Control of field-induced localization in superlattices: Coherence and relaxation

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(Received 4 November 2004; published 11 May 2005)

We consider a d-dimensional conductor (a semiconductor/optical superlattice, quantum/molecular wire) with arbitrary electron dispersion within the independent-electron one-band approach. Its nonperturbative nonstationary response to arbitrary time-periodic electric fields is studied (a) in the quantum coherent "dynamic" (or short-time) regime and (b) in the "kinetic" (or long-time) regime under the influence of weak scattering. We provide a classification and analysis of field-induced dynamic localization and response through the dc/ac current and mean square displacement of electrons. We demonstrate that the overall localization increases in passing from the periodic regime through the commensurate to the incommensurate one (governed by the relation of field period and Bloch frequency) both in the dynamic and kinetic cases. Simultaneously, exceptional localization (for some particular values of field parameters or symmetries) typically retains its order in the small relaxation rate, but on the background of increasing overall localization becomes less pronounced, both in dynamic and kinetic regimes. In the dynamic regime exceptional localization is manifested through diffusion and dc response, in the kinetic—through diffusion and ac response. In the commensurate case with long-range overlap the leading responses are formed by "resonant" neighbors only; within nearest-neighbor approximation the commensurate regime becomes qualitatively analogous to the incommensurate one. Ways of controlling localization/response by the applied field and the reasons for the similarity/difference of dynamic and kinetic regimes are discussed.

DOI: 10.1103/PhysRevB.71.195311 PACS number(s): 68.65.Cd, 05.60.Gg, 73.21.Cd, 78.67.Pt

I. INTRODUCTION

For several years experimental and theoretical studies of electron kinetics in semiconductor superlattices have attracted considerable attention.[1–6] More than three decades ago Esaki and Tsu[7] suggested that electrons in a constant electric field, undergoing Bloch oscillations,[8] should produce terahertz radiation. However, in experiments, coherence effects in superlattices have been observed only recently.[9–11]

Electronic properties of crystalline conductors in the kinetic regime under the action of an electric field have been studied extensively using various approaches, including band theory, the Boltzmann kinetic equation, BBGKY hierarchy for the density matrix with various truncation schemes, Wannier-Stark hopping, sequential tunneling, diagrammatic methods for linear response, nonequilibrium Green's function technique, etc. (see Refs. 2, 5–8, and 12–16 and references therein). However, the investigation of the nearly coherent electron dynamics in band conductors with slow scattering in strong and nonstationary electric fields is not complete yet, except for Bloch oscillations and some other particular cases.[5–8,17–29] The Wannier-Stark ladder states in a constant electric field have been considered in Refs. 6–8, 17, and 21–26. The effect of dynamic localization under the action of a harmonic field has been found[19] and studied within various approaches.[23–28,31–33,35–37] Electron dynamics in a dc and ac field and a rectangular alternating field have been considered.[19,21,22] A classification of localized/delocalized states in arbitrary time-periodic electric fields has been presented.[27,28] In Ref. 31 it has been shown that dynamic localization persists in a nonlinear system. General aspects of the theory of electronic properties in multi-band superlattices have been addressed.[24,32] The quasienergy spectrum and its relation to dynamic localization has been considered in Refs. 23 and 32. A first-principles treatment of the motion of a quantum particle in a crystal interacting with a thermostat, followed by destruction of coherence effects, has been provided in Ref. 15. The effect of relaxation processes on dynamic localization in an ac field has been investigated.[18,20,28] Multiphoton absorption in a superlattice in the presence of a static electric field has been studied.[24] Coherent electronic and excitonic Bloch oscillations in the Stark ladder have been analyzed and compared to the semiclassical results.[25] Terahertz emission and four-wave-mixing signals from Bloch oscillations in a semiconductor superlattice have been computed, and the effect of Coulomb interactions upon dynamic localization, and of the latter upon the effective dimensionality of excitons, has been considered.[26] Coherent time-dependent transport, dynamic localization effect, and linear absorption spectra of multiband semiconductor superlattices in THz field have been analyzed with the help of the density matrix and nonequilibrium Green functions.[32] The validity of different theories for inelastic quantum transport in superlattices, including the Boltzmann equation approach, has been investigated on the basis of Keldysh diagram technique.[34] Degenerate four-wave-mixing signals from dc-and ac-driven semiconductor superlattices, and their relation to dynamic localization, have been analyzed.[35] The relation of linear optical properties of superlattices and dynamic localization has been discussed.[36] The delocalization transition in coupled minibands as function of ac field parameters has been studied.[37] Optical absorption and sideband generation in quantum wells driven by a terahertz electric field has been calculated.[38] Nonperturbative dynamic electro-optical effects in semiconductor quantum wires under the action of a strong THz field have been considered.[39] A new phasorlike interpretation of electronic motion in constant and time-periodic electric fields and of dynamic localization has been constructed.[40] Electron transport in a tight-binding model of a field-driven molecular wire within Floquet approach has been considered.[41] Nonlinear optical properties of semiconductors under an intense terahertz field have been studied by diagrammatic methods on the basis of Floquet states.[42]

