Free transverse vibrations of double-walled carbon nanotubes using a theory of nonlocal elasticity

Y. Q. Zhang,[1] G. R. Liu,[1] and X. Y. Xie[2]
[1]*Centre for Advanced Computations in Engineering Science, Department of Mechanical Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260, Singapore*
[2]*Department of Civil Engineering, Zhejiang University, Hangzhou 310027, People's Republic of China*
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Based on theory of nonlocal elasticity, a nonlocal double-elastic beam model is developed for the free transverse vibrations of double-walled carbon nanotubes. The effect of small length scale is incorporated in the formulation. With this nonlocal double-elastic beam model, explicit expressions are derived for natural frequencies and associated amplitude ratios of the inner to the outer tubes for the case of simply supported double-walled carbon nanotubes. The effect of small length scale on the properties of vibrations is discussed. It is demonstrated that the natural frequencies and the associated amplitude ratios of the inner to the outer tubes are dependent upon the small length scale. The effect of small length scale is related to the vibrational mode and the aspect ratio.

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I. INTRODUCTION

Among nanostructured materials, the carbon nanotubes discovered in 1991 (Ref. 1) have received tremendous attention from various branches of science. By the use of varieties of experimental, theoretical, and computer simulation approaches, extensive research studies of the properties of carbon nanotubes have been carried out.[2–10] Carbon nanotubes are cylindrical macromolecules composed of carbon atoms in a periodic hexagonal arrangement. As they are found to have remarkable mechanical, physical, and chemical properties, carbon nanotubes hold exciting promise as structural elements in nanoscale devices or reinforcing element in superstrong nanocomposites.[11,12]

As a thorough understanding of the mechanical responses of individual carbon nanotubes is of great importance for their potential applications,[13,14] the study of vibrational behavior of carbon nanotubes is of practical interest. For the sake of the difficulties in experimental characterization of nanotubes and time consuming and computationally expensive for atomistic simulations, elastic continuum models have been widely used to study the vibrational behavior of carbon nanotubes.[15–20] In these continuum models, the single-elastic beam model[15,16] assumes that all originally concentric tubes of a multiwalled carbon nanotube remain coaxial during vibration while the multiple-elastic beam model[19,20] considers the intertube radial displacements of multiwalled carbon nonotubes which give rise to complicated intertube resonant frequencies and noncoaxial vibrational modes. Though the classic continuum models are relevant to some extent, the length scales associated with nanotechnology are often sufficiently small to call the applicability of classical continuum models into question. The main reason is that at small length scales the material microstructure (such as lattice spacing between individual atoms) becomes increasingly important and its effect can no longer be ignored.[21] This has raised a major challenge to the classic continuum mechanics. It is a possible solution to extend the classic continuum approach to smaller length scales by incorporating information regarding the behavior of material microstructure. It is accomplished quite easily by the use of the theory of nonlocal continuum mechanics.

While the classical (local) continuum mechanics assumes that the stress state at a given point is dependent uniquely on the strain state at that same point, the nonlocal continuum mechanics regards the stress state at a given point as a function of the strain states of all points in the body. Thus, the theory of nonlocal continuum mechanics contains information about the long-range forces between atoms, and the internal length scale is introduced into the constitutive equations simply as a material parameter. It has been applied to a wide variety of fields such as lattice dispersion of elastic waves, fracture mechanics, dislocation mechanics, wave propagation in composites, and surface tension in fluids, among others. Recently, Peddieson *et al.*[22] pointed out that nanoscale devices would exhibit nonlocal effects and the nonlocal continuum mechanics could potentially play a useful role in analysis related to nanotechnology applications. On the basis of the theory of nonlocal continuum mechanics, Sudak[23] presented a multiple-elastic column model to study column buckling of multiwalled carbon nanotubes which demonstrated that small scale effects contribute significantly to the mechanical behavior of multiwalled carbon nanotubes. Zhang *et al.*[24] put forward a nonlocal multiple-elastic shell model for the axially compressed buckling of multiwalled carbon nanotubes and discussed the effect of small length scale on the axial buckling strain.

