Fluctuation-dissipation considerations and damping models for ferromagnetic materials

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The role of fluctuation-dissipation relations (theorems) for the magnetization dynamics with Landau-Lifshitz-Gilbert and Bloch-Bloembergen damping terms are discussed. We demonstrate that the use of the Callen-Welton fluctuation-dissipation theorem, which was proven for Hamiltonian systems, can give an inconsistent result for magnetic systems with dissipation.

I. INTRODUCTION

The study of linear stochastic magnetization dynamics is of great importance in applications to nanomagnetic devices and ultrathin films. In general, gyromagnetic magnetization motion around an effective field is randomly forced by fluctuations on spins by means of interaction with a thermal bath (phonons, magnons, conduction electrons, impurities, etc.). Fluctuation-dissipation relations provide a very useful tool to find a correspondence between the dynamic variable fluctuations, temperature, and dissipation for a given magnetic dynamic system (see, e.g., Refs. 1–5).

The most frequently used fluctuation-dissipation relations are known as the Callen-Welton fluctuation-dissipation theorem (FDT).[6,7] There are two integrals that express this theorem. The first one gives the correlation function of the dynamic variables in terms of an integral of the system susceptibility. The second form relates the correlation function of the applied noise to an integral of the inverse susceptibility. This theorem is proved for rather general assumptions (we will discuss them later) and therefore appears to be very attractive for general problems. The aim of this paper is to discuss the inapplicability of the second relation to magnetization dynamics in ferromagnetic systems. In particular, we demonstrate that Callen-Welton FDT gives mathematical inconsistency and cannot provide an argument to discriminate between different damping models.

II. FLUCTUATION-DISSIPATION THEOREM

The Callen-Welton fluctuation-dissipation theorem is proved under very general assumptions for *Hamiltonian systems* of an arbitrary type (see, e.g., Refs. 8 and 9). These systems have no internal dissipation at all. The energy $\mathcal{V}$ of the external perturbation acting on the system is taken in the form

$$\mathcal{V} = -\sum_j x_j(t) f_j(t), \tag{1}$$

where $x_j(t)$ is the jth component of the dynamic variable and $f_j(t)$ is the jth component of the external alternating field. All dynamic variables are assumed to be equal to zero in the absence of the perturbation, and the linear responses are defined by

$$x_j(t) = \sum_k \int_{-\infty}^t K_{jk}(t-\tau) f_k(\tau) d\tau, \tag{2}$$

where $K_{jk}(t)$ is a memory function that depends on the properties of the dynamic system.[10] A "thermal bath" in this theorem is modeled as a set of periodic external fields, which are absorbed by the dynamic system. On the other hand, the external fields stimulate radiation from the system and cause a loss of energy. For a periodic external field

$$f_j(t) = \tfrac{1}{2}[f_{0j}e^{-i\omega t} + f_{0j}^* e^{i\omega t}], \tag{3}$$

one can rewrite Eq. (2) in the form

$$x_j(t) = \frac{1}{2}\sum_k [\chi_{jk}(\omega)f_{0k}e^{-i\omega t} + \chi_{jk}^*(\omega)f_{0k}^* e^{i\omega t}], \tag{4}$$

where

$$\chi_{jk}(\omega) = \int_0^\infty K_{jk}(t)\exp(i\omega t)dt \tag{5}$$

is the susceptibility tensor.

The change of the total energy $\mathcal{E}$ (which includes the perturbation energy $\mathcal{V}$) in the dynamic system that plays an important part in this theory is expressed as

$$\frac{d\mathcal{E}}{dt} = -\sum_j x_j(t)\frac{df_j(t)}{dt}. \tag{6}$$

Averaging this equation over the period of the external fields (3) and taking into account Eq. (4), one obtains the following expression for the dissipated power:

$$Q = \frac{i\omega}{4}\sum_{j,k}[\chi_{jk}^*(\omega) - \chi_{kj}(\omega)]f_{0j}f_{0k}^*. \tag{7}$$

In the derivation both the absorbed and radiated averaged powers are usually expressed quantum mechanically in terms of transitions probabilities (per unit time) between infinitely narrow energy levels, i.e., no linewidth or damping is taken into account. Temperature is introduced by the thermodynamic weights of the energy levels. This gives a thermal averaging over a thermal bath. Summing up over frequencies ω, one can obtain the balance between absorption and irradiation. This balance, expressed in terms of dynamic suscep-

tibility, gives the Callen-Welton FDT. In the classical limit it can be written as

$$\langle x_j(\tau)x_k(0)\rangle = \frac{k_B T}{2\pi}\int_{-\infty}^{\infty}\frac{\chi_{kj}(\omega)-\chi_{jk}^*(\omega)}{i\omega}e^{-i\omega\tau}d\omega. \quad (8)$$

Here k_B is the Boltzman constant, and T is the temperature. Angular brackets denote both thermal averaging and averaging over random phases in the periodic fields.

