormalization group techniques. Our motivation for studying this simplified model is as follows. First, it is the simplest model that one can construct that captures the instability that occurs when the resistance becomes negative. Second, we expect this model to be appropriate for describing physics on length scales where terms involving the magnitude of Φ play a relatively unimportant role. This will be quantified below. Third, understanding the properties of the transition in this minimal model will allow us to better understand the effects of adding in terms that violate the $\Phi \to \Phi + \text{const}$ symmetry, which will be done in Sec. IV.

A. Ordered state and linearized theory of fluctuations

When $r < 0$, the ordered state within a model with $\Phi \to \Phi + \text{const}$ symmetry consists of a uniform current

$$\mathbf{j}_0 = \sqrt{|r|/u}\,\hat{\mathbf{x}}, \tag{24}$$

where the direction $\hat{\mathbf{x}}$ is spontaneously picked out. Balancing the associated Lorentz force requires an electric field given by

$$\mathbf{E}_0 = -\nabla\Phi_0 = (\tilde{\omega}_c/\mu)\mathbf{j}_0 \times \hat{\mathbf{z}}. \tag{25}$$

We have assumed here that j_0 is small in some sense so that, for instance, terms in the equation of motion proportional to $|\mathbf{j}|^4\mathbf{j}$ can be neglected compared to the $u|\mathbf{j}|^2\mathbf{j}$ term. To further simplify things, terms such as $(\nabla\Phi)^2\mathbf{j}$ that would arise from expanding the longitudinal resistivity to higher order in the potential have also been neglected. Their presence alters only quantitative properties of the ordered state. For instance, a uniform current still develops, but with a modified magnitude.

To establish a connection with the experiments, note that the longitudinal resistance at zero wave vector and frequency is proportional to $-|r| + u|\mathbf{j}|^2$ (neglecting higher-order terms in $\mathbf{j}$ and $\nabla\Phi$). The spontaneous current j_0 therefore corresponds to a vanishing longitudinal resistance as seen experimentally. This is also consistent with the results of Andreev *et al.*[17] which show that a stable state must have spontaneous currents corresponding to a vanishing longitudinal resistivity.

To analyze the stability of the ordered state, we consider fluctuations about the uniform current state by writing $\mathbf{j} = \mathbf{j}_0 + \delta\mathbf{j}$ and $n = n_0 + \delta n$, where $\mathbf{j}_0$ is given in Eq. (24) and n_0 corresponds to the potential Φ_0 given in Eq. (25). In the linearized equations of motion for $\delta\mathbf{j}$ and δn, there are two damped modes in the $k \to 0$ limit with frequencies

$$\omega'_\pm = -i|r| \pm \sqrt{\tilde{\omega}_c^2 - |r|^2}. \tag{26}$$

Since the ordered state breaks rotational symmetry, there is also a Goldstone mode with frequency ω_G whose real and imaginary parts are given by

$$\text{Re}\,\omega_G = \frac{\mu j_0 \nu_1}{\tilde{\omega}_c^2} k_\parallel k^2 V(k), \tag{27}$$

$$\text{Im}\,\omega_G = -\frac{\mu}{\tilde{\omega}_c^2}(2uj_0^2 k_\perp^2 + \eta_1 k_\parallel^4)V(k), \tag{28}$$

where $k_\perp$ and $k_\parallel$ are the components of $\mathbf{k}$ perpendicular and parallel to $\mathbf{j}_0$, respectively. Note that the damping within this mode is anisotropic. In particular, fluctuations with wave vector parallel to $\mathbf{j}_0$, which produce long-wavelength variations in the direction of $\mathbf{j}_0$, relax much more slowly than fluctuations with wave vector perpendicular to $\mathbf{j}_0$.

To see if the ordered state is stable to fluctuations, one needs to calculate the mean-squared fluctuations of $\delta\mathbf{j}(\mathbf{r},t)$ and $\delta n(\mathbf{r},t)$, averaged over the noise. A divergence of either of these quantities would signal the destruction of the ordered state. We compute these quantities within the linearized theory, focusing only on fluctuations arising from the Goldstone mode for simplicity. (Long-wavelength fluctuations arising from the $\omega'_\pm$ modes will be finite since they have a nonzero damping rate as $k \to 0$; consequently, these modes can be neglected.) Denoting the current fluctuations parallel and perpendicular to $\mathbf{j}_0$ by $\delta\mathbf{j}_\parallel$ and $\delta\mathbf{j}_\perp$, respectively, we find

$$\langle \delta\mathbf{j}_\parallel^2(\mathbf{r},t)\rangle \approx \frac{g}{\tilde{\omega}_c^4}\int_{\mathbf{k}} \frac{\mu V(k) k_\perp^2(\tilde{\omega}_c^2 k^2 - \alpha^2 k_\perp^2)}{\alpha k_\perp^2 + \eta_1 k^4}, \tag{29}$$

$$\langle \delta \mathbf{j}_{\perp}^2(\mathbf{r},t)\rangle \approx \frac{g}{\tilde{\omega}^6}\int_{\mathbf{k}} \frac{\mu V(k)[\tilde{\omega}_c^4 k^2 k_{\parallel}^2 - \alpha^2(\tilde{\omega}_c^2+\alpha^2)k_{\perp}^4]}{\alpha k_{\perp}^2 + \eta_1 k^4}, \tag{30}$$

where $\alpha = 2uj_0^2$ and g is the noise strength. Equation (29) is obviously finite with either short- $[V(k)\sim\text{const}]$ or long-range interactions $[V(k)\sim 1/k]$ since the integrand itself is not infrared divergent. With long-range interactions, the integrand in Eq. (30) is infrared divergent. However, this divergence is integrable in 2D, leading to finite transverse current fluctuations. The mean-squared density fluctuations are given by

$$\langle \delta n^2(\mathbf{r},t)\rangle \approx \frac{g}{\tilde{\omega}_c^2}\int_{\mathbf{k}} \frac{\tilde{\omega}_c^2 k^2 + \alpha^2 k_{\perp}^2 - 2\alpha\tilde{\omega}_c k_{\parallel} k_{\perp}}{\mu V(k)(\alpha k_{\perp}^2 + \eta_1 k^4)}. \tag{31}$$

Again, since the infrared divergence in Eq. (31) is integrable with either short- or long-range interactions, the density fluctuations are also finite. Hence we conclude that, for sufficiently low noise g, the spontaneous current-carrying state is stable to current and density fluctuations with either short- or long-range interactions.