In this paper, based on the theory of nonlocal elasticity, a double-elastic beam model is developed for the free transverse vibrations of double-walled carbon nanotubes, which considers the effect of small length scale in the formulation. Explicit expressions are derived for natural frequencies and associated amplitude ratios of the inner to the outer tubes for the case of simply supported double-walled carbon nanotubes, and the influences of small length scale on them are investigated.

II. NONLOCAL CONTINUUM BEAM MODEL

A. Brief introduction of the Erigen nonlocal elasticity model

The theory of nonlocal continuum mechanics was formally initiated by the papers of Erigen[25] and Eringen and Edelen[26] on nonlocal elasticity. In the Eringen nonlocal elasticity model,[27] the stress state at a reference point **x** in the body is regarded to be dependent not only on the strain state at **x** but also on the strain states at all other points of the body. This is in accordance with atomic theory of lattice dynamics and experimental observations on phonon dispersion. The most general form of the constitutive equation for nonlocal elasticity involves an integral over the entire region of interest. This integral contains a kernel function which describes the relative influences of strains at various locations on the stress at a given location. In the limit when the effects of strains at points other than **x** are neglected, the nonlocal theory of elasticity reverts to the classic (local) theory.

For homogeneous and isotropic elastic solids, the linear theory of nonlocal elasticity is given by the set of equations

$$\sigma_{kl,k} + \rho(f_l - \ddot{u}_l) = 0, \tag{1}$$

$$\sigma_{kl}(\mathbf{x}) = \int_V \alpha(\mathbf{x},\mathbf{x}')\tau_{kl}(\mathbf{x}')dV(\mathbf{x}'), \tag{2}$$

$$\tau_{kl}(\mathbf{x}') = \lambda\varepsilon_{mm}(\mathbf{x}')\delta_{kl} + 2\mu\varepsilon_{kl}(\mathbf{x}'), \tag{3}$$

$$\varepsilon_{kl}(\mathbf{x}') = \frac{1}{2}\left(\frac{\partial u_k(\mathbf{x}')}{\partial x_l'} + \frac{\partial u_l(\mathbf{x}')}{\partial x_k'}\right), \tag{4}$$

where σ_{kl}, ρ, f_l, and u_l are, respectively, the nonlocal stress tensor, mass density, body force density, and the displacement vector at a reference point **x** in the body. $\tau_{kl}(\mathbf{x}')$ denotes the macroscopic (classical) stress tensor at any point $\mathbf{x}'$ in the body, and $\varepsilon_{kl}(\mathbf{x}')$ is the strain tensor. The two parameters λ and μ are Lamé constants. The kernel function $\alpha(\mathbf{x},\mathbf{x}')$ is the attenuation function which incorporates into the constitutive equations the nonlocal effects. The volume integral in Eq. (2) is over the region V occupied by the body. In addition, it is seen that the only difference between Eqs. (1)–(4) and the corresponding equations of classical elasticity is in the constitutive equations (2) which replaces Hooke's law (3) by Eq. (2).

While the constitutive equation of classical elasticity is an algebraic relationship between the stress and strain tensors, that of nonlocal elasticity involves spatial integrals which represent weighted averages of the contributions of the strain tensors of all points in the body to the stress tensor at the given point. Though it is difficult mathematically to get the solution of nonlocal elasticity problems due to the involved spatial integrals in the constitutive equations, these integral constitutive equations can be converted to equivalent differential constitutive equations under certain conditions. This provides a great deal of simplicity and convenience for the application of the theory of nonlocal elasticity.

B. Nonlocal double-elastic beam model

The treatment of beam flexure which is developed here is on the basis of the Bernoulli-Euler theory. This theory is based upon the assumption that plane cross sections of a beam remain plane during flexure and that the radius of curvature of a bent beam is large compared with the beam's depth. Using the Bernoulli-Euler beam theory, the general equation for transverse vibrations of an elastic beam under distributed transverse pressure is expressed by[28–30]

$$p(x) = EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2}, \tag{5}$$

where x is the axial coordinate, t is time, $p(x)$ is the distributed transverse pressure per unit axial length (measured positive in the direction of the deflection), w is the deflection of the beam, I and A are the moment of inertia and the area of the cross section of the beam, and E and ρ are Young's modulus and the mass density. Thus, EI denotes the bending stiffness of the beam, and ρA represents the mass density per unit axial length.