Qualitatively analogous coherent effects are also revealed by cold atoms in optical lattices.[11,43,44]

Another interesting and promising field of research is coherent control of properties of matter with the help of specially tailored laser pulses.[2,4,14,30,45–53] In solid state systems typically it implies control over the magnitude and direction of the electric current through phase relationships of the applied coherent ac fields. Most often a laser field ω and its generated second harmonic 2ω with some phase shift φ are used, without any dc field component. Changing the phase φ and the amplitudes of the components $\mathbf{E}_1$ and $\mathbf{E}_2$, one controls the magnitude and even the polarity of the produced direct current. Many aspects of the effect have been studied lately for semiconductors, superlattices, optical lattices, and molecular and quantum wires—both theoretically and experimentally.[2,5,13,14,26–30,43–53] Phase-coherent control of the direct current, generated in shallow-level doped semiconductors by multi-frequency laser excitation, has been considered in Ref. 49. Coherent control through the carrier photoexcitation from the ground state in the quantum well of a superlattice up to the continuum has been studied for the interpretation of the performed experiment in Ref. 50. Calculations of phase-controlled interband transitions in bulk semiconductors have been performed in Ref. 51. Rectification of the harmonic-mixing field in a single-band tight-binding system with quantum dissipation in the sequential tunneling regime has been studied in Ref. 52. Laser-assisted conductance of molecular wires and the switching effect have been considered.[53] The space-time symmetry aspects of the directed diffusion and direct current in ac fields, both in the classical and quantum framework, have been investigated in Ref. 30.

In our previous paper[29] (see also Ref. 55) we demonstrated that phase (coherent) control of electric currents is possible within the one-band model of a semiconductor superlattice in the nearly coherent regime (slow relaxation, low temperatures). In the present paper our goal is to demonstrate and to study in detail the possibility of coherent control of electron localization within the same model—through field effect upon intraband evolution. We provide a classification of localized/delocalized behavior and response of electrons in arbitrary time-periodic electric fields, and analyze the effect of scattering in the nearly coherent regime.

II. MODEL AND GENERAL SOLUTION

We consider a d-dimensional crystalline conductor within an independent-electron one-band approach. The lattice structure is arbitrary (possibly anisotropic), so that the electron dispersion $H^{(0)}(\mathbf{k})$ is of general type, and the overlap between all sites may be taken into account. The basis set in the real-space lattice we denote by $\mathbf{a}_i$, the basis in the reciprocal lattice by $\mathbf{b}^j$. Site positions are $\mathbf{n}=n^i\mathbf{a}_i$, while the wave vector in the reciprocal space is $\mathbf{k}=k_j\mathbf{b}^j$ with orthonormality condition $\mathbf{a}_i\mathbf{b}^j=\delta_i^j$. The system is exposed to an arbitrary time-periodic (period T, basic frequency $\omega=2\pi/T$) space-homogeneous electric field $\mathbf{E}(t)=E_j(t)\mathbf{b}^j$, which may be strong. In the Wannier basis the quantum Hamiltonian reads

$$H(t) = \sum H^{(0)}_{\mathbf{n},\mathbf{n}'}|\mathbf{n}\rangle\langle\mathbf{n}'| + e\mathbf{E}(t)\sum \mathbf{n}|\mathbf{n}\rangle\langle\mathbf{n}|. \tag{1}$$

The matrix $H^{(0)}_{\mathbf{n},\mathbf{n}'}$ is translationally invariant, and in some cases we will consider lattices with inversion symmetry, when additionally $H^{(0)}_{\mathbf{0},\mathbf{n}}=H^{(0)}_{\mathbf{0},-\mathbf{n}}$. The sign of the electron charge is incorporated in the equations here and below, so that e is its modulus.

