

Diagrammatic analysis of the unitary group for double-barrier ballistic cavities: Equivalence with circuit theory

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We derive a set of coupled nonlinear algebraic equations for the asymptotics of the Poisson kernel distribution describing the statistical properties of a two-terminal double-barrier chaotic billiard (or ballistic quantum dot). The equations are calculated from a diagrammatic technique for performing averages over the unitary group, proposed by Brouwer and Beenakker [J. Math. Phys. **37**, 4904 (1996)]. We give strong analytical evidences that these equations are equivalent to a much simpler polynomial equation calculated from a recent extension of Nazarov's circuit theory [A. M. S. Macêdo, Phys. Rev. B **66**, 033306 (2002)]. These results offer interesting perspectives for further developments in the field via the direct conversion of one approach into the other.

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I. INTRODUCTION

Random-matrix theory has proved to be a powerful tool for describing generic features of quantum chaotic systems.^{1,2} For closed systems it offers an accurate statistical description of both energy levels and wave-functions characteristics with overwhelming numerical and experimental confirmation. In this case, the central hypothesis is to replace the Hamiltonian of the system by a matrix with random independent Gaussian entries obeying certain fixed exact symmetries, such as time reversal and spin rotation. The resulting ensembles are known in the literature as the Gaussian ensembles. For open systems, such as ballistic chaotic cavities and disordered wires attached to external leads, the major role is shifted from the Hamiltonian to the appropriate matrix associated with the description of electron transport, viz., the scattering matrix or the transfer matrix. Here, the justification of the specific ensemble is a subtle procedure, because, unlike the previous case, one needs to take into account subtle correlations between the entries of the matrix induced by flux conservation and quantum diffusion (for disordered wires).

For ballistic chaotic cavities, a general procedure to obtain such an ensemble was developed by Mello and Baranger.³ It consists of evoking the *maximum information entropy principle* along with certain generic assumptions, such as analyticity, unitarity, and some specific symmetries that are exactly preserved in the presence of chaotic dynamics. The resulting S -matrix distribution $P(S)$ turns out to be a multidimensional generalization of the Poisson kernel, usually found in two-dimensional electrostatics. Its general form reads

$$P(S) \propto |\det(1 - \bar{S}^\dagger S)|^{-(\beta M + 2 - \beta)}, \quad (1)$$

where $\beta \in \{1, 2, 4\}$ is a parameter identifying distinct symmetry classes,⁴ $M = 2N$ is the total number of open scattering channels (N for each waveguide) and \bar{S} is a subunitary matrix, i.e., the eigenvalues of $\bar{S}\bar{S}^\dagger$ may be less than unity. For a

two-terminal system, consisting of a ballistic chaotic cavity or quantum dot coupled to two waveguides, the scattering matrix can be conveniently written as³

$$S = \begin{pmatrix} r & t \\ t' & r' \end{pmatrix}, \quad (2)$$

where t, t' and r, r' are, respectively, transmission and reflection matrices. The presence of barriers of arbitrary transparencies at the interface between the chaotic cavity and the waveguides can be accounted for by specifying the average (or optical part) of the scattering matrix.^{1,3}

$$\bar{S} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad (3)$$

where r_1 and r_2 are reflection matrices for barriers 1 and 2, respectively.

In mesoscopic physics, one is usually concerned with the description of a well-defined measurement, such as the full counting statistics (FCS) of a two-terminal system, whose cumulant generating function is given by the Levitov-Lesovik formula.⁵

$$\Phi(\lambda) = -M_0 \text{Tr} \ln[1 + (e^{i\lambda} - 1)t t^\dagger], \quad (4)$$

where $M_0 = eVT_0/h \gg 1$ is the number of attempts to transmit an electron during the observation time T_0 , t is the transmission matrix and V is the voltage. Physical observables can be obtained from the series expansion

$$\Phi(\lambda) = - \sum_{k=1}^{\infty} \frac{(i\lambda)^k}{k!} q_k, \quad (5)$$

where

$$q_k = - \frac{d^k}{d(i\lambda)^k} \Phi(\lambda) \Big|_{\lambda=0} \quad (6)$$

are the irreducible cumulants of the FCS. Some well-known examples are the dimensionless conductance and shot-noise power, which are given, respectively, by

$$g = q_1/M_0 = \text{Tr}(tt^\dagger), \quad (7)$$

which is the Landauer formula, and

$$p = q_2/M_0 = \text{Tr}[tt^\dagger(1 - tt^\dagger)]. \quad (8)$$

The third cumulant has also attracted some recent interest, including an experimental observation in tunnel junctions.⁶ It is given by

$$\kappa = q_3/M_0 = \text{Tr}[tt^\dagger(1 - tt^\dagger)(1 - 2tt^\dagger)]. \quad (9)$$