Another assumption behind the Bernoulli-Euler beam model is that the beam consists of fibers parallel to the x axis, each in a state of uniaxial tension or compression. Adopting the theory of nonlocal elasticity, the classic Hooke's law for a uniaxial stress state is replaced by[22,23]

$$\sigma - (e_0 a)^2\frac{\partial^2 \sigma}{\partial x^2} = E\varepsilon, \tag{6}$$

where σ is the axial stress, ε is the axial strain, e_0 is a constant appropriate to each material, and a is an internal characteristic length (e.g., length of C—C bond, lattice spacing, granular distance). In addition, it should be noted that the value of e_0 needs to be determined for each material.

When the beam is vibrating transversely, the equation of motion perpendicular to the x axis is obtained in the form[29,31]

$$\frac{\partial S}{\partial x} = -p + \rho A\frac{\partial^2 w}{\partial t^2}, \tag{7}$$

where S is the shear force, and the moment equilibrium condition gives

$$S = \frac{\partial M}{\partial x}, \tag{8}$$

where M is the bending moment which can be obtained by[32]

$$M = \int_A y\sigma dA, \tag{9}$$

where y is the transverse coordinate measured positive in the direction of deflection. In addition, for small deflections we have[32]

$$\varepsilon = -y\frac{\partial^2 w}{\partial x^2}. \tag{10}$$

Combination of Eqs. (6), (9), and (10) results in

$$M - (e_0a)^2\frac{\partial^2 M}{\partial x^2} = -EI\frac{\partial^2 w}{\partial x^2}. \tag{11}$$

It follows from Eqs. (7), (8), and (11) that

$$p = EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - (e_0a)^2\left(\rho A\frac{\partial^4 w}{\partial x^2\partial t^2} - \frac{\partial^2 p}{\partial x^2}\right) \tag{12}$$

which is the general equation for transverse vibrations of an elastic beam under distributed transverse pressure on the basis of nonlocal elasticity. It is noted that when the small scale parameter a vanishes, the above equation reduces to the classical Bernoulli-Euler expression (5).

It is known that double-walled carbon nanotubes are distinguished from traditional elastic beams by their hollow two-layer structure and associated intertube van der Waals forces. Equation (12) can be used to each of the inner and outer tubes of the double-walled carbon nanotubes. Assuming that the inner and outer tubes have the same thickness and effective material constants, we have

$$p_{12} = EI_1\frac{\partial^4 w_1}{\partial x^4} + \rho A_1\frac{\partial^2 w_1}{\partial t^2} - (e_0a)^2\left(\rho A_1\frac{\partial^4 w_1}{\partial x^2\partial t^2} - \frac{\partial^2 p_{12}}{\partial x^2}\right), \tag{13a}$$

$$-p_{12} = EI_2\frac{\partial^4 w_2}{\partial x^4} + \rho A_2\frac{\partial^2 w_2}{\partial t^2} - (e_0a)^2\left(\rho A_2\frac{\partial^4 w_2}{\partial x^2\partial t^2} + \frac{\partial^2 p_{12}}{\partial x^2}\right), \tag{13b}$$

where subscripts 1 and 2 are used to denote the quantities associated with the inner and the outer tubes, respectively, and p_{12} denotes the van der Waals pressure per unit axial length exerted on the inner tube by the outer tube.

For small-deflection linear vibration, the van der Waals pressure at any point between two tubes should be a linear function of the jump in deflection at that point. Thus, the interaction pressure per unit axial length is given by[23,33]

$$p_{12} = c(w_2 - w_1), \tag{14}$$

where c is the intertube interaction coefficient per unit length between two tubes, which can be estimated by[23,30]

$$c = \frac{320(2R_1)\text{erg/cm}^2}{0.16d^2}\ (d = 0.142\ \text{nm}),$$

where R_1 is the radius of the inner tube.