A second form of the Callen-Welton FDT is also introduced

$$\langle f_j(\tau)f_k(0)\rangle = \frac{k_B T}{2\pi}\int_{-\infty}^{\infty}\frac{\chi_{kj}^{*-1}(\omega)-\chi_{jk}^{-1}(\omega)}{i\omega}e^{-i\omega\tau}d\omega. \quad (9)$$

This form results from the reversibility of the linear relation

$$x_j(\omega)=\chi_{jk}(\omega)f_k(\omega) \quad (10)$$

to

$$f_j(\omega)=\chi_{jk}^{-1}(\omega)x_k(\omega). \quad (11)$$

It should be noted that the first (8) and second (9) relations can be interpreted differently. Namely, in the first case we have a reasonable correspondence between the correlations of dynamic variables and the susceptibility of the dynamic system. Changing the dynamic system properties and, hence, the susceptibility, we change the correlations of the dynamic variable. A specific external noise mechanism is excluded, and the role of the thermal bath is included implicitly by assuming a system in thermal equilibrium. This gives a linear dependence on the temperature T.

On the other hand, in the second case the noise correlations are associated with the inverse susceptibility of the dynamic system. In other words, two principally different physical characteristics, the noise (which results from a thermal bath dynamics) and the susceptibility(which describes just the properties of the dynamic system) should correspond to each other. From Eq. (9) it follows that by changing properties of the dynamic system, one can change the noise correlations.

However, physically it seems impossible, in general, that the dynamic susceptibility determines the noise variance. It is not clear that the correspondence between (10) and (11) applies for general random processes. This follows from the fact that the integral (2) depends on the form of stochastic integration because $f_k(\tau)$ are random variables. For example, the Ito and Stratonovich approaches to stochastic integration have no general relationship between each other in the case of multiplicative noise.[11]

In applications to systems with dissipation, the Callen-Welton fluctuation-dissipation theorem may be used just as an approximation, which must be validated. For some dissipative systems the Callen-Welton FDT gives reasonable estimates. It covers, for example, the Einstein relation for Brownian motion and the classical Nyquist formula for voltage fluctuations. However, these particular cases do not prove that the theorem is applicable for any linear system with dissipation. Some criticism of the Callen-Welton FDT is given by Klimontovich.[9,12]

III. FDT AND MAGNETIC DAMPING

A commonly used theoretical approach to magnetic dynamics, which is purely phenomenological, is based on the Landau-Lifshitz equation,[13] or its modification in the Gilbert form.[14] Local and isotropic phenomenological damping terms and corresponding local random fields are assumed to describe a "thermal bath" in a magnetic material. On the other hand, a microscopic approach predicts nonlocal field fluctuations[15] and magnetization dynamics in the form of the Bloch-Bloembergen equations; the Landau-Lifshitz-Gilbert equations occur only for uniaxial symmetry.[16,17]

Recently Smith,[18] using the second form of the Callen-Welton theorem (9), claimed that fluctuation-dissipation relations provide a means for discriminating between alternate phenomenological magnetic damping models. Here we demonstrate that, by using Eq. (9), the fluctuation-dissipation relations for stochastic dynamics with Landau-Lifshitz-Gilbert damping and Bloch-Bloembergen damping are inconsistent.