The field-induced localization and the nonlinear nonstationary electric properties of the model are characterized by the evolution of the electron average and mean-square displacements $\langle\mathbf{R}(t)\rangle=\Sigma\mathbf{n}\rho_{\mathbf{n},\mathbf{n}}(t)$, $\Delta\langle\mathbf{R}(t)\rangle=\langle\mathbf{R}(t)\rangle-\langle\mathbf{R}(0)\rangle$, $\langle\mathbf{R}^2(t)\rangle=\Sigma\mathbf{n}^2\rho_{\mathbf{n},\mathbf{n}}(t)-N_e^{-1}[\Sigma\mathbf{n}\rho_{\mathbf{n},\mathbf{n}}(t)]^2$, $\Delta\langle\mathbf{R}^2(t)\rangle=\langle\mathbf{R}^2(t)\rangle-\langle\mathbf{R}^2(0)\rangle$ (ρ is the density matrix, N_e is the number of electrons, and $Tr(\rho)=N_e$), the induced polarization $\Delta\mathbf{d}(t)=-e\Delta\langle\mathbf{R}(t)\rangle$, and the electric current $\mathbf{j}(t)=(d/dt)\Delta\mathbf{d}(t)$. By localization we mean an electron evolution such that the changes of both the average displacement and the mean-square displacement are bounded in time (for any initial conditions), when the wavepacket breathes and/or oscillates without propagation. We will not address directly the spatial radius of localization here (which would require the specification of particular initial conditions), but rely on the definition through the electric current and diffusion coefficient, or through the changes $\Delta\langle\mathbf{R}(t)\rangle$, $\Delta\langle\mathbf{R}^2(t)\rangle$. The angular brackets here and below denote quantum averaging in the coherent relaxation-free case, while in the kinetic regime averaging with the density operator over scattering and fluctuations is implied. Double angular brackets denote additional time-average over the period.

The electron wavepacket velocity and the diffusion coefficient are defined in the usual way: $\langle\mathbf{v}\rangle=(d/dt)\Delta\langle\mathbf{R}(t)\rangle$ and $D=(1/2dt)\Delta\langle\mathbf{R}^2(t)\rangle$, where d in the second expression denotes dimension. In the following discussions we speak interchangeably of the electric current or of the wavepacket velocity, summed over the band filling. Clearly, these quantities are proportional to each other with coefficient $-e$.

The Schrödinger equation of the system without relaxation (1) can be solved exactly to give (Ref. 27, generalization of Ref. 19)

$$\Delta\langle\mathbf{R}(t)\rangle = \int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\int_0^t dt'\mathbf{v}_{\mathbf{k}(t')},$$

$$\mathbf{v}_{\mathbf{k}(t)} = \frac{1}{\hbar}\nabla_{\mathbf{k}} H^{(0)}(\mathbf{k}(t)), \quad \mathbf{k}(t) = \mathbf{k} + \frac{e}{\hbar c}\mathbf{A}(t), \tag{2}$$

$$\Delta\langle \mathbf{R}^2(t)\rangle = \int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\left(\int_0^t dt'\mathbf{v}_{\mathbf{k}(t')}\right)^2 + \int d\mathbf{k}\frac{2\mathbf{R}_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\int_0^t dt'\mathbf{v}_{\mathbf{k}(t')} - \frac{1}{N_e}\langle \mathbf{R}(t)\rangle^2. \tag{3}$$

Here $\mathbf{A}(t) = -c\int_0^t dt\mathbf{E}(t)$ is the vector potential, so that $\mathbf{A}(0) = 0$. The scalar potential is identically zero. $H^{(0)}(\mathbf{k})$ is the electron dispersion in the absence of the field, $\rho_{\mathbf{k},\mathbf{k}}(0)$—the initial density matrix with the norm $V_{BZ}^{-1}\int\rho_{\mathbf{k},\mathbf{k}}d\mathbf{k} = N_e$, and $\mathbf{R}_{\mathbf{k},\mathbf{k}}(0) = \Sigma_{\mathbf{n},\mathbf{n}'}\rho_{\mathbf{n},\mathbf{n}'}(0)[(\mathbf{n}+\mathbf{n}')/2]\exp[-i\mathbf{k}(\mathbf{n}-\mathbf{n}')]$ is the Fourier component of the initial average displacement. V_{BZ} is the volume of the Brillouin zone. We take the intersite distance a as length unit throughout.