Introduction of Eq. (14) into Eqs. (13a) and (13b) yields

$$c(w_2 - w_1) = EI_1\frac{\partial^4 w_1}{\partial x^4} + \rho A_1\frac{\partial^2 w_1}{\partial t^2} - (e_0a)^2 \times\left[\rho A_1\frac{\partial^4 w_1}{\partial x^2\partial t^2} - c\frac{\partial^2}{\partial x^2}(w_2 - w_1)\right], \tag{15a}$$

$$-c(w_2 - w_1) = EI_2\frac{\partial^4 w_2}{\partial x^4} + \rho A_2\frac{\partial^2 w_2}{\partial t^2} - (e_0a)^2 \times\left[\rho A_2\frac{\partial^4 w_1}{\partial x^2\partial t^2} + c\frac{\partial^2}{\partial x^2}(w_2 - w_1)\right]. \tag{15b}$$

With the effect of small length scale included, these two differential equations describe the free transverse vibrations of double-walled carbon nanotubes, and they are coupled together by the van der Walls interaction. When the effect of small length scale is ignored, Eqs. (15a) and (15b) reduce to the classic (local) result for double-walled carbon nanotubes.[30]

III. SOLUTION OF THE PROBLEM

Let us consider a double-walled nanotube of length L. Suppose that its ends are simply supported, the boundary conditions are given by

$$w_1(0,t) = \frac{\partial^2 w_1(0,t)}{\partial x^2} = w_1(L,t) = \frac{\partial^2 w_1(L,t)}{\partial x^2} = 0, \tag{16a}$$

$$w_2(0,t) = \frac{\partial^2 w_2(0,t)}{\partial x^2} = w_2(L,t) = \frac{\partial^2 w_2(L,t)}{\partial x^2} = 0. \tag{16b}$$

The homogeneous partial differential equations (15a) and (15b) with the governing boundary conditions (16a) and (16b) can be solved by the Bernoulli-Fourier method assuming the solutions in the form

$$w_1(x,t) = \sum_{n=1}^{\infty} X_n(x)T_{1n}(t), \tag{17a}$$

$$w_2(x,t) = \sum_{n=1}^{\infty} X_n(x)T_{2n}(t), \tag{17b}$$

where $T_{1n}(t)$ and $T_{2n}(t)$ are the unknown time functions, and $X_n(x)$ is the known mode shape function for simply supported single beam, which is expressed as

$$X_n(x) = \sin(k_n x), \quad k_n = \frac{n\pi}{L},\ n = 1,2,3,\ldots\ .$$

Substitution of Eqs. (17a) and (17b) into Eqs. (15a) and (15b) produces

$$\sum_{n=1}^{\infty}\left[\rho A_1(1 + e_0^2a^2k_n^2)\frac{\partial^2 T_{1n}}{\partial t^2} + (EI_1k_n^4 + c + ce_0^2a^2k_n^2)T_{1n} - c(1 + e_0^2a^2k_n^2)T_{2n}\right]X_n = 0,$$

$$\sum_{n=1}^{\infty}\left[\rho A_2(1+e_0^2a^2k_n^2)\frac{\partial^2 T_{2n}}{\partial t^2}+(EI_2k_n^4+c+ce_0^2a^2k_n^2)T_{2n}-c(1+e_0^2a^2k_n^2)T_{1n}\right]X_n=0.$$

It follows from the above that

$$\frac{\partial^2 T_{1n}}{\partial t^2}+\left(\frac{F_1}{1+e_0^2a^2k_n^2}+H_1\right)T_{1n}-H_1T_{2n}=0, \quad (18a)$$

$$\frac{\partial^2 T_{2n}}{\partial t^2}+\left(\frac{F_2}{1+e_0^2a^2k_n^2}+H_2\right)T_{2n}-H_2T_{1n}=0, \quad (18b)$$

where

$$F_1=\frac{EI_1k_n^4}{\rho A_1}, \quad H_1=\frac{c}{\rho A_1},$$

$$F_2=\frac{EI_2k_n^4}{\rho A_2}, \quad H_2=\frac{c}{\rho A_2}.$$

The solutions of Eqs. (18a) and (18b) can be expressed by

$$T_{1n}(t)=C_{1n}e^{i\omega_n t}, \quad T_{2n}(t)=C_{2n}e^{i\omega_n t}, \ i=\sqrt{-1}, \quad (19)$$

where ω_n denotes the natural frequency of the double-walled carbon nanotube, and C_{1n} and C_{2n} represent the amplitude coefficients of the inner and outer tubes, respectively. Substituting Eq. (19) into Eqs. (18a) and (18b), we obtain

$$\left(\frac{F_1}{1+e_0^2a^2k_n^2}+H_1-\omega_n^2\right)C_{1n}-H_1C_{2n}=0, \quad (20a)$$

$$\left(\frac{F_2}{1+e_0^2a^2k_n^2}+H_2-\omega_n^2\right)C_{2n}-H_2C_{1n}=0. \quad (20b)$$