The state characterized by Eqs. (24) and (25) can clearly not exist in arbitrarily large samples since the density would eventually become negative on one side of the sample. For a given spontaneous current $\mathbf{j}_0$, one can estimate the maximum sample length L_{c1} below which this is a sensible ordered state by finding how large the sample can be before the density change becomes comparable to the mean density. We do this by assuming that the electron-electron interactions are screened so that the electric field is given by the gradient in the electrochemical potential μ. If $\Delta\mu$ is the change in μ between the edges of a sample of length L, then the magnitude of the electric field is $E = \Delta\mu/eL$, where e is the electron charge. Regions of density variation comparable to the mean density will appear if $\Delta\mu \sim E_F$, where E_F is the Fermi energy. Setting $\Delta\mu = E_F$ and using Eq. (25), we get

$$L_{c1} \sim \frac{E_F}{e\rho_H j_0}, \tag{32}$$

where we have identified $\tilde{\omega}_c/\mu = \rho_H$.

Note that as the mean-field critical point is approached, $j_0 \to 0$ and so L_{c1} diverges. One might therefore be tempted to conclude that the model with $\Phi \to \Phi + \text{const}$ symmetry correctly describes the physics at the transition at all length scales. We stress that this is not necessarily the case. In computing L_{c1}, we have only demanded that no unphysical features such as negative density arise in this minimal model. What we have *not* done is compute the characteristic length (which can be smaller than L_{c1} above) below which terms that depend on the magnitude of Φ play a negligible role. We will elaborate further on this in the following subsection.

We now estimate L_{c1} using parameters measured by Willett *et al.*[7] in order to get a feel for this length scale. In their experiments, carried out on a GaAs/Al_xGa_{1-x}As samples, the density is $n \approx 2\times 10^{11}$ cm^{-2}, from which we estimate $E_F \sim 5$ meV. In a 20 GHz microwave field the primary zero-resistance state occurs at $B \approx 0.4$ kG, where $\rho_H = B/ne \approx 125\ \Omega$. From spontaneous voltages that develop in this zero-resistance region, they estimate a spontaneous current of roughly 5 μA flowing between the center and edge in square samples of length 0.4 mm. Assuming a single domain between these contacts, we find $j_0 \approx 25\ \mu$A mm^{-1}. Putting these parameters together, we estimate the critical length to be $L_{c1} \sim 1$ mm. In samples with dimension larger than L_{c1}, terms in the equation of motion involving the magnitude of Φ must be taken into account to produce a sensible ordered state. Such terms would prevent the density from becoming arbitrarily large and negative, and would lead to inhomogeneous currents and densities. Determining the corresponding current-carrying ordered state on these longer length scales is an interesting problem that we do not address here.

B. Transition with short-range interactions

Having discussed an example of a stable ordered state that arises from a model with $\Phi \to \Phi + \text{const}$ symmetry when $r < 0$, we now turn to the critical properties of the system at the phase transition. We begin with the simplest case of short-range interactions.

As discussed in Sec. II, our analysis is greatly simplified by assuming that near the critical point circulating-current modes remain damped while the diffusive mode becomes critical. Focusing only on the diffusive mode enables us to write the current in terms of the density. To do this, we start with Eq. (2) and construct the current for a static density configuration, setting all time derivatives to zero. Next, assuming that gradients and nonlinear terms involving $\mathbf{j}$, Φ are "small," we reexpress order by order the current in terms of Φ. To leading order, this is achieved by keeping only the lowest-order terms in Eq. (2) and obtaining $\mathbf{j}$ by inverting

$$0 = -r\mathbf{j} + \tilde{\omega}_c \hat{\mathbf{z}} \times \mathbf{j} - \mu \nabla \Phi + \boldsymbol{\zeta}. \tag{33}$$

Finally, inserting the expression for the current into Eq. (1) (the continuity equation) yields a density-only model.

Imposing the symmetry $\Phi \to \Phi + \text{const}$ in the initial Eq. (2), the resulting equation of motion takes the form

$$\begin{aligned} 0 = {} & \partial_t n - \tilde{r}\nabla^2\Phi + D\nabla^4\Phi - \nabla\cdot\boldsymbol{\zeta}' + \lambda_1 \nabla\cdot(\nabla^2\Phi\,\nabla\Phi) \\ & + \lambda_2 \hat{\mathbf{z}}\cdot(\nabla\Phi \times \nabla^3\Phi) + \lambda_3 \nabla\cdot([\hat{\mathbf{z}}\times\nabla\Phi)\cdot\nabla \\ & \times(\hat{\mathbf{z}}\times\nabla\Phi)] + \mu_1 \nabla\cdot[\nabla\Phi(\nabla\Phi)^2] \\ & + \mu_2 \hat{\mathbf{z}}\cdot[\nabla\Phi \times \nabla(\nabla\Phi)^2], \end{aligned} \tag{34}$$

where $D > 0$ and $\boldsymbol{\zeta}'$ is a Gaussian noise source with variance $2g'$. The transition occurs at $\tilde{r} = 0$ (within mean-field theory), so we will take $\tilde{r} = 0$ from now on. Since we are considering short-range interactions here, $\Phi[n] = n$. Quartic and higher-order terms in $\nabla\Phi$ have been neglected since they contain at least five gradient operators and are therefore irrelevant in two dimensions.