Scattering by phonons within the stochastic Liouville equation is introduced by imposing dephasing of the wave function (relaxation of nondiagonal elements of the density matrix $\rho_{\mathbf{n},\mathbf{n}'}$) at a constant rate α.[54] Besides, relaxation of site populations (diagonal elements $\rho_{\mathbf{n},\mathbf{n}}$) to the thermal equilibrium distribution function $f(\varepsilon(\mathbf{k}))$ at rate α is introduced.[20] Diagonal relaxation is required as in its absence [or for equilibrium distribution, independent of wave vector $\mathbf{k}$, $f(\varepsilon(\mathbf{k})) = V_{BZ}^{-1}$] the distribution function ρ would relax to uniform band filling $\rho_{\mathbf{k},\mathbf{k}} = N_e$ (equivalent to high-temperature approximation), which is both logically unsatisfactory and leads to the long-time current vanishing identically in any field.[20,27,28] This is discussed in more detail in Ref. 20. The formal equation for the density matrix $\rho(t)$ in the mixed $\mathbf{n}-\mathbf{k}$ representation then takes the form

$$i\hbar\frac{\partial}{\partial t}\rho_{\mathbf{n},\mathbf{n}'}(t) = [H^{(0)},\rho(t)]_{\mathbf{n},\mathbf{n}'} + e\mathbf{E}(t)(\mathbf{n}-\mathbf{n}')\rho_{\mathbf{n},\mathbf{n}'}(t) - i\alpha\hbar(1-\delta_{\mathbf{n},\mathbf{n}'})\rho_{\mathbf{n},\mathbf{n}'}(t) - i\alpha\hbar\,\delta_{\mathbf{n},\mathbf{n}'}\left(\rho_{\mathbf{n},\mathbf{n}'}(t) - \frac{N_e}{N_s}V_{BZ}f(\varepsilon(\mathbf{k}))\right). \tag{4}$$

The density matrix is normalized to the total number of electrons $\Sigma\rho_{\mathbf{n},\mathbf{n}}(t) = N_e$, and N_s is the number of sites in the lattice. The equilibrium distribution function is normalized as $\int f(\varepsilon(\mathbf{k}))d\mathbf{k} = 1$ and is assumed symmetric in $\pm\mathbf{k}$.

In the solution of Eq. (4) we follow Ref. 20. The resulting expressions for the average displacement and the mean-square displacement are

$$\Delta\langle \mathbf{R}(t)\rangle = \int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\int_0^t dt' e^{-\alpha t'}\mathbf{v}_{\mathbf{k}+(e/\hbar c)\mathbf{A}(t')} + \alpha N_e\int d\mathbf{k} f(\varepsilon(\mathbf{k}))\int_0^t dt' e^{-\alpha t'} \times\int_0^{t'} dt'' e^{\alpha t''}\mathbf{v}_{\mathbf{k}+(e/\hbar c)\mathbf{A}(t')-(e/\hbar c)\mathbf{A}(t'')}, \tag{5}$$

$$\Delta\langle \mathbf{R}^2(t)\rangle = 2\int d\mathbf{k}\frac{1}{V_{BZ}}\int_0^t dt' e^{-\alpha t'}\mathbf{v}_{\mathbf{k}+(e/\hbar c)\mathbf{A}(t')}\left(\mathbf{R}_{\mathbf{k},\mathbf{k}}(0) + \int_0^{t'} dt''\mathbf{v}_{\mathbf{k}+(e/\hbar c)\mathbf{A}(t'')}[\rho_{\mathbf{k},\mathbf{k}}(0) + N_e(e^{\alpha t''}-1)]\right) - \frac{1}{N_e}\langle \mathbf{R}(t)\rangle^2. \tag{6}$$

Despite the different appearance, Eqs. (6) and (5) are in fact generalizations of Eqs. (2.20) and (4.5) of Ref. 20 to the case of arbitrary lattice structure, long-distance overlap, and arbitrary initial conditions. Another point is that the definition of the density matrix in Refs. 19 and 20 differs from the conventional one by taking a complex conjugate of it. The form of Eqs. (2)–(6) presented here is more convenient for analytical analysis, and it makes the relation to conventional formulas of solid-state theory more transparent. As in Ref. 20, Eq. (6) has been obtained in the high-temperature limit or uniform equilibrium band filling. Equation (5) is equivalent to the solution of the kinetic Boltzmann equation. Equations (2), (3), (5), and (6) form the starting point of the present analysis.