Nontrivial solutions for the constants C_{1n} and C_{2n} can be obtained only when the determinant of the coefficients in Eqs. (20a) and (20b) vanishes. In this manner we have

$$\omega_n^4-\left(\frac{F_1+F_2}{1+e_0^2a^2k_n^2}+H_1+H_2\right)\omega_n^2+\frac{F_1F_2}{(1+e_0^2a^2k_n^2)^2}+\frac{F_1H_2+F_2H_1}{1+e_0^2a^2k_n^2}=0 \quad (21)$$

which is the frequency characteristic equation. It is found that the discriminant of this biquadratic algebraic equation is positive

$$\Delta=\left(\frac{F_1-F_2}{1+e_0^2a^2k_n^2}+H_1-H_2\right)^2+4H_1H_2>0$$

and the relationships in the following are also satisfied:

$$\frac{F_1F_2}{(1+e_0^2a^2k_n^2)^2}+\frac{F_1H_2+F_2H_1}{1+e_0^2a^2k_n^2}>0,$$

$$\frac{F_1+F_2}{1+e_0^2a^2k_n^2}+H_1+H_2>\sqrt{\Delta}.$$

Thus the characteristic equation (21) has two different, real, and positive roots

$$\omega_{nI}^2=\frac{1}{2}\left[\frac{F_1+F_2}{1+e_0^2a^2k_n^2}+H_1+H_2-\sqrt{\left(\frac{F_1-F_2}{1+e_0^2a^2k_n^2}+H_1-H_2\right)^2+4H_1H_2}\right], \quad (22a)$$

$$\omega_{nII}^2=\frac{1}{2}\left[\frac{F_1+F_2}{1+e_0^2a^2k_n^2}+H_1+H_2+\sqrt{\left(\frac{F_1-F_2}{1+e_0^2a^2k_n^2}+H_1-H_2\right)^2+4H_1H_2}\right], \quad (22b)$$

where ω_{nI} is the lower natural frequency, and ω_{nII} is the higher natural frequency. For each of the natural frequencies, the associated amplitude ratio of vibrational modes of the inner to the outer tubes is given by

$$B_n=\frac{C_{1n}}{C_{2n}}=\frac{H_1(1+e_0^2a^2k_n^2)}{F_1+(H_1-\omega_n^2)(1+e_0^2a^2k_n^2)}=\frac{F_2+(H_2-\omega_n^2)(1+e_0^2a^2k_n^2)}{H_2(1+e_0^2a^2k_n^2)}. \quad (23)$$

Introducing Eqs. (22a) and (22b) into Eq. (23), respectively, we obtain

$$B_{nI}=\frac{1}{2H_2}\left[\frac{F_2-F_1}{1+e_0^2a^2k_n^2}+H_2-H_1+\sqrt{\left(\frac{F_1-F_2}{1+e_0^2a^2k_n^2}+H_1-H_2\right)^2+4H_1H_2}\right], \quad (24a)$$

$$B_{nII}=\frac{1}{2H_2}\left[\frac{F_2-F_1}{1+e_0^2a^2k_n^2}+H_2-H_1-\sqrt{\left(\frac{F_1-F_2}{1+e_0^2a^2k_n^2}+H_1-H_2\right)^2+4H_1H_2}\right]. \quad (24b)$$

It can be observed that the amplitude ratio $B_{n\mathrm{I}}$ dependent on the lower natural frequency ω_{nI} is always positive, which indicates that the inner and outer tubes execute synchronous vibrations, while the amplitude ratio B_{nII} dependent on the higher frequency ω_{nII} is always negative, which indicates that the inner and outer tubes execute asynchronous vibrations.