The upper critical dimension for this model is $d_{\mathrm{UC}} = 2$ for the case of short-range interactions. We use dynamical renormalization group techniques[19,24] to deduce whether the nonlinearities appearing in Eq. (34) are marginally relevant or irrelevant in two dimensions. This procedure is facilitated by the use of the Martin-Siggia-Rose (MSR) formalism.[25] The essence of this formalism is that one introduces a "partition function" Z that is useful for obtaining various correlation functions, namely,

$$Z = \int \mathcal{D}n\ \delta(\partial_t n - \tilde{r}\nabla^2\Phi + D\nabla^4\Phi - \boldsymbol{\nabla}\cdot\boldsymbol{\zeta} + \cdots). \quad (35)$$

This imposes the equation of motion as a constraint on all possible spacetime "trajectories" of $n(\mathbf{r},t)$. The ellipsis indicates all nonlinearities appearing in Eq. (34). This functional δ-function constraint is implemented through an auxiliary field $\tilde{n}(\mathbf{r},t)$ so that Z can be written as

$$Z = \int \mathcal{D}n\ \mathcal{D}\tilde{n} e^{iS[n,\tilde{n}]}, \quad (36)$$

$$S = \int_{\mathbf{r},t} \tilde{n}[\partial_t n - \tilde{r}\nabla^2\Phi + D\nabla^4\Phi - \boldsymbol{\nabla}\cdot\boldsymbol{\zeta}' + \cdots], \quad (37)$$

where constants have been absorbed into the integration measure for $\tilde{n}$. A useful feature of this method is that the noise averaging can now be easily performed, with the result that

$$\int_{\mathbf{r},t} \tilde{n}[-\boldsymbol{\nabla}\cdot\boldsymbol{\zeta}'] \to ig' \int_{\mathbf{r},t} (\boldsymbol{\nabla}\tilde{n})^2 \quad (38)$$

in the "action" S. This leads to a MSR action expressed in terms of the fields $n,\tilde{n}$ and no noise terms. One can then implement the renormalization group transformation using standard field-theory techniques as follows. First, the action S is written in Fourier space with an ultraviolet cutoff Λ reflecting the coarse graining of the fields. One then integrates out fields with wave vectors q such that $\Lambda/s < q < \Lambda$, where $s > 1$. This results in an effective action with a reduced cutoff Λ/s. To restore the initial cutoff, the wave vectors, frequencies, and fields are rescaled according to

$$k' = sk, \quad (39)$$

$$\omega' = s^z\omega, \quad (40)$$

$$n'(\mathbf{k}',\omega') = s^{-\chi}n(\mathbf{k},\omega), \quad (41)$$

$$\tilde{n}'(\mathbf{k}',\omega') = s^{-\tilde{\chi}}\tilde{n}(\mathbf{k},\omega). \quad (42)$$

By setting $s=1+d\ell$, one arrives at differential recursion relations that specify how the effective coupling constants for the long-scale degrees of freedom "flow" as short-scale degrees of freedom are integrated out. In the present paper these recursion relations will be calculated to one-loop order.

In anticipation of finding a stable Gaussian fixed point, we choose the rescaling exponents to take on their mean-field values: $z=4$, $\chi=6$, and $\tilde{\chi}=4$. (Since there is no small parameter at our disposal, the only possible controlled fixed point *must* be Gaussian.) These exponents keep the noise strength g' fixed under renormalization since diagrammatic corrections to g' vanish at one-loop order. To simplify the flow equations for the remaining coupling constants, we define the following dimensionless parameters:

$$\Lambda_1 = (\alpha/D)\lambda_3(4\lambda_1 - 3\lambda_3),$$

$$\Lambda_2 = -(\alpha/D)\lambda_3\lambda_2,$$

$$\Lambda_3 = (\alpha/D)\lambda_3^2,$$

$$\Lambda_4 = -\alpha\mu_1,$$

$$\Lambda_5 = \alpha\mu_2, \quad (43)$$

where $\alpha = g/4\pi D^2$. The flow equations in terms of these parameters are

$$\partial_\ell D = \frac{1}{2}\Lambda_1 D, \quad (44)$$

$$\partial_\ell \Lambda_1 = -\left(\frac{3}{2}\Lambda_1 + 13\Lambda_4\right)\Lambda_1 - 3\Lambda_3\Lambda_4 + 4\Lambda_2\Lambda_5, \quad (45)$$

$$\partial_\ell \Lambda_2 = -\left(\frac{3}{2}\Lambda_1 + 7\Lambda_4\right)\Lambda_2 + \frac{1}{4}(3\Lambda_3 - 7\Lambda_1)\Lambda_5, \quad (46)$$

$$\partial_\ell \Lambda_3 = -\left(\frac{3}{2}\Lambda_1 + 12\Lambda_4\right)\Lambda_3, \quad (47)$$

$$\partial_\ell \Lambda_4 = -(\Lambda_1 + 9\Lambda_4)\Lambda_4 + \Lambda_5^2, \quad (48)$$