The first term in Eq. (5) characterizes the quantum evolution of the electron, exponentially damped by relaxation from the initial coherent oscillations. At low scattering $\alpha T \ll 1$ (T is the period of the field, as defined above) and short time $\alpha t \ll 1$ the first term governs the evolution—it is independent of α, while the second one is small with αT. This limit in the main term is equivalent to the coherent relaxation-free case (2) and (3). The second term describes the kinetic regime plus the transition process: a small fraction αN_e of the electrons is scattered, relaxes towards the equilibrium distribution $f(\varepsilon(\mathbf{k}))$, and then again evolves quantum mechanically in the field. This term becomes dominant at long time $\alpha t \gg 1$, when the first term fades. The first regime we call dynamic (short time), the second one kinetic (long time). Note that the dynamic (either short time or purely coherent) regime cannot be obtained from the long-time kinetic one even in the limit of vanishing scattering. Indeed, by going to the long-time limit first at whatever small but finite damping we imply that the electron during that time has suffered many scatterings. This precludes us from going back to the purely coherent (dynamic, relaxation-free) regime of electron evolution by subsequent transition $\alpha T \to 0$.

III. FIELD-INDUCED LOCALIZATION

Let us analyze the electron propagation/localization in the applied field with the account of relaxation.

First, we note that strong scattering $\alpha T \gg 1$ leads to the conventional kinetic regime, which at present is not of interest to us.

For the study of coherence effects and induced localization the limiting case of slow scattering $\alpha T \ll 1$ (nearly coherent regime) is of major interest, and it will be addressed in the rest of the paper. At short time $\alpha t \ll 1$ and for low scattering the purely coherent (dynamic) results with no relaxation, Eqs. (2) and (3), studied previously,[19,27] are reproduced. The most interesting results, however, deal with the kinetic propagation/localization regimes at long time $\alpha t \gg 1$ and low scattering $\alpha T \ll 1$, when the electron has undergone many scatterings and thus has thermalized, but the probability of a scattering event during one period of the applied ac field is low.

As in the study of dynamic localization in the absence of relaxation (see Ref. 27), we separate the applied field into a constant dc component and a time-periodic one: $\mathbf{E}(t)=\mathbf{E}_0+\mathbf{E}_p(t)$. The governing parameter is composed of the magnitude of the constant component $\mathbf{E}_0$ and the frequency ω (or period T) of the oscillating part $\mathbf{E}_p(t)$. Indeed, $\mathbf{E}_0$ determines the average growth rate of $\mathbf{k}(t)$, which enters $\mathbf{v}_{\mathbf{k}(t)}$, periodic in the reciprocal lattice, while ω characterizes the periodicity, imposed by the external field. The interrelation of these two periodicities determines the evolution of $\mathbf{k}(t)$ in the Brillouin zone: periodic, commensurate, or incommensurate. Equivalently, the governing parameter characterizes the gain of the vector potential during the period of the field $\Delta\mathbf{A}=-c\mathbf{E}_0T$, compared to vectors of the reciprocal lattice $\mathbf{Q}$. Yet another interpretation of the governing parameter is the relation of two characteristic frequencies ω_0/ω: frequency of Bloch oscillations $\omega_0=2\pi eE_0/\hbar Q$, where E_0 is the projection of $\mathbf{E}_0$ upon the considered axis $\mathbf{Q}$, and frequency ω of the periodic field.

In the subsequent consideration we will incorporate the coefficient $e/\hbar c$ into the vector potential for brevity: $\tilde{\mathbf{A}}(t)=(e/\hbar c)\mathbf{A}(t)$.

A. Periodic case

The periodic case takes place, when $\mathbf{E}_0$ and T satisfy either one of the following equivalent equations with some integer μ:

$$\mathbf{E}_0T=-\frac{\hbar}{e}\mu\mathbf{Q} \quad \text{or} \quad \frac{\omega_0}{\omega}=|\mu|. \tag{7}$$

Here $\mathbf{Q}$ is some vector (2π-integer) of the reciprocal lattice. The quantum evolution of the electronic system during the period of the external field drives it back to the same states in $\mathbf{k}$ space it occupied before. However, in position space the electron may shift during that time. Equation (7) signifies that the gain of the wavevector $\Delta\mathbf{k}$ during the period of the field T is an integer multiple of some vector of the reciprocal lattice $\mathbf{Q}$, or the projection of the "vector" Bloch frequency $\boldsymbol{\omega_0}=e\mathbf{E}_0/\hbar$ upon $\mathbf{Q}$ is an integer multiple of the field frequency ω.