IV. DISCUSSION

When the effect of small length scale is ignored, Eqs. (22a), (22b), and (23) reduce to the classical (local) results[19]

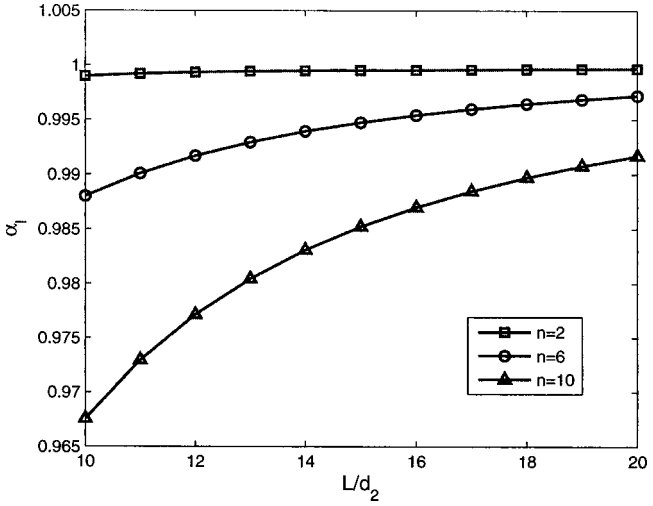

FIG. 1. Effects of small length scale on the lower natural frequency ω_{nI} for various aspect ratios L/d_2 and vibrational mode numbers n.

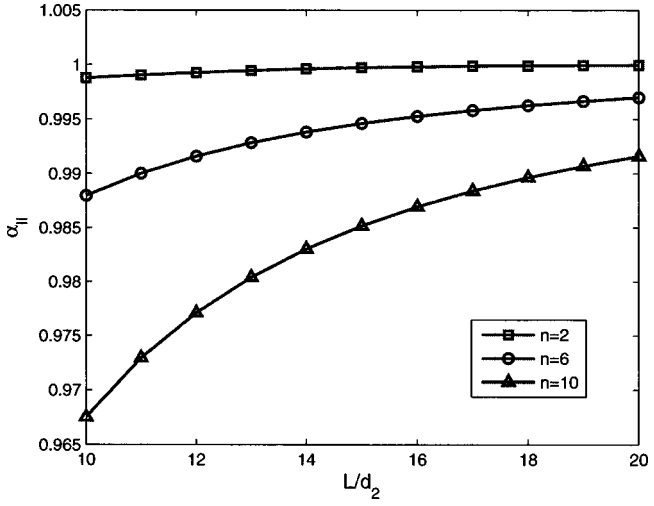

FIG. 2. Effects of small length scale on the higher natural frequency ω_{nII} for various aspect ratios L/d_2 and vibrational mode numbers n.

$$\omega_{nI}^2 = \tfrac{1}{2}[M_1 + M_2 - \sqrt{(M_1 - M_2)^2 + 4H_1H_2}], \quad (25a)$$

$$\omega_{nII}^2 = \tfrac{1}{2}[M_1 + M_2 + \sqrt{(M_1 - M_2)^2 + 4H_1H_2}], \quad (25b)$$

$$B_n = \frac{M_2 - \omega_n^2}{H_2}, \quad (26)$$

where

$$M_1 = F_1 + H_1, \quad M_2 = F_2 + H_2.$$

Substituting Eqs. (25a) and (25b) into Eq. (26), respectively, we have

$$B_{nI} = \frac{1}{2H_2}[M_2 - M_1 + \sqrt{(M_1 - M_2)^2 + 4H_1H_2}], \quad (27a)$$

$$B_{nII} = \frac{1}{2H_2}[M_2 - M_1 - \sqrt{(M_1 - M_2)^2 + 4H_1H_2}]. \quad (27b)$$

To examine the influence of the small length scale on vibrations of double-walled nanotubes, let us compare the local and nonlocal results. It follows that the ratios of the nonlocal results to the corresponding local results are, respectively, given by

$$(\omega_{nI})_{\mathrm{NL}} = \alpha_I(\omega_{nII})_{\mathrm{LC}}, \quad (\omega_{nII})_{\mathrm{NL}} = \alpha_{II}(\omega_{nII})_{\mathrm{LC}},$$

$$(B_{nI})_{\mathrm{NL}} = \beta_I(B_{nI})_{\mathrm{LC}}, \quad (B_{nII})_{\mathrm{NL}} = \beta_{II}(B_{nII})_{\mathrm{LC}},$$

where the subscripts NL and LC refer to nonlocal and local, respectively.