The above expressions for the cumulants of the FCS are sample specific, which means for a chaotic cavity that an average over the Poisson kernel distribution is necessary in order to make comparisons with real experimental data. Such calculations can in principle be performed exactly for arbitrary values of M and β using the method of supersymmetry^{7,8} and some very general results are in fact already available in the literature.^{9,10} Albeit powerful and general, this method can become very cumbersome and sometimes unwieldy. Fortunately, if one is interested only in the semiclassical regime, where one neglects quantum interference contributions, defined mathematically by the asymptotic condition $M \gg 1$, much simpler alternative techniques exist.

Two well-known methods to deal with the semiclassical regime are the diagrammatic analysis of the unitary group, presented by Brouwer and Beenakker¹¹ and Nazarov's circuit theory.¹² In its original form circuit theory does not allow for a direct comparison with the diagrammatic approach in the entire range of parameters, i.e., for arbitrary values of the barrier's transparencies, because of the intrinsic difficulty to determine the average pseudocurrent-voltage characteristics of an arbitrary circuit element. This is related to the well-known problem of breakdown of semiclassical transport equations close to boundaries and interfaces, where pure quantum effects become dominant. This difficulty was recently removed by a systematic treatment presented in Ref. 13, in which circuit theory is combined with the supersymmetric nonlinear sigma model yielding a powerful technique with a very convenient algebraic structure. This extended version of circuit theory raises the natural question as to whether its information content coincides with that of the diagrammatic method in all ranges of the input parameters. This is a highly nontrivial question, because the semiclassical concatenation principle, which is the basis of circuit theory, does not have a direct representation in the diagrammatic formulation. In particular, there is no obvious way how to extend this concatenation principle to account for quantum corrections, such as weak localization. We remark that the semiclassical concatenation principle is directly related to the independence of the leading asymptotic contribution on the symmetry parameter β . By contrast, the weak localization correction is strongly β dependent¹¹ and, in particular, vanishes for $\beta=2$, i.e., for systems with broken time-reversal symmetry. Interestingly, for the particular case of symmetric barriers a direct comparison between both approaches was presented in Ref. 13 and complete agreement was found.

Motivated by this result, we present in this work a detailed comparison between the diagrammatic approach and the extended version of circuit theory for the case of asymmetric barriers with arbitrary transparencies, thus completing the analysis of Ref. 13 and providing strong evidence for the full equivalence between these semiclassical techniques. This result is particularly relevant in practical applications because of the great algebraic advantage of the calculations in circuit theory (in our case study the problem is reduced to a polynomial equation of fourth order) in comparison with alternative approaches. In fact, we believe that circuit theory equations reach the ultimate irreducible form in terms of simplifying the description. Furthermore, when combined with the scaling theory presented in Ref. 13 for the ballistic-diffusive crossover in phase-coherent metallic conductors, in which the present results enters as an initial condition, we end up with a very powerful formalism for performing concrete calculations in realistic conductors. Applications might include spintronics¹⁴ and normal-superconducting hybrid systems.¹⁵

II. DIAGRAMMATIC TECHNIQUE

In this section we present a detailed account of the application of the diagrammatic method to the calculation of the semiclassical limit of a double barrier chaotic ballistic cavity. It contains the central result of this paper and complements a previous analysis by Brouwer and Beenakker.¹¹

We start by noting that all cumulants of the FCS can be generically written in the form

$$A = \text{Tr}[a(tt^\dagger)], \quad (10)$$

where $a(x)$ is an arbitrary smooth function. It means that the ensemble averages of transport observables can be fully described by the following density function:

$$\rho(\tau) = \langle \text{Tr} \delta(\tau - tt^\dagger) \rangle, \quad (11)$$

so that

$$\langle A \rangle = \int_0^1 d\tau a(\tau) \rho(\tau). \quad (12)$$

The problem is thus reduced to the calculation of $\rho(\tau)$ in the asymptotic regime $M=2N \gg 1$.

In the diagrammatic formalism, one starts by introducing the generating functions

$$F_1(z) = \langle C_1(z - S^\dagger C_2 S C_1)^{-1} \rangle, \quad (13)$$

and

$$F_2(z) = \langle C_2(z - S C_1 S^\dagger C_2)^{-1} \rangle, \quad (14)$$

where S is the $M \times M$ scattering matrix defined in Eq. (2) and the auxiliary matrices C_1 and C_2 are defined as

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_N \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1_N & 0 \\ 0 & 0 \end{pmatrix}, \quad (15)$$

in which 1_N is the $N \times N$ unit matrix. It can be easily verified that

$$\rho(\tau) = -\frac{1}{\pi} \text{Im}[f_1(\tau + i0^+)] = -\frac{1}{\pi} \text{Im}[f_2(\tau + i0^+)], \quad (16)$$

where $f_i(z) \equiv \text{Tr}[F_i(z)]$, $i=1, 2$.