$$\partial_\ell \Lambda_5 = -(\Lambda_1 + 10\Lambda_4)\Lambda_5. \quad (49)$$

At this point we would like to identify the basin of attraction for the Gaussian fixed point under consideration. That is, for a given set of initial conditions for Λ_i, we would like to know whether these parameters all flow to zero as $\ell \to \infty$. While it is straightforward to check this numerically, it is difficult to draw general conclusions either analytically or from the numerics due to the five-dimensional parameter space and the fact that Eqs. (45)–(49) are all coupled. In the subspace with $\lambda_3=0$, one can show analytically that the Gaussian fixed point is stable to all perturbations within that subspace. In the full parameter space with $\lambda_3 \neq 0$, we have shown that a finite-volume region of initial conditions corresponds to stable trajectories where each Λ_i flows to zero. The asymptotic solution for such trajectories is given by $\Lambda_1 \sim (2/11)\ell^{-1}$, $\Lambda_2 \sim (5c_2/242)\ell^{-10/11}$, $\Lambda_3 \sim c_1\ell^{-15/11}$, $\Lambda_4 \sim (1/11)\ell^{-1}$, and $\Lambda_5 \sim (1/c_2)\ell^{-12/11}$, where $c_{1,2}$ are arbitrary constants. One can verify the stability of these flows by perturbing around this solution. According to Eq. (44), the subdiffusion constant grows asymptotically as $D \sim D_0\ell^{1/11}$ along these trajectories, where D_0 is a constant. The asymptotic behavior of the original coupling constants is given by $\lambda_1 \sim \ell^{-2/11}$, $\lambda_2 \sim \ell^{-1/11}$, $\lambda_3 \sim \ell^{-6/11}$, $\mu_1 \sim \ell^{-9/11}$, and $\mu_2 \sim \ell^{-10/11}$, demonstrating marginal irrelevance of all nonlinearities, and therefore the stability of the Gaussian fixed point.

Flows that terminate along the above asymptotic trajectories correspond to marginally irrelevant couplings that reside in the basin of attraction for the Gaussian fixed point. In such cases mean-field theory should be a reasonable starting point for analyzing the transition to the density-ordered phase within this model. In particular, as we saw in the previous subsection mean-field theory predicts a *continuous* transition since the spontaneous current and the associated density gradient develop smoothly from zero as r becomes negative [see Eqs. (24) and (25)]. Another important mean-field prediction we can make is that at the transition there is a single subdiffusive mode for density fluctuations with frequency

$$\omega = -iDk^4. \tag{50}$$

This slow relaxation of long-wavelength fluctuations should be accompanied by large voltage fluctuations near the transition. Equal-time density-density correlations, which should mimic voltage correlations, are given within mean-field theory by

$$\langle n(\mathbf{r},t)n(\mathbf{0},t)\rangle = \frac{g'}{D}\int_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{k^2}. \tag{51}$$

The integral diverges logarithmically at small k. To regulate the integral, we restrict the range of integration to $2\pi/L < k < \Lambda$, where L is the system size. In the limit $r\Lambda \gg 1$ and $r/L \ll 1$, we obtain

$$\langle n(\mathbf{r},t)n(\mathbf{0},t)\rangle \approx \frac{g'}{2\pi D}\ln(L/r). \tag{52}$$

Equal-time current-current correlations in mean-field theory are given by

$$\langle \mathbf{j}(\mathbf{r},t)\mathbf{j}(\mathbf{0},t)\rangle = \frac{g'\mu\Lambda^2}{2\pi D\tilde{\omega}_c^2}\frac{J_1(r\Lambda)}{r\Lambda}, \tag{53}$$

where $J_1(x)$ is a Bessel function of the first kind. Note that the current becomes δ-function correlated in the limit that $\Lambda \to \infty$. Since the interactions are only marginally irrelevant, they will give rise to logarithmic corrections to these correlation functions, which will not be computed here.

So far we have focused on the case where the coupling constants flow to zero upon renormalization. Even outside of this marginally stable region, the couplings are still only marginally relevant, and therefore grow only logarithmically with length scale. In fact, we have seen numerically that many trajectories that initially flow toward the Gaussian fixed point eventually diverge from it, but do so only after many renormalization group iterations. In these instances, it may be very difficult to resolve deviations from mean-field theory either numerically or experimentally, and the transition may appear continuous even in the presence of the marginally relevant couplings.

C. Is this model valid near the transition?

We will now discuss when the model with $\Phi \to \Phi + \text{const}$ symmetry and short-range interactions is expected to be appropriate for describing the physics at the transition. Consider adding the term $\lambda \boldsymbol{\nabla}\cdot(\Phi\boldsymbol{\nabla}\Phi)$ to Eq. (34), which is the most relevant nonlinearity that violates this symmetry. This term can be traced back to the $\gamma_2\Phi\mathbf{j}$ term in Eq. (2). We define a dimensionless coupling constant $\tilde{\lambda} \equiv \lambda/D$, where D is the subdiffusion constant. If we interpret the equation of motion as arising from a Taylor expansion of the resistivity, then we can write

$$\tilde{\lambda} = \frac{\partial\rho_D/\partial n}{\partial\rho_D/\partial k^2}. \tag{54}$$

One expects $\tilde{\lambda}$ to be small in a not-too-dirty electron gas, since in a pure system ρ_D is already nonvanishing at any nonzero wave vector (contributing to the denominator), while the numerator vanishes by Galilean invariance in this limit. However, this term is strongly relevant in two dimensions. Hence, even if one starts with $\tilde{\lambda} \ll 1$, in an infinite system this coupling constant will eventually become much greater than unity under renormalization. Ignoring this term will certainly not be valid in this case, so one would need to appeal to the full equation of motion to describe the transition. In a finite system, however, one is interested in reducing the cutoff to roughly $1/L$, where L is the system size, so the growth of the coupling constant will be bounded. The model with $\Phi \to \Phi + \text{const}$ symmetry will provide a reasonable description of the transition as long as L is sufficiently small that $\tilde{\lambda}$ does not become of order 1. Under a tree-level renormalization group iteration, the renormalized coupling constant $\tilde{\lambda}'$ grows according to $\tilde{\lambda}' = \tilde{\lambda}s^2$, with $s > 1$. In terms of the reduced cutoff $\Lambda' = \Lambda/s$, where Λ is the initial cutoff, this can be expressed as

$$\frac{\tilde{\lambda}'}{\tilde{\lambda}} = \left(\frac{\Lambda}{\Lambda'}\right)^2. \tag{55}$$

We take $\Lambda' = 1/L_{c2}$ and $\Lambda = 1/l_{\text{in}}$, where l_{in} is the inelastic mean free path. For the samples used in the experiments, $l_{\text{in}} \sim \hbar v_F E_F/(k_B T)^2 \sim 100\ \mu\text{m}$,[15] and is comparable to the transport mean free path estimated from the mobility at a temperature of 1 K. This is about an order of magnitude smaller than the sample lengths.