In the investigation of the effect of small length scale, it is of great importance to determine the value of parameter e_0 as its magnitude has a noticeable influence on the effect of small length scale.[23,24] Although the value of this parameter for carbon nanotubes is not available at present, it may be estimated by fitting theoretical results obtained on the basis of nonlocal elasticity to those from experiments, molecular dynamics (MD) or molecular mechanics (MM) simulations. Due to the lack of experimental data, this study uses the result from MM simulation[34] for the critical axial buckling strain of a single-walled carbon nanotube to predict the value of parameter e_0. By MM simulation, Sears and Batra[34] studied a short single-walled carbon nanotube with length $L=1.618$ nm and radius $R=0.5937$ nm under axial compression. It was found that the tube began to buckle locally at the critical strain of 0.098 with the axial half wave number $m_1=2$ and the circumferential wave number $m_2=2$ while the axis remained straight. Based on the Donnell shell theory, the classical (local) result for the critical buckling strain of this tube is about 0.133 with $m_1=2$ and $m_2=2$. (It should be mentioned that Sears and Batra[34] gave the wrong value 0.147for the critical strain.) Then the value of the ratio of the theoretical local result to that obtained by MM simulation is about 1.36. On the other hand, it is known that the ratio of the local result to nonlocal result for the axial buckling strain of single-walled carbon nanotubes can be expressed as[24]

$$\chi = 1 + \eta e_0^2 a^2 \quad (28)$$

with

$$\eta = m_1^2\pi^2/L^2 + m_2^2/R^2.$$

Let the ratio $\chi=1.36$, from Eq. (28) the value of parameter e_0 can be obtained. Through calculation, we find $e_0 \approx 0.82$.

We suppose that the Young's modulus $E=1$ TPa with the effective thickness of single-walled carbon nanotubes taken to be 0.35 nm, and the mass density $\rho=2.3$ g/cm^3.[35] In addition, the value of parameter a is chosen to be 0.142 nm, which is the length of a C—C bond. Since the value of parameter e_0 is determined, for a simply supported double-walled carbon nanotube with the inner diameter $d_1=0.7$ nm and the outer diameter $d_2=1.4$ nm,[36] the effects of small length scale represented by the ratios of the nonlocal results to the corresponding local results are shown in Figs. 1–4, respectively. It is observed from Figs. 1–4 that the effects of

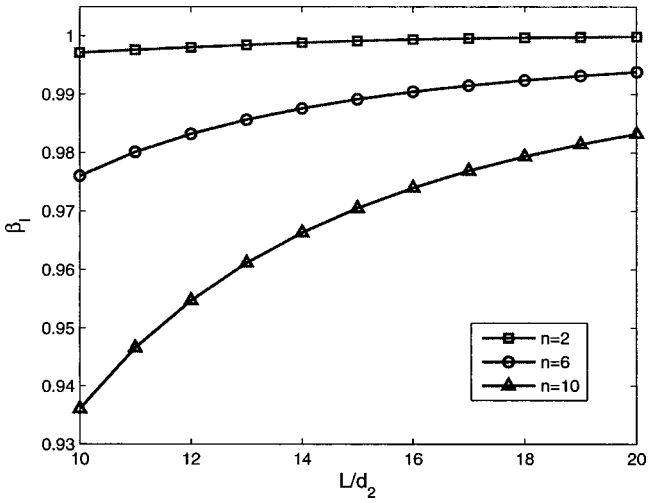

FIG. 3. Effects of small length scale on the amplitude ratio B_{nI} dependent on the lower natural frequency for various aspect ratios L/d_2 and vibrational mode numbers n.