The next step is to define the matrix generating function

$$\hat{F}(z) = \begin{pmatrix} 0 & F_1(z) \\ F_2(z) & 0 \end{pmatrix}, \quad (17)$$

which can be written as a sum of two terms that can be averaged separately, $\hat{F}(z) = (2z)^{-1}[\hat{F}_+(z) + \hat{F}_-(z)]$. Each term is given by

$$\hat{F}_\sigma(z) = \hat{C} + \sigma \hat{A}(\hat{1} - \hat{\Sigma}_\sigma \hat{G}_\sigma^0)^{-1} \hat{\Sigma}_\sigma \hat{B}, \quad (18)$$

where $\hat{A} = \hat{C}\hat{L}$, $\hat{B} = \alpha\hat{T}\hat{C}$, with $\alpha \equiv z^{-1/2}$ and the following matrices have been defined:

$$\hat{C} = \begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix}, \quad \hat{L} = \begin{pmatrix} L & 0 \\ 0 & L^\dagger \end{pmatrix}, \quad (19)$$

and

$$\hat{T} = \begin{pmatrix} T & 0 \\ 0 & T^\dagger \end{pmatrix}, \quad \hat{R} = \begin{pmatrix} R & 0 \\ 0 & R^\dagger \end{pmatrix}. \quad (20)$$

The submatrices L , T , and R describe transmission and reflection coefficients of the barriers and are related to \bar{S} of Eq. (3) via the condition that

$$\hat{U} = \begin{pmatrix} \bar{S} & L \\ T & R \end{pmatrix}, \quad (21)$$

be unitary. The remaining matrices in Eq. (18) are the ‘‘free propagator’’

$$\hat{G}_\sigma^0 = \hat{R} + \sigma \alpha \hat{T} \hat{C} \hat{L} \quad (22)$$

and the ‘‘self-energy’’ matrix $\hat{\Sigma}_\sigma$. They satisfy a Dyson equation

$$\hat{G}_\sigma = \hat{G}_\sigma^0 + \hat{G}_\sigma^0 \hat{\Sigma}_\sigma \hat{G}_\sigma = \hat{G}_\sigma^0 + \hat{G}_\sigma^0 \hat{\Sigma}_\sigma \hat{G}_\sigma^0. \quad (23)$$

Summation over planar diagrams¹¹ yields the following expression for the self-energy matrix:

$$\hat{\Sigma}_\sigma = \hat{P}_\sigma \zeta(\hat{P}_\sigma^2), \quad (24)$$

where

$$\hat{P}_\sigma = \begin{pmatrix} 0 & \text{Tr}(G_\sigma^{12}) \\ \text{Tr}(G_\sigma^{21}) & 0 \end{pmatrix} \otimes 1_M, \quad (25)$$

in which G_σ^{12} and G_σ^{21} are off-diagonal blocks of the full Green’s function

$$\hat{G}_\sigma = \begin{pmatrix} G_\sigma^{11} & G_\sigma^{12} \\ G_\sigma^{21} & G_\sigma^{22} \end{pmatrix}. \quad (26)$$

The function $\zeta(z)$ is defined by the planar series, $\zeta(z) = \sum_{n=1}^{\infty} w_n z^{n-1}$, in which w_n are symmetry coefficients given by

$$w_n = \frac{(-1)^{n+1} (2n-2)!}{n!(n-1)! M^{2n-1}}. \quad (27)$$

Evaluating the sum we obtain the closed expression

$$z^2 \zeta(z^2) = \frac{1}{2} (\sqrt{4z^2 + M^2} - M). \quad (28)$$

Using Eq. (28) we may write Eq. (24) in the form

$$\hat{\Sigma}_\sigma \hat{P}_\sigma \hat{\Sigma}_\sigma + M \hat{\Sigma}_\sigma = \hat{P}_\sigma. \quad (29)$$