To estimate the critical length scale L_{c2} below which the model is valid, we set $\tilde{\lambda}' = 1$, leading to

$$L_{c2} = \frac{l_{\text{in}}}{\sqrt{|\tilde{\lambda}|}}. \tag{56}$$

We note that if $|\tilde{\lambda}| \ll 1$, say around 0.01, then the critical length L_{c2} would already be comparable to the sample sizes studied in the experiments. A serious estimate of this length would require a microscopic calculation of the resistivity $\rho_D(\mathbf{k},\omega)$ in the presence of microwaves to compute the bare value of $\tilde{\lambda}$ via Eq. (54) and would be valuable.

D. Transition with long-range interactions

We have seen in the case of short-range interactions above that a finite-volume region of initial couplings is marginally irrelevant and flows to zero upon coarse graining. Next, we discuss the fate of these flows when long-range interactions are turned on. This case is relevant experimentally due to the absence of metallic gates in the experiments conducted so far, leading to unscreened Coulomb interactions.

Consider again the equation of motion given in Eq. (34), with $\Phi[n]=\int_{\mathbf{r}'}V(\mathbf{r}-\mathbf{r}')n(\mathbf{r}')$. With long-range Coulomb interactions, the Fourier-transformed interaction potential (in two dimensions) is $V(k)\sim 1/k$. The upper critical dimension in this case is $d_{\mathrm{UC}}=3$. Since we are interested in the transition in $d=2$ dimensions, one option is to carry out an ϵ expansion in $d=3-\epsilon$ dimensions. This approach is complicated by the need to generalize the interactions in Eq. (34) to higher dimensions. Alternatively, one can perform an ϵ expansion by writing $V(k)=1/k^{\epsilon}$, with $\epsilon \ll 1$. The upper critical dimension is then $d_{\mathrm{UC}}=2+\epsilon$. We will adopt the latter approach since we can then work directly in $d=2$ dimensions and thereby avoid generalizing the equation of motion.

We use the dynamical renormalization group as outlined above to calculate the flow equations at one-loop order and to lowest order in ϵ. As in the short-range case, there are no diagrammatic corrections to the noise strength g' at one loop. To keep g' fixed under renormalization, we take the rescaling exponent $\tilde{\chi}=(4+z)/2$. Similarly, we choose the exponent $\chi=3z/2$ to fix the coefficient of $\partial_t n$ to be unity in Eq. (34). To simplify the flow equations for the remaining parameters, we again use the dimensionless coupling constants defined in Eq. (43) (with $\alpha=g/4\pi D^2\Lambda^{\epsilon}$). The subdiffusion constant then flows according to

$$\partial_l D=(z-4+\epsilon+\Lambda_1/2)D. \tag{57}$$

For convenience we choose $z=4-\epsilon-\Lambda_1/2$ to keep D fixed.

With this choice of rescaling exponents, the flow equations for the parameters Λ_i are

$$\partial_\ell \Lambda_1=\epsilon\Lambda_1-\left(\frac{3}{2}\Lambda_1+13\Lambda_4\right)\Lambda_1-3\Lambda_3\Lambda_4+4\Lambda_2\Lambda_5, \tag{58}$$

$$\partial_\ell \Lambda_2=\epsilon\Lambda_2-\left(\frac{3}{2}\Lambda_1+7\Lambda_4\right)\Lambda_2+\frac{1}{4}(3\Lambda_3-7\Lambda_1)\Lambda_5, \tag{59}$$

$$\partial_\ell \Lambda_3=\epsilon\Lambda_3-\left(\frac{3}{2}\Lambda_1+12\Lambda_4\right)\Lambda_3, \tag{60}$$

$$\partial_\ell \Lambda_4=\epsilon\Lambda_4-(\Lambda_1+9\Lambda_4)\Lambda_4+\Lambda_5^2, \tag{61}$$

$$\partial_\ell \Lambda_5=\epsilon\Lambda_5-(\Lambda_1+10\Lambda_4)\Lambda_5. \tag{62}$$

In the case of short-range interactions we found that there are stable trajectories where all the coupling constants go asymptotically to zero. This clearly cannot happen in the case of finite-range interactions due to the $\epsilon\Lambda_i$ terms above. Instead, we search for fixed points of the form $\Lambda_i=a_i\epsilon$, where a_i are constants. One can easily show that all such fixed points are unstable. We interpret this lack of a stable fixed point as signaling a first-order transition. Thus, we conclude that *the continuous transition that can occur with short-range interactions is driven first order by the presence of long-range interactions of the form* $V(k)=1/k^{\epsilon}$, *with* $\epsilon\ll 1$.

This result may seem surprising initially since one might expect long-range interactions to suppress density fluctuations and thereby further stabilize the Gaussian fixed point. For instance, in the linearized equation of motion the density diffuses faster with long-range interactions. A competing effect, however, is that density fluctuations can interact nonlocally through the nonlinear terms. Thus, density fluctuations in one region of the sample can further induce fluctuations over long distances. This can lead to positive feedback of these density fluctuations via the nonlinearities, which evidently drives the transition first order.