For simplicity we shall consider a single-domain particle with the magnetic energy $\mathcal{E}$ in the vicinity of equilibrium

$$\frac{\mathcal{E}}{M_s V}=\frac{H_x}{2}m_x^2+\frac{H_y}{2}m_y^2-h_x m_x-h_y m_y. \quad (12)$$

Here m_x and m_y are the transverse components of the unit vector **m** oriented along the magnetization, M_s is the saturation magnetization, V is the volume of the sample, and H_x and H_y are the Kittel stiffness fields (in general, $H_x\neq H_y$, for example, in a thin film). For a general stochastic problem $h_x(t)$ and $h_y(t)$ are random variables arising from the interaction with a thermal bath. To illustrate the utilization of Callen-Welton FDT one assumes that $h_x(t)$ and $h_y(t)$ are equivalent to the external alternating fields f_j in Eq.(1) that leads to the forms of Eqs. (8) and (9).

The linearized Landau-Lifshitz-Gilbert equation can be written in the form

$$\frac{1}{\gamma}(\dot{m}_x+\alpha\dot{m}_y)=-H_y m_y+h_y(t),$$

$$\frac{1}{\gamma}(\dot{m}_y-\alpha\dot{m}_x)=H_x m_x-h_x(t), \quad (13)$$

or, equivalently, as

$$\frac{1}{\tilde{\gamma}}\dot{m}_x=-\alpha H_x m_x-H_y m_y+h_y(t)+\alpha h_x(t),$$

$$\frac{1}{\tilde{\gamma}}\dot{m}_y=-\alpha H_y m_y+H_x m_x-h_x(t)+\alpha h_y(t), \quad (14)$$

where $\tilde{\gamma}=\gamma/(1+\alpha^2)$, γ is the gyromagnetic ratio, and α is a dimensionless damping parameter. From Eq. (13) one can obtain the inverse susceptibility

$$\chi_{jk}^{-1}(\omega)=\frac{1}{\gamma M_s V}\begin{pmatrix}-i\alpha\omega+\gamma H_x & i\omega \\ -i\omega & -i\alpha\omega+\gamma H_y\end{pmatrix}, \quad (15)$$

where $h_j(\omega)=\chi_{jk}^{-1}(\omega)m_k(\omega)M_s V; m_k(\omega)M_s V$ is the kth component of the magnetic moment. Substituting Eq. (15) and its

Hermitian conjugate into Eq. (9), the second form of Callen-Welton FDT can be expressed as

$$
\begin{pmatrix} \langle h_x(\tau)h_x(0)\rangle & \langle h_x(\tau)h_y(0)\rangle \\ \langle h_y(\tau)h_x(0)\rangle & \langle h_y(\tau)h_y(0)\rangle \end{pmatrix} = \frac{2k_B T\alpha}{\gamma M_s V}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta(\tau).
$$
(16)

The case of linearized Bloch-Bloembergen magnetization dynamics is described by

$$
\frac{1}{\gamma}\dot m_x = -\frac{1}{\gamma T_2}m_x - H_y m_y + h_y(t),
$$

$$
\frac{1}{\gamma}\dot m_y = -\frac{1}{\gamma T_2}m_y + H_x m_x - h_x(t),
$$
(17)

where T_2 is the relaxation time. From Eq. (17) we can obtain the following inverse susceptibility:

$$
\chi_{jk}^{-1}(\omega) = \frac{1}{\gamma M_s V}\begin{pmatrix} \gamma H_x & i\omega - T_2^{-1} \\ -i\omega + T_2^{-1} & \gamma H_y \end{pmatrix}.
$$
(18)

Application of Eq. (9) yields FDT

$$
\begin{pmatrix} \langle h_x(\tau)h_x(0)\rangle & \langle h_x(\tau)h_y(0)\rangle \\ \langle h_y(\tau)h_x(0)\rangle & \langle h_y(\tau)h_y(0)\rangle \end{pmatrix} = \frac{k_B T}{\gamma M_s V}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\frac{\mathrm{sgn}(\tau)}{T_2}.
$$
(19)

Note that the difference between Eqs. (16) and (19) lies in the tensor on the right-hand side of each equation. Because of the diagonal tensor, Eq. (16) implies no correlations between h_x and h_y. On the other hand, Eq. (19) does show such correlations. One can argue that the Bloch-Bloembergen form of damping is not physical because of the nondiagonal form of (19). However, as we show below, there is an inconsistency in the use of the second form of the Callen-Welton FDT.