To proceed, we introduce the variables

$$\theta_{1\sigma} = \frac{1}{N} \text{Tr}(G_\sigma^{12}) = \sigma \theta_1 \quad (30)$$

and

$$\theta_{2\sigma} = \frac{1}{N} \text{Tr}(G_\sigma^{21}) = \sigma \theta_2, \quad (31)$$

which implies the following simple form for the matrix \hat{P}_σ :

$$\hat{P}_\sigma = N \sigma \begin{pmatrix} 0 & \theta_1 \\ \theta_2 & 0 \end{pmatrix} \otimes 1_M. \quad (32)$$

Using the ansatz

$$\hat{\Sigma}_\sigma = \sigma \begin{pmatrix} 0 & \beta_1 \\ \beta_2 & 0 \end{pmatrix} \otimes 1_M \quad (33)$$

in Eq. (29), we obtain the following coupled system:

$$\begin{cases} \beta_1^2 \theta_2 + 2\beta_1 = \theta_1 \\ \beta_2^2 \theta_1 + 2\beta_2 = \theta_2. \end{cases} \quad (34)$$

At this stage we need to further specify the nature of the contacts between the cavity and the leads. We assume the presence of asymmetric barriers with N equivalent transmitting channels. The adequate choices for the matrices T , L , and R are therefore given by

$$T = \begin{pmatrix} i\sqrt{\Gamma_1} & 0 \\ 0 & i\sqrt{\Gamma_2} \end{pmatrix} \otimes 1_N = L \quad (35)$$

and

$$R = \begin{pmatrix} \sqrt{1-\Gamma_1} & 0 \\ 0 & \sqrt{1-\Gamma_2} \end{pmatrix} \otimes 1_N = \bar{S}, \quad (36)$$

in which Γ_1 and Γ_2 represent the transmission coefficients of each channel in barriers 1 and 2, respectively. Combining these expressions with the ansatz (33) for the self-energy matrix, we may calculate explicitly the full Green’s function, from which [after using Eq. (25)] we obtain

$$\theta_1 = \frac{\alpha \Gamma_2 + (1-\Gamma_2)\beta_1}{1 - \alpha \beta_2 \Gamma_2 - \beta_1 \beta_2 (1-\Gamma_2)} + \frac{(1-\Gamma_1)\beta_1}{1 - \alpha \beta_1 \Gamma_1 - \beta_1 \beta_2 (1-\Gamma_1)} \quad (37)$$

and

$$\theta_2 = \frac{\alpha\Gamma_1 + (1 - \Gamma_1)\beta_2}{1 - \alpha\beta_1\Gamma_1 - \beta_1\beta_2(1 - \Gamma_1)} + \frac{(1 - \Gamma_2)\beta_2}{1 - \alpha\beta_2\Gamma_2 - \beta_1\beta_2(1 - \Gamma_2)}. \quad (38)$$

Combining Eqs. (37) and (38) with Eq. (34), we obtain the following nonlinear algebraic system:

$$\begin{aligned} \alpha(1 - \Gamma_1)\Gamma_2\beta_1\beta_2^3 + \{[\alpha^2\Gamma_1\Gamma_2 + (2\Gamma_1 - 1)\Gamma_2 - \Gamma_1]\beta_1 - \alpha(1 \\ + \Gamma_1)\Gamma_2\}\beta_2^2 + [(\Gamma_1 - 2\Gamma_1\Gamma_2)\alpha\beta_1 + \alpha^2\Gamma_1\Gamma_2 + \Gamma_1 + \Gamma_2]\beta_2 \\ - \alpha\Gamma_1 = 0 \end{aligned} \quad (39)$$

and

$$\begin{aligned} \alpha(1 - \Gamma_2)\Gamma_1\beta_2\beta_1^3 + \{[\alpha^2\Gamma_1\Gamma_2 + (2\Gamma_2 - 1)\Gamma_1 - \Gamma_2]\beta_2 - \alpha(1 \\ + \Gamma_2)\Gamma_1\}\beta_1^2 + [(\Gamma_2 - 2\Gamma_1\Gamma_2)\alpha\beta_2 + \alpha^2\Gamma_1\Gamma_2 + \Gamma_1 + \Gamma_2]\beta_1 \\ - \alpha\Gamma_2 = 0. \end{aligned} \quad (40)$$

Finally, inserting the ansatz Eq. (24) into Eq. (18) yields

$$f_1(z) = \alpha^2 N \left[1 - \frac{\alpha\Gamma_2\beta_2}{1 - (1 - \Gamma_2)\beta_1\beta_2} \right]^{-1}, \quad (41)$$

and

$$f_2(z) = \alpha^2 N \left[1 - \frac{\alpha\Gamma_1\beta_1}{1 - (1 - \Gamma_1)\beta_1\beta_2} \right]^{-1}. \quad (42)$$

Equations (39), (40), (42), and (41) are the central results of this section. Together with Eq. (16) they represent the complete solution of the problem. It extends the calculations of Brouwer and Beenakker¹¹ to include the case of asymmetric barriers (although in their solution of the symmetric case the channels in the barriers were considered nonequivalent). Before presenting a complete analysis of the generic situation, we shall discuss below some important particular cases.