IV. TRANSITION TO DENSITY-ORDERED STATE AT ZERO WAVE VECTOR IN THE FULL ROTATIONALLY INVARIANT MODEL

The model considered above with $\Phi\to\Phi+\text{const}$ symmetry is only appropriate for describing physics up to a certain length scale. For instance, in the ordered state with a given uniform current, regions of negative density appear if the sample is too large. We estimated this length scale to be roughly ~ 1 mm using parameters from Willett's experiments.[7] On larger scales, terms in the equation of motion depending on the magnitude of Φ, which prevent the density from becoming negative, must be taken into account. As discussed above, at the transition, the leading term involving the magnitude of Φ [i.e., the $\gamma_2\Phi\mathbf{j}$ term in Eq. (2)] is strongly relevant in two dimensions. The dimensionless coupling constant for this term therefore grows under renormalization. In a finite system, the growth of this coupling is limited by the system size L since one only reduces the wave vector cutoff to of order $1/L$. Neglecting terms depending on the magnitude of Φ becomes an invalid approximation when the system size is sufficiently large that this renormalized dimensionless coupling becomes of order unity.

To describe physics in samples with linear dimensions larger than these length scales, one must therefore relax the $\Phi\to\Phi+\text{const}$ symmetry and appeal to the full equation of motion in Eq. (2) with no additional symmetries. This is the subject of the present section. Identifying the ordered state that develops in this case is nontrivial, so we will focus only on the transition to the ordered state, considering both short- and long-range interactions.

A. Transition with short-range interactions

When we relax the $\Phi\to\Phi+\text{const}$ symmetry, Eq. (34) generalizes to

$$0=\partial_t n-\tilde{r}\nabla^2\Phi+D\nabla^4\Phi-\nabla\cdot\boldsymbol{\zeta}'-\lambda\nabla\cdot(\Phi\nabla\Phi), \tag{63}$$

where $D>0$ and $\boldsymbol{\zeta}'$ is a Gaussian noise source with variance $2g'$. In this subsection we consider short-range interactions, so $\Phi=n$. The transition in the linearized theory occurs at $\tilde{r}=0$ since the diffusive mode becomes unstable when $\tilde{r}<0$. Other nonlinearities are in principle present in Eq. (63), but they are less relevant than the λ term and can be neglected provided we work near the upper critical dimension.

To derive Eq. (63), we solved for the current in terms of the density assuming two spatial dimensions. However, the λ interaction is strongly relevant in two dimensions, so to

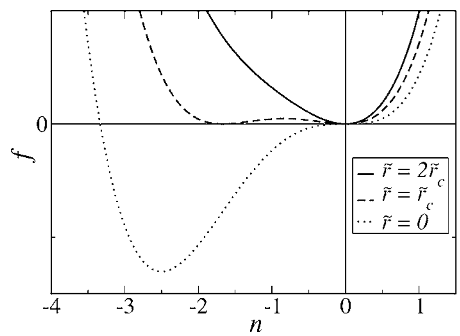

FIG. 2. "Free-energy" density f as a function of constant density n for $\lambda=1$, $\tilde{u}=1/5$, with $\tilde{r}=2\tilde{r}_c$, $\tilde{r}_c$, 0 (see text for details).

study its effects we need to to continue this model to higher dimensions. We initially adopt the most naive way of doing this continuation; namely, we simply assert that Eq. (63) is valid in d dimensions. In the case of short-range interactions the upper critical dimension for the λ nonlinearity is then $d_{\mathrm{UC}}=6$.

We have carried out an ϵ expansion in $d=6-\epsilon$ dimensions to obtain the renormalization group flow equations to one-loop order. Rather than go through the details of the calculation, we will merely state that these equations lack a stable fixed point, which we interpret as signaling a first-order transition. A more direct route to this conclusion can be obtained by observing that Eq. (63) is identical to the equation of motion for an *equilibrium* model with a conserved order parameter. That is, it [Eq. (63)] can be rewritten as

$$\partial_t n = \nabla^2 \frac{\delta F}{\delta n} + \nabla \cdot \boldsymbol{\zeta}', \tag{64}$$

with the "free energy" given by

$$F = \frac{1}{2}\int_r \left[\tilde{r} n^2 + D(\nabla n)^2 + \frac{\lambda}{3} n^3 + \frac{\tilde{u}}{2} n^4 \right], \tag{65}$$

The term proportional to $\tilde{u}$ that results from Eq. (64) is irrelevant at the upper critical dimension and has therefore been excluded from Eq. (63). We will assume $\tilde{u}>0$ for simplicity, although this is not essential. Figure 2 depicts the free-energy density f as a function of uniform density n for three different values of $\tilde{r}$. When $\tilde{r}>\lambda^2/18\tilde{u}\equiv\tilde{r}_c$, the free energy is minimized when $n=0$ as illustrated by the solid curve. At $\tilde{r}=\tilde{r}_c$, the free energy has two degenerate minima as shown in the dashed curve. Below $\tilde{r}_c$, the free energy is minimized by a nonzero value of n. This situation is represented by the dotted line for $\tilde{r}=0$. When $\tilde{r}$ decreases below $\tilde{r}_c$, the density will therefore jump discontinuously from zero to minimize the "free energy." This signals the onset of a first-order transition, consistent with our renormalization group results. Note that the transition occurs at a finite value of $\tilde{r}$, preempting the apparent transition (a "spinodal") at $\tilde{r}=0$ expected from the linear theory. As Fig. 2 demonstrates, the point $\tilde{r}=0$ actually corresponds to a spinodal decomposition where the system goes from being metastable to globally unstable at $n=0$.