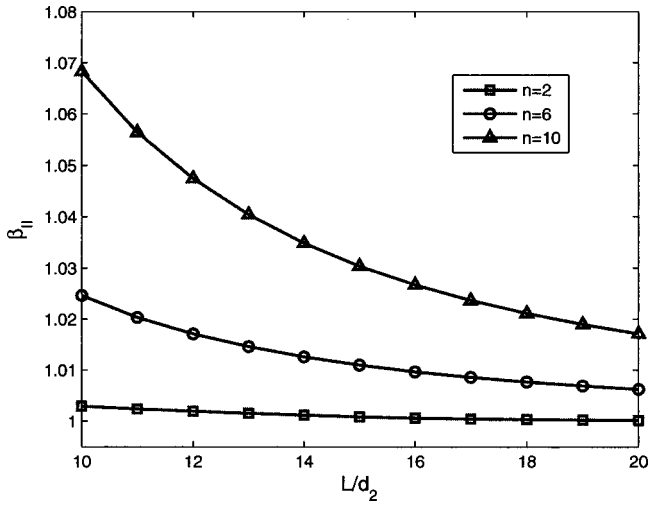

FIG. 4. Effects of small length scale on the amplitude ratio B_{nII} dependent on the higher natural frequency for various aspect ratios L/d_2 and vibrational mode numbers n.

small length scale are dependent on the aspect ratio L/d_2 and vibrational mode number n. With the same aspect ratio, the effects of small length scale on the lower natural frequency ω_{nI}, the higher natural frequency ω_{nII}, and the associated amplitude ratios B_{nI} and B_{nII} of vibrational modes of the inner to the outer tubes get larger with the increase of the vibrational mode number n. It is mainly because when the tube length remains constant, the wavelength in axial direction diminishes with increasing the number n, which makes the effects of small length scale more significant. With the same vibrational mode number, the effects of small length scale on ω_{nI}, ω_{nII}, $B_{n\mathrm{I}}$, and B_{nII} decrease with the increase of the aspect ratio L/d_2. For example, when $n=10$ the relative error between the nonlocal and local results for the lower natural frequency ω_{nI} is about 3.3% with $L/d_2=10$ and about 0.8% with $L/d_2=20$, which also holds true for the higher natural frequency ω_{nII}. On the other hand, when $n=10$ the relative error between the nonlocal and local results for the amplitude ratio $B_{n\mathrm{I}}$ is close to 6.4% with $L/d_2=10$ and 1.7% with $L/d_2=20$ while 6.8% with $L/d_2=10$ and 1.7% with $L/d_2=20$ for the amplitude ratio B_{nII}. The main reason is that when the number n remains constant, the wavelength becomes larger as the tube length increases due to the increment of the aspect ratio, which causes the diminishment of the effects of small length scale. Consequently, it suggests that at large aspect ratios (say $L/d_2 \geqslant 20$) the classic (local) double-elastic beam model may be directly applied to study the properties of transverse vibration of a double-walled carbon nanotube with very small relative errors. Moreover, it is seen from Figs. 1–4 that the nonlocal solutions of the two natural frequencies ω_{nI} and ω_{nII} and the amplitude ratio B_{nI} are smaller than the corresponding classic (local) solutions whereas the nonlocal solution of the amplitude ratio B_{nII} is larger than the local result. As a consequence, it is concluded that the classic (local) double-elastic beam model[30] could overestimate the two natural frequencies ω_{nI} and ω_{nII} and the amplitude ratio $B_{n\mathrm{I}}$ but could underestimate the amplitude ratio B_{nII}.

V. CONCLUSIONS

Based on the Bernoulli-Euler beam theory and nonlocal elasticity, the general equation for transverse vibrations of an elastic beam under distributed transverse pressure is formulated. Following this general equation, a nonlocal double-elastic beam model is developed for the free transverse vibrations of double-walled carbon nanotubes, which takes the effect of small length scale into account. For the case of simply supported double-walled carbon nanotubes, the natural frequencies and the associated amplitude ratios of the inner to the outer tubes are determined.

As the magnitude of the parameter e_0 is essential for the effect of small length scale, its value is predicted by matching the nonlocal theoretical results with those obtained by MM simulations for the carbon nanotubes. Based on this value, the effect of small length scale on the properties of vibrations is discussed. It is demonstrated that the natural frequencies and the associated amplitude ratios of the inner to the outer tubes are dependent on the small length scale. The effect of small length scale is related to the vibrational mode and the aspect ratio. With the vibrational mode increasing and the aspect ratio becoming smaller, the small length scale has a more significant influence on the natural frequencies and the associated amplitude ratios of the inner to the outer tubes. Moreover, we can conclude that the classic (local) double-elastic beam model could overestimate the two natural frequencies and the amplitude ratio dependent upon the lower natural frequency but could underestimate the amplitude ratio dependent upon the higher natural frequency.

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