A. Chaotic cavity with symmetric barriers

The case of symmetric barriers is described by setting $\Gamma_1 = \Gamma = \Gamma_2$ and $\beta_1 = \beta = \beta_2$ in Eqs. (39), (40), (42), and (41), so that we find

$$(\alpha\beta^2 - 2\beta + \alpha)[(1 - \Gamma)\beta^2 + \alpha\Gamma\beta - 1] = 0, \quad (43)$$

and

$$f_1(z) = \alpha^2 N \left[1 - \frac{\alpha\Gamma\beta}{1 - (1 - \Gamma)\beta^2} \right]^{-1} = f_2(z). \quad (44)$$

The physical root of Eq. (43) is given by $\beta = (1 - \sqrt{1 - \alpha^2})/\alpha = \sqrt{z} - \sqrt{z - 1}$, which yields $f_1(z) = f(z) = f_2(z)$, where

$$\frac{f(z)}{N} = \frac{2(\sqrt{z} - \sqrt{z - 1})(1 - \Gamma) + \Gamma/\sqrt{z - 1}}{2z(\sqrt{z} - \sqrt{z - 1})(1 - \Gamma) + \Gamma\sqrt{z}}. \quad (45)$$

Inserting Eq. (45) into Eq. (16) we find the average density

$$\rho(\tau) = \frac{N}{\pi} \frac{\Gamma(2 - \Gamma)}{(\Gamma^2 - 4\Gamma\tau + 4\tau)\sqrt{\tau(1 - \tau)}}, \quad (46)$$

in agreement with Ref. 13. The solution obtained by Brouwer and Beenakker¹¹ is, in fact, a generalization of Eq. (46) for nonequivalent channels. It reads

$$\rho(\tau) = \sum_{n=1}^N \frac{\Gamma_n(2 - \Gamma_n)}{\pi(\Gamma_n^2 - 4\Gamma_n\tau + 4\tau)\sqrt{\tau(1 - \tau)}}, \quad (47)$$

which, of course, reproduces Eq. (46) when $\Gamma_n = \Gamma$ for all n .

B. Chaotic cavity with two tunnel junctions

This system is described by applying the conditions $\Gamma_1, \Gamma_2 \ll 1$ in Eqs. (39), (40), (42), and (41). We get

$$\beta_2^2\alpha\Gamma_2 - \beta_2(\Gamma_1 + \Gamma_2) + \alpha\Gamma_1 = 0,$$

$$\beta_1^2\alpha\Gamma_1 - \beta_1(\Gamma_1 + \Gamma_2) + \alpha\Gamma_2 = 0, \quad (48)$$

together with

$$f_1(z) = \alpha^2 N \left[1 - \frac{\alpha\Gamma_1\beta_1}{1 - \beta_1\beta_2} \right]^{-1}, \quad (49)$$

and

$$f_2(z) = \alpha^2 N \left[1 - \frac{\alpha\Gamma_2\beta_2}{1 - \beta_1\beta_2} \right]^{-1}. \quad (50)$$

From the physical roots of Eq. (48), we obtain $f_1(z) = f(z) = f_2(z)$, where

$$f(z) = \frac{N}{z} \left[1 + \frac{\Gamma_1\Gamma_2}{(\Gamma_1 + \Gamma_2)\sqrt{z(z - \tau_0)}} \right], \quad (51)$$

in which $\tau_0 = 4\Gamma_1\Gamma_2/(\Gamma_1 + \Gamma_2)^2$. Inserting Eq. (51) into Eq. (16) we obtain

$$\rho(\tau) = \frac{N\Gamma_1\Gamma_2}{\pi(\Gamma_1 + \Gamma_2)} \frac{1}{\tau^{3/2}\sqrt{\tau_0 - \tau}}, \quad (52)$$

in agreement with Refs. 12 and 16.