These results hold only near $d=6$. We can reduce the upper critical dimension of the model by considering a spatially anisotropic continuation of Eq. (63) to higher dimensions. To do this, we can start by continuing Eq. (2) to d dimensions and taking the resistance at zero wave vector and frequency to be anisotropic. That is, write

$$r\mathbf{j} \to r_\perp \mathbf{j}_\perp + r_\parallel \mathbf{j}_\parallel \tag{66}$$

in Eq. (2), where $\mathbf{j}_\perp$ is the current in the x-y plane and $\mathbf{j}_\parallel$ represents the current in the additional $d-2$ dimensions. We will be interested in tuning the resistance $r_\perp$ in the x-y plane to zero while leaving the resistance $r_\parallel$ for the remaining directions positive. We can then eliminate the current in favor of the density as before to obtain

$$0 = \partial_t n - \tilde{r}_\parallel \nabla_\parallel^2 \Phi + D_\perp \nabla_\perp^4 \Phi - \nabla_\perp \cdot \boldsymbol{\zeta}'_\perp - \lambda_\perp \nabla_\perp \cdot (\Phi \nabla_\perp \Phi), \tag{67}$$

where $\tilde{r}_\parallel$, $D_\perp>0$ and we have set $r_\perp=0$. We have also only retained the x-y (in-plane) components of the noise $\boldsymbol{\zeta}'_\perp$ (with strength $g_\parallel$), since noise components in the additional $(d-2)$ $\parallel$ dimensions are irrelevant. Similarly, we have omitted nonlinear terms involving $\nabla_\parallel$ since, due to the high anisotropy of the harmonic terms, these are clearly less relevant than the corresponding terms involving only $\nabla_\perp$ derivatives.

The upper critical dimension for this model is $d_{\mathrm{UC}}=4$. We have performed an ϵ expansion in $d=4-\epsilon$ dimensions to one-loop order. Upon integrating out high-energy modes, the momenta are anisotropically rescaled according to $k'_\parallel=sk_\parallel$ and $k'_\perp=s^\mu k_\perp$, and the exponents z, χ, $\tilde{\chi}$ are defined as in Eqs. (40)–(42). Requiring that the $\partial_t n$ term remains invariant under rescaling leads to the relation $\chi+\tilde{\chi}=2(1+\mu-\epsilon+z)$. Defining

$$\bar{\lambda}_\perp^2 = \frac{\lambda_\perp^2 g_\perp}{48\pi^2 D_\perp^2 \tilde{r}_\parallel}, \tag{68}$$

and setting $s=1+d\ell$, the flow equations are given by

$$\partial_\ell \tilde{r}_\parallel = (z-2\mu)\tilde{r}_\parallel, \tag{69}$$

$$\partial_\ell D_\perp = (z-4+\bar{\lambda}_\perp^2) D_\perp, \tag{70}$$

$$\partial_\ell \lambda_\perp = (2z-2-\tilde{\chi}+3\bar{\lambda}_\perp^2)\lambda_\perp, \tag{71}$$

$$\partial_\ell g_\perp = (2\tilde{\chi}-z-4-2\mu+2\epsilon) g_\perp. \tag{72}$$

We let $\tilde{r}_\parallel, D_\perp, g_\perp$ flow to fixed points, and follow the flow for $\lambda_\perp$. Once again, we find that the model lacks a stable fixed point. Thus, even near $d=4$ dimensions, the transition still appears to be first order.

It seems quite likely that the transition is first order in $d=2$ dimensions as well. In the model with $\Phi\to\Phi+\text{const}$ symmetry and short-range interactions, we showed in the previous section that one *could* have a continuous transition in $d=2$ dimensions. Terms that violate the $\Phi\to\Phi+\text{const}$ symmetry appear then to always drive the transition first order.

We propose the following physical interpretation for this. We have been analyzing the transition at zero wave vector and zero frequency, where one expects long-wavelength fluctuations that become critical at the transition to give rise to an ordered state with uniform, static current. Such an ordered state must be accompanied by a density gradient transverse to the current to balance the Lorentz force. We have already argued that such a state cannot exist in the thermodynamic limit because the density would become arbitrarily large and negative at the edges of the sample. The only terms in the equation of motion that sense these unphysical features are precisely those terms that depend on the magnitude of the density. In the thermodynamic limit, these terms must therefore induce a first-order transition into some other state, such as a state ordered at finite wave vector or a phase-separated state. A direct transition from a uniform isotropic liquid to a modulated (finite wave vector) smectic state can also be argued to be first order on quite general grounds.[26–28]

B. Transition with long-range interactions

Finally, let us consider the effect of long-range interactions. We saw in the model with $\Phi \rightarrow \Phi+\text{const}$ symmetry that turning on long-range interactions drove the transition first order. In the present case, the transition is already first order with short-range interactions, so it seems rather likely that the transition will remain so with long-range interactions. This is indeed what we find based on a renormalization group analysis. We will therefore only outline the calculation and state the results.

Consider Eq. (63) with $\Phi[n]=\int_{\mathbf{r}'}V(\mathbf{r}-\mathbf{r}')n(\mathbf{r}')$ and $V(k)=1/k$. Once again, the λ interaction is strongly relevant in two dimensions, so we would like to continue Eq. (63) to d dimensions. We will only consider the simplest isotropic continuation and assert that Eq. (63) holds in d dimensions. The upper critical dimension is then $d_{\text{UC}}=7$. We have performed a one-loop ϵ expansion in $d=7-\epsilon$ dimensions, and find that the model lacks a stable fixed point. Thus, as expected, the transition remains first order when long-range interactions are included.

V. DISCUSSION AND SUMMARY

The focus of this paper has been on the physics near the transition to a zero-resistance state in 2DEGs driven with microwave radiation. Our goal was to understand the long-distance, long-time properties of the system taking into account noise and fluctuation effects within a nonequilibrium hydrodynamic theory involving the electron current and density. We specifically focused on the transition to a time-independent, density-ordered state that occurs when the microscopic resistance first becomes negative at $(\mathbf{k}=\mathbf{0},\omega=0)$. The long-wavelength subdiffusive density fluctuations are the only critical modes at this transition. We analyzed two models involving the density mode: (i) model I, characterized by an imposed symmetry under a global uniform shift of the density, valid only on sufficiently small length scales, and (ii) model II, which is most general rotationally invariant model with no additional symmetries.