Let us now consider the uniaxial case when both dynamics should coincide. We examine the dynamics of a soft micromagnetic particle (no anisotropy) in an external magnetic field H_0 and small damping approximation $\alpha \ll 1$ (the most interesting case), when terms $\sim \alpha^2$ can be neglected. Thus $H_x = H_y = H_0$, and the Landau-Lifshitz-Gilbert equation (14) is reduced to

$$
\frac{1}{\gamma}\frac{dm_x}{dt} = -\alpha H_0 m_x - H_0 m_y + h_y(t) + \alpha h_x(t),
$$

$$
\frac{1}{\gamma}\frac{dm_y}{dt} = -\alpha H_0 m_y + H_0 m_x - h_x(t) + \alpha h_y(t).
$$
(20)

Thus far, as $h_x(t)$ and $h_y(t)$ represent two independent random fields, their linear combinations are also random. We can consider the following orthogonal transformation:

$$
\tilde h_y(t) = \frac{h_y(t) + \alpha h_x(t)}{\sqrt{1+\alpha^2}} \simeq h_y(t) + \alpha h_x(t),
$$

$$
-\tilde h_x(t) = \frac{-h_x(t) + \alpha h_y(t)}{\sqrt{1+\alpha^2}} \simeq -h_x(t) + \alpha h_y(t).
$$
(21)

Thus, random fields $\tilde h_x(t)$ and $\tilde h_y(t)$ are independent and Eq. (20) can be rewritten in the form

$$
\frac{1}{\gamma}\frac{dm_x}{dt} = \alpha H_0 m_x - H_0 m_y + \tilde h_y(t),
$$

$$
\frac{1}{\gamma}\frac{dm_y}{dt} = \alpha H_0 m_y + H_0 m_x - \tilde h_x(t).
$$
(22)

From Eqs. (16) and (21) one can easily calculate pair correlations for $\tilde h_x(t)$ and $\tilde h_y(t)$ (the terms $\sim \alpha^2$ are neglected)

$$
\begin{pmatrix} \langle \tilde h_x(\tau)\tilde h_x(0)\rangle & \langle \tilde h_x(\tau)\tilde h_y(0)\rangle \\ \langle \tilde h_y(\tau)\tilde h_x(0)\rangle & \langle \tilde h_y(\tau)\tilde h_y(0)\rangle \end{pmatrix} = \frac{2kT\alpha}{\gamma M_s V}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta(\tau).
$$
(23)

Note that the Eq. (22) is an exact mathematical analog of the Bloch-Bloembergen equation (17) $(1/T_2 = \alpha\gamma H_0)$ for the transverse magnetic components with random fields $\tilde h_x(t)$ and $\tilde h_y(t)$. Thus, we see that the fluctuation-dissipation relations (23) and (19) are principally different in the case when they must coincide (Gilbert and Bloch-Bloembergen dynamics are the same). The second form of the Callen-Welton FDT cannot be used to verify the validity of one form of damping versus another.

The origin of the inconsistency is in the use of the second form (9) of Callen-Welton FDT, applied to systems with dissipation (Landau-Lifshitz-Gilbert and Bloch-Bloembergen equations). Without dissipation ($\alpha=0$) this inconsistency disappears. It should be pointed out that Eqs. (20) and (22) describe stochastic magnetization dynamics with $\langle h_x(t)\rangle = \langle h_y(t)\rangle = \langle m_x(t)\rangle = \langle m_y(t)\rangle = 0$ in accordance with thermodynamics.

Note that for no damping the dynamic susceptibility of the Landau-Lifshitz-Gilbert equation (15) and the Bloch-Bloembergen equation (18) are identical in form. In the stochastic case with random fields $h_x(t)$ and $\langle h_y(t)\rangle$ at first glance Eqs. (14) and (17) appear to be different. However, we have shown that by a simple transformation (21) both equations are equivalent in the case of uniaxial anisotropy.