Using Eq. (52) we can compute the average value of several physical observables. Of particular interest are the first three cumulants of the FCS, defined in Eqs. (7)–(9). We find

$$\langle g \rangle_{\text{TJ}} = N \frac{\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2}, \quad (53)$$

for the average conductance

$$\left\langle \frac{\langle p \rangle}{\langle g \rangle} \right\rangle_{\text{TJ}} = \frac{\Gamma_1^2 + \Gamma_2^2}{(\Gamma_1 + \Gamma_2)^2}, \quad (54)$$

for the Fano factor, and

$$\left\langle \frac{\langle \kappa \rangle}{\langle p \rangle} \right\rangle_{\text{TJ}} = \frac{\Gamma_1^4 - 2\Gamma_1^3\Gamma_2 + 6\Gamma_1^2\Gamma_2^2 - 2\Gamma_1\Gamma_2^3 + \Gamma_2^4}{(\Gamma_1 + \Gamma_2)^2(\Gamma_1^2 + \Gamma_2^2)}, \quad (55)$$

for the ratio between the average third cumulant and the average shot-noise power. The subscript TJ stands for *Tunnel*

Junction. Equations (53)–(55) are well known in the literature.¹⁷

III. GENERAL ANALYSIS AND COMPARISON WITH CIRCUIT THEORY

In this section we compare the predictions of the above diagrammatic approach with those of the extended version of circuit theory. The exact agreement that we found for various quantities strongly suggests the full equivalence of these semiclassical techniques in the entire range of input parameters.

A. Circuit theory

Circuit theory was invented by Nazarov¹² and represents a very powerful tool for computing ensemble averages of quantum chaotic systems in the semiclassical regime. It consists of a finite element approach in which the spatial support of the system is partitioned into a network, containing edges (or connectors), and vertices (nodes or terminals). In its simplest version, there are only two terminals and the system is subject to a fictitious pseudo-potential Φ with fixed values at the terminals and unknown values at the internal nodes. The basic principles are (1) a general law of pseudocurrent-voltage (I - V) characteristics for a connector (i, j) subject to a pseudopotential drop $\Delta\Phi_{ij}$

$$I_{ij}(\Delta\Phi_{ij}) = \int_0^1 d\tau \frac{\tau \rho_{ij}(\tau) \sin(\Delta\Phi_{ij})}{1 - \tau \sin^2(\Delta\Phi_{ij}/2)}, \quad (56)$$

where $I_{ij}(\Delta\Phi_{ij})$ is a pseudocurrent and $\rho_{ij}(\tau) = \langle \text{Tr} \delta(\tau - t_{ij}^\dagger t_{ij}) \rangle$ is the average transmission eigenvalue density of the connector. We defined t_{ij} as the sample specific transmission matrix of the connector. (2) A generalized Kirchhoff law for pseudocurrent conservation at each node.

The power of this approach depends crucially on the appropriate choice of the connectors, which in turn depends on our ability to calculate the correspondent pseudo- I - V characteristics. This is the point where the extended approach, put forward in Ref. 13, differs from Nazarov's original formulation. In Ref. 13 the function $I_{ij}(\Delta\Phi_{ij})$ was calculated directly from the supersymmetric nonlinear σ model. The advantage of this procedure is the possibility to take full account of quantum effects in the description, whenever necessary, such as near barriers and interfaces, without the need to introduce additional assumptions for performing the ensemble averages.

As an example, consider the system studied in previous section, i.e., a double barrier chaotic billiard with N equivalent channels at each contact. In the extended circuit theory, we end up with only two equations¹³

$$I(\phi) = I_1(\phi - \theta) = I_2(\theta), \quad (57)$$

where

$$I_j(\phi) = \frac{N\Gamma_j \sin(\phi)}{1 - \Gamma_j \sin^2(\phi/2)} \quad (58)$$

is the pseudo- I - V characteristics of barrier j , interpreted as a connector. We remark that these equations contain all quan-

tum information that remain relevant in the semiclassical regime after ensemble averaging. They are therefore a direct consequence of the maximum entropy principle. In Ref. 16 it was found to be convenient to introduce the following modified pseudocurrent:

$$K(x) = \frac{i}{2} I(-2ix), \quad (59)$$

which yields the conservation law

$$K(x) = K_1(x - y) = K_2(y), \quad (60)$$

where

$$K_j(x) = \frac{N}{2} \left[\tanh\left(x + \frac{1}{2}\alpha_j\right) + \tanh\left(x - \frac{1}{2}\alpha_j\right) \right], \quad (61)$$

in which we introduced the constants α_j via the relation $\Gamma_j = \text{sech}^2(\alpha_j/2)$. Inserting (61) into Eq. (60) yields

$$\begin{aligned} & \tanh\left(x - y + \frac{1}{2}\alpha_1\right) + \tanh\left(x - y - \frac{1}{2}\alpha_1\right) \\ &= \tanh\left(y + \frac{1}{2}\alpha_2\right) + \tanh\left(y - \frac{1}{2}\alpha_2\right), \end{aligned} \quad (62)$$