The ordered state in model I consists of a uniform current and a transverse Hall electric field that balances the Lorentz force. This state was shown to be stable within a linearized theory of fluctuations about the ordered state.

We argued that the uniform-current steady state in model I cannot exist in arbitrarily large samples since the uniform Hall field would eventually lead to regions of negative density. Using parameters from Willett *et al.*'s experiments,[7] we estimated that samples with dimension smaller $L_{c1}\sim 1$ mm can support this state. To describe the ordered state in larger samples, one must include terms that depend on the magnitude of the density, which would prevent the density from becoming arbitrarily large and negative.

Since the ordered state in model I breaks continuous rotational symmetry, there is an associated Goldstone mode corresponding to long-wavelength fluctuations of the current transverse to the uniform-current flow direction. Surface acoustic waves in the zero-resistance regime[29,30] at the right wavelength and frequency should couple to the Goldstone mode, opening up the possibility of detecting this mode as a signature of symmetry breaking.

The transition to this ordered state in model I was analyzed in both the cases of short- and long-range interactions using dynamical renormalization group methods. This model is valid for describing the transition on length scales $L<L_{c2}$, which we think could be comparable to sample sizes in current experiments as discussed in Sec. III C, although it would be valuable to have an estimate from microscopic calculations. With long-range interactions, we showed that model I undergoes a first-order transition. However, with short-range interactions, we showed that in two dimensions the Gaussian fixed point in model I has a finite-volume basin of attraction. That is, a finite-volume region of initial nonlinear couplings all flow to zero upon renormalization. The transition in these cases is of the mean-field type. In particular, mean-field theory predicts a *continuous* transition to the ordered state. Additionally, the density subdiffuses at the critical point, with a frequency given by $\omega\propto -ik^4$. This subdiffusion should lead to large density fluctuations and hence large voltage fluctuations at the transition. It may be interesting to observe this in samples with metallic gates, so that the Coulomb interactions are screened. Although it may be difficult to quantitatively test the mean-field predictions, one could perhaps measure voltage correlations at contacts placed along the perimeter of the sample. These voltages should behave similarly to the density-density correlations given in Eq. (51). Qualitatively, one should at least observe large voltage fluctuations since the density is critical and subdiffusive at the transition.

We next turned to an analysis of the transition in the more generic model II, which includes terms that depend explicitly on the magnitude of the density. We found that the transition within this model is always first order independent of whether interactions are short- or long-range, at least near the upper critical dimension of the theory. The physical mechanism for this first-order transition is as follows. If the resistance minimum occurs at $k=\omega=0$, then one would expect long-wavelength fluctuations to give rise to a time-independent state with a uniform current and transverse Hall field. As mentioned above, such a state cannot exist in arbitrarily large samples since regions of negative density will eventually appear. The role of terms that depend on the magnitude of the density is to prevent such unphysical features from arising. These terms consequently force a first-order transition into a more complicated ordered state.

The experiments conducted so far were carried out using samples without metallic gates, leading to unscreened Coulomb interactions. The transition in these systems is therefore predicted to be first order, which should have measurable consequences. One possible controlled way of detecting a first-order signature might be to measure the critical current above which the zero-resistance state disappears.[17] If one approaches the transition from the ordered state (by, say, changing the magnetic field) then the critical current should drop discontinuously from some finite value to zero if the transition is indeed first order.

There are several future directions one could pursue with the theory presented here that we believe would be interesting and provide further insight into the remarkable physics of driven 2DEGs. Regarding the transition to zero resistance, we have considered only the simplest case where the resistance minimum occurs at $k=\omega=0$. It may be interesting to generalize our results for this case to include static disorder to see how it affects the transition. One could also analyze the transition at finite frequency where a time-dependent state such as circulating currents would arise. Additionally, one could consider the transition at zero frequency but nonzero wave vector k_0. In this case one would be interested in wave vectors k such that $|k-k_0|<\Lambda$ for some cutoff Λ. If the equation of motion was derivable from a free energy of the form

$$F=\int_{\mathbf{q}} r(\mathbf{q})n(\mathbf{q})n(-\mathbf{q}) + \int_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} \lambda(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)n(\mathbf{q}_1)n(\mathbf{q}_2)n(\mathbf{q}_3) + \int_{\mathbf{q}_1\cdots\mathbf{q}_4} u(\mathbf{q}_1,\ldots,\mathbf{q}_4)n(\mathbf{q}_1)n(\mathbf{q}_2)n(\mathbf{q}_3)n(\mathbf{q}_4) \quad (73)$$

then we know that the cubic term drives the transition first order based on analogies with the solidification of an isotropic liquid.[26] Even in the case where the cubic term vanishes, the transition is still driven first order by fluctuations.[27] Due to the presence of nonequilibrium terms, however, the equation of motion will *not* be derivable from a free energy. Nonequilibrium effects could cause dramatic deviations from the equilibrium theory, and at present it is unclear what effect such terms will have on the transition.

Another avenue one could pursue with this theory is to address the properties of the ordered state away from the transition in model II. Numerical studies may be best suited for this purpose especially since the ordered state is likely to be inhomogeneous and not analytically tractable. One possible route is to generalize the numerics on the flocking transition done by Vicsek *et al.*[31] to include a magnetic field and interactions.

Finally, we note that while early numerical work on the flocking transition in the absence of a magnetic field indicated a continuous phase transition,[31] some recent simulations on larger system sizes hint at a weak first-order transition.[22] If true, this would be consistent with the transition continuing to be first order in the presence of a magnetic field as argued in this paper.

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