Application of the first FDT (8) to both Landau-Lifshitz-Gilbert and Bloch-Bloembergen stochastic dynamics does not give such a strong inconsistency as does the second form. Using (8) with (15), we obtain the following averages for the Gilbert dynamics:

$$\begin{pmatrix} \langle m_x(\tau)m_x(0)\rangle & \langle m_x(\tau)m_y(0)\rangle \\ \langle m_y(\tau)m_x(0)\rangle & \langle m_y(\tau)m_y(0)\rangle \end{pmatrix}$$

$$= \frac{\gamma\alpha k_B T}{\pi M_s V} \int_{-\infty}^{\infty} \frac{d\omega}{|D_G(\omega)|^2} e^{-i\omega\tau}$$

$$\times \begin{pmatrix} \omega^2(1+\alpha^2)+(\gamma H_y)^2 & -i\omega\gamma(H_x+H_y) \\ i\omega\gamma(H_x+H_y) & \omega^2(1+\alpha^2)+(\gamma H_x)^2 \end{pmatrix}, \quad (24)$$

where $D_G(\omega) = \omega_0^2 - \omega^2(1+\alpha^2) - i\alpha\omega\gamma(H_x+H_y)$ and $\omega_0^2 = \gamma^2 H_x H_y$ is the ferromagnetic resonance frequency.

On the other hand, the use of (8) with (18) gives

$$\begin{pmatrix} \langle m_x(\tau)m_x(0)\rangle & \langle m_x(\tau)m_y(0)\rangle \\ \langle m_y(\tau)m_x(0)\rangle & \langle m_y(\tau)m_y(0)\rangle \end{pmatrix}$$

$$= \frac{\gamma T_2^{-1} k_B T}{\pi M_s V} \int_{-\infty}^{\infty} \frac{d\omega}{i\omega |D_{BB}(\omega)|^2} e^{-i\omega\tau}$$

$$\times \begin{pmatrix} i2\omega\gamma H_y & \omega_0^2+\omega^2+T_2^{-2} \\ -\omega_0^2-\omega^2-T_2^{-2} & i2\omega\gamma H_x \end{pmatrix}, \quad (25)$$

where $D_{BB}(\omega) = \omega_0^2 - \omega^2 + T_2^{-2} - 2i\omega T_2^{-1}$.

In general, magnetic correlations (24) and (25) differ from each other. However, in the most important case of the noise power ($\tau=0$) both (24) and (25) are reduced to

$$\begin{pmatrix} \langle m_x^2(0)\rangle & \langle m_x(0)m_y(0)\rangle \\ \langle m_y(0)m_x(0)\rangle & \langle m_y^2(0)\rangle \end{pmatrix} = \frac{k_B T}{M_s V} \begin{pmatrix} H_x^{-1} & 0 \\ 0 & H_y^{-1} \end{pmatrix}. \quad (26)$$

This equation is completely consistent with thermodynamics, namely, with energy equipartion

$$\frac{\langle \mathcal{E}\rangle}{2} = \frac{M_s V H_x}{2}\langle m_x^2\rangle = \frac{M_s V H_y}{2}\langle m_y^2\rangle = \frac{k_B T}{2}. \quad (27)$$

IV. DISCUSSION

Other forms of the fluctuation-dissipation relations, which are similar to the first Callen-Welton FDT, have been derived by Kubo[19] for the permeability of a dynamic system and by White[2] for the susceptibility of a general magnetic system. A standard method to study stochastic dynamics in systems with dissipation is the Langevin approach.[9,20] Application of the Langevin approach, utilizing specific dissipation mechanisms, can be found in.[15–17]

The analysis of damping in magnetic systems has had a long history (e.g., Refs. 21,22). Each spin wave (magnon) interacts with a so-called thermal bath, which consists of magnons, phonons, conduction electrons, impurities, etc. Analyzing the processes of relaxation, one can find the spin-wave damping (see, e.g., Ref. 21) and the corresponding thermal noise. The microscopic stochastic differential equation (SDE) is shown to be of the Langevin form for a damped harmonic oscillator for a wide class of relaxation processes.[17] Note that the collective description works even in the case of local interactions. A local defect or impurity perturbs the band structure of the magnetic crystal (see, e.g., Ref. 2) and affects the spin-wave spectrum, whose imaginary part is responsible for the damping of collective magnetic excitations.

The occurrence of delocalized damping has been directly demonstrated in the problem of two coupled magnetic grains with local thermal baths.[15] Stochastic forces are uncorrelated in the spin-wave coordinates, but become correlated when the SDEs are expressed in the original magnetization coordinates. Because of the system interactions, even though the coupling to the thermal bath may be purely local, there is no physical requirement that the stochastic forces in the magnetization coordinates be uncorrelated.

In summary, we have argued that the second form of the Callen-Welton fluctuation dissipation theorem does not correctly apply to damped systems, has inconsistencies in its application and thus cannot distinguish between relaxation models.

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