which after using the trigonometric identity

$$\tanh(x - y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}, \quad (63)$$

yields the following polynomial equation for the variable $\xi = \tanh y$

$$\begin{aligned} & [\Gamma_1(1 - \Gamma_2)\tanh x] \xi^4 - (3\Gamma_1\Gamma_2 \tanh x) \xi^2 + [(\Gamma_1\Gamma_2 + \Gamma_2 - \Gamma_1) \\ & \times \tanh^2 x + 2\Gamma_1\Gamma_2 - \Gamma_1 - \Gamma_2] \xi^3 + [(\Gamma_1\Gamma_2 + \Gamma_1 - \Gamma_2) \\ & \times \tanh^2 x + \Gamma_1 + \Gamma_2] \xi - \Gamma_1 \tanh x = 0. \end{aligned} \quad (64)$$

This equation must be supplemented by

$$K(x) = \frac{N\Gamma_2 \xi}{1 - (1 - \Gamma_2)\xi^2}. \quad (65)$$

Equations (64) and (65) were presented in Ref. 16 and are the circuit theory equations for the double barrier chaotic billiard. In the next section we shall compare them with Eqs. (39), (40), (42), and (41), obtained from the diagrammatic method.

B. Comparison between circuit theory and the diagrammatic approach

In order to facilitate comparison, let us first establish a direct relation between the generating function $f_j(z) = \text{Tr} F_j(z)$ of the diagrammatic technique and the modified pseudo-current $K(x)$ of circuit theory. From Eq. (13) we obtain

$$F_1(z) = \sum_{n=0}^{\infty} \langle C_1(S^\dagger C_2 S C_1)^n \rangle \frac{1}{z^{n+1}}, \quad (66)$$

and

$$F_2(z) = \sum_{n=0}^{\infty} \langle C_2 (S C_1 S^\dagger C_2)^n \rangle \frac{1}{z^{n+1}}. \quad (67)$$

Inserting Eqs. (2) and (15) into the above equations we get

$$F_1(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \begin{pmatrix} 0 & 0 \\ 0 & (t^\dagger t)^n \end{pmatrix}, \quad (68)$$

and

$$F_2(z) = \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \begin{pmatrix} (tt^\dagger)^n & 0 \\ 0 & 0 \end{pmatrix}.$$

Performing the trace, we obtain $f_1(z) = f(z) = f_2(z)$, where

$$z^2 f(z) = Nz + h(1/z), \quad (69)$$

and we introduced the function $h(z)$, defined as

$$h(z) = \left\langle \text{Tr} \left(\frac{tt^\dagger}{1 - ztt^\dagger} \right) \right\rangle = \int_0^1 d\tau \frac{\tau \rho(\tau)}{1 - z\tau}. \quad (70)$$

From Eq. (56), one realizes that

$$I(\phi) = h[\sin^2(\phi/2)] \sin \phi, \quad (71)$$

which when combined with Eq. (69) yields

$$f(z) = \frac{1}{z} \left(N + \frac{I(\phi)}{2\sqrt{z-1}} \right) \Bigg|_{\sin(\phi/2)=1/\sqrt{z}}. \quad (72)$$

Using the relation between $K(x)$ and $I(\phi)$, Eq. (59), we may rewrite the above equation as

$$f(z) = \frac{1}{z} (N - K(x) \tanh x) \Big|_{\sinh x = 1/\sqrt{z}}. \quad (73)$$

Let us consider two simple applications of Eq. (73). For a chaotic cavity with symmetric barriers, Eqs. (64) and (65) yield

$$K(x) = \frac{N\Gamma \sinh x}{2 - \Gamma + \Gamma \cosh x}. \quad (74)$$

Inserting Eq. (74) into Eq. (73) we find

$$f(z) = \frac{N}{\sqrt{z}} \left(\frac{(2 - \Gamma)\sqrt{z-1} + \Gamma\sqrt{z}}{\sqrt{z-1}[(2 - \Gamma)\sqrt{z} + \Gamma\sqrt{z-1}]} \right), \quad (75)$$

which can be easily shown to agree with Eq. (45). For a chaotic cavity with two tunnel junctions, we obtain from Eqs. (64) and (65) the following expression:

$$K(x) = \frac{N\Gamma_1\Gamma_2 \sinh 2x}{2\sqrt{\Gamma_1^2 + \Gamma_2^2 + 2\Gamma_1\Gamma_2} \cosh 2x}. \quad (76)$$

Inserting Eq. (76) into Eq. (73) we obtain

$$f(z) = \frac{N}{z} \left[1 + \frac{\Gamma_1\Gamma_2}{(\Gamma_1 + \Gamma_2)\sqrt{z}(z - \tau_0)} \right], \quad (77)$$

in agreement with Eq. (51).

We can also invert Eq. (73) to obtain $K(x)$ from the diagrammatic approach

$$K(x) = \coth(x) (N - zf(z)) \Big|_{z = -\sinh^{-2}(x)}. \quad (78)$$

In particular, we can calculate the average of the cumulants of the FCS using the formula

$$h_{k+1} \equiv \langle \text{Tr}[(tt^\dagger)^{k+1}] \rangle = \frac{(-1)^k 2^k}{k!} \frac{d^k H(x)}{d[\cosh(2x)]^k} \Big|_{x=0}, \quad (79)$$

where $H(x) \equiv 2K(x)/\sinh(2x)$. The first three cumulants are then given by

$$\langle g \rangle = h_1, \quad \langle p \rangle = h_1 - h_2, \quad \langle \kappa \rangle = h_1 - 3h_2 + 2h_3. \quad (80)$$

We remark that the function $H(x)$ can also be obtained directly from $f(z)$ via the relation

$$H(x) = z(zf(z) - N) \Big|_{z = -\sinh^{-2}(x)}. \quad (81)$$

Motivated by strong numerical evidences,¹⁶ our basic conjecture is that one obtains the *same* function $H(x)$ either from Eqs. (39), (40), (42), and (41) or from Eqs. (64) and (65). Although explicit analytic evaluation of $H(x)$ in both approaches, for general Γ_1 and Γ_2 , is too cumbersome (albeit possible in principle), we can still make some analytical progress in verifying our conjecture by expanding $H(x)$ in powers of x , and evaluating the coefficients as explicit functions of Γ_1 and Γ_2 . This procedure yields, *in both approaches*, the following closed expressions:

$$\langle g \rangle = N \frac{\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} = \langle g \rangle_{\text{TJ}} \quad (82)$$

for the conductance

$$\frac{\langle p \rangle}{\langle g \rangle} = \frac{\Gamma_1 + \Gamma_2 - \Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \left(\frac{\langle p \rangle}{\langle g \rangle} \right)_{\text{TJ}} \quad (83)$$

for the Fano factor, and

$$\frac{\langle \kappa \rangle}{\langle p \rangle} = \frac{\Gamma_1 + \Gamma_2 - 2\Gamma_1\Gamma_2}{\Gamma_1 + \Gamma_2} \left(\frac{\langle \kappa \rangle}{\langle p \rangle} \right)_{\text{TJ}} \quad (84)$$

for the ratio between the average third cumulant $\langle \kappa \rangle$ and the average shot-noise power $\langle p \rangle$. We also found agreement for the fourth cumulant. Note that for tunnel junctions, we have $\Gamma_1, \Gamma_2 \ll 1$, and Eqs. (82)–(84) reduce to Eqs. (53)–(55), as expected. The above expressions together with the fourth cumulant are in complete agreement with the semiclassical cascade approach presented in Ref. 18. They represent strong analytical evidences for the mathematical equivalence between Eqs. (39), (40), (42), and (41) of the diagrammatic technique and Eqs. (64) and (65) of circuit theory, in the description of the asymptotic semiclassical domain of the Poisson kernel distribution.

IV. SUMMARY AND PERSPECTIVES

We presented a detailed comparison between two well known semiclassical approaches, the diagrammatic analysis of the unitary group and circuit theory, in the description of quantum transport through double-barrier chaotic billiards.

The problem was reduced to a comparison between a pair of coupled non-linear algebraic equations (diagrammatic technique) and a polynomial equation of fourth order (circuit theory). Exact agreement was found for a variety of quantities, such as the first four average cumulants of the full counting statistics and the average transmission eigenvalue density for symmetric barriers and tunnel junctions. The complete equivalence of these approaches is a non-trivial result, because the semiclassical concatenation principle, used to derive circuit theory equations, has no obvious counterpart in the diagrammatic method and leads to a substantial algebraic simplification of the whole problem. It is quite amusing to observe that circuit theory turns out to yield a irreducible description that intuitively satisfies the ‘‘Occam’s razor’’ principle of descriptive simplicity.¹⁹

An interesting consequence of our result would be the extension of circuit theory to deal with quantum corrections, which can be systematically treated in both the supersymmetric non-linear σ -model and in the diagrammatic technique. From a broader perspective, we expect our result to help establishing a direct connection between several recent independent developments of both circuit theory and the diagrammatic S -matrix approach, such as those in magnetoelectronics²⁰ with obvious technical advantages.

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