In the periodic regime the electron wavepacket velocity at arbitrary time $t=mT+\Delta t$ (where m is integer, and $0<\Delta t<T$) can be obtained by presenting the time integral in Eq. (5) as a sum of m integrals over period T, and summing up the resultant series:[27,28]

$$\begin{aligned}
&\langle\mathbf{v}(t=mT+\Delta t)\rangle \\
&\quad = e^{-\alpha(mT+\Delta t)}\int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t)} \\
&\qquad + \alpha N_e e^{-\alpha\Delta t}\int d\mathbf{k} f(\varepsilon(\mathbf{k})) \\
&\qquad \times\left[\frac{1-e^{-\alpha mT}}{1-e^{-\alpha T}}\int_0^T d(\Delta t')e^{-\alpha(T-\Delta t')}\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t)-\tilde{\mathbf{A}}(\Delta t')}\right. \\
&\qquad \left. + \int_0^{\Delta t} d(\Delta t')e^{\alpha\Delta t'}\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t)-\tilde{\mathbf{A}}(\Delta t')}\right],
\end{aligned} \tag{8}$$

where $\mathbf{v}_{\mathbf{k}(t)}$ is determined in Eq. (2).

First we consider the short time case $\alpha t \ll 1$ with slow-scattering $\alpha T \ll 1$. Under condition (7) the electron evolution belongs then to the periodic dynamic regime (PD) (cf. Ref. 27). The PD ac response is provided by the integral in the first term of Eq. (8) and is not small in relaxation $\sim(\alpha T)^0$. The dc response: in the main (first) term of Eq. (8) the electrons are typically delocalized and propagate with the average velocity

$$\langle\langle\mathbf{v}\rangle\rangle=\frac{1}{T}\int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\int_0^T dt\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)} \tag{9}$$

with superimposed periodic oscillations. The leading term of the PD diffusion coefficient follows from either Eq. (3) or (6), and also typically corresponds to delocalization (propagation):

$$\begin{aligned}
D = {} & \frac{t}{2d}\int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\left[\frac{1}{T}\int_0^T dt'\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t')}\right]^2 \\
& + \frac{1}{dT}\int d\mathbf{k}\frac{\mathbf{R}_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\int_0^T dt'\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t')}-\frac{t}{2dN_e}\langle\langle\mathbf{v}\rangle\rangle^2.
\end{aligned} \tag{10}$$

Thus the leading terms of all the PD responses are not small in the relaxation rate. In the relaxation-free formulation (2) and (3), these equations are valid in the entire time domain. The diffusion coefficient (10) is always positive, and the dc velocity (9) is typically nonzero even in the absence of the dc component in the applied field. However, in some cases the average velocity (9), summed over symmetric (in $\pm k$) band filling, vanishes. It happens if the field satisfies the following equation:

$$\int_0^T dt\,\sin[\tilde{\mathbf{A}}(t)\mathbf{n}]=0 \tag{11}$$

for all sites $\mathbf{n}$ with nonzero overlap $H^{(0)}_{\mathbf{0},\mathbf{n}}$. A sufficient (but not necessary) condition for it is provided, for example, by the time-inversion symmetry of the field at some time t_0:

$$\mathbf{E}(t_0+\Delta t)=\mathbf{E}(t_0-\Delta t), \quad \mathbf{A}(t_0)=0. \tag{12}$$

Of importance[27,28,30] is another case of antisymmetry with respect to a time shift by $T/2$:

$$\mathbf{E}(t)=-\mathbf{E}\left(t+\frac{T}{2}\right), \quad \mathbf{A}\left(\frac{T}{2}\right)=0. \tag{13}$$

However, the vanishing of the average current does not necessarily mean localization, as the wavepacket still may spread due to diffusion or propagate in parts simultaneously in opposite directions. The criterion of localization demands that the mean-square displacement remains bounded (the diffusion coefficient vanishes), or the average velocity of each **k** mode in the Brillouin zone vanishes independently:[19–23,26,27]

$$\langle\langle \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)}\rangle\rangle = 0. \tag{14}$$

The latter is an equation for the parameters of the applied field, which holds under the condition of Eq. (11) and

$$\int_0^T dt \cos[\tilde{\mathbf{A}}(t)\mathbf{n}] = 0 \tag{15}$$

for all sites $\mathbf{n}$ with nonzero overlap $H^{(0)}_{\mathbf{0},\mathbf{n}}$ (Ref. 27) (lattices with inversion symmetry). This localization is in this case an exception (a rather rare occasion) and is typically called “dynamic” localization.[19–23,26,27]

Here we would like to note the following. With the account of distant neighbors the electrons get localized under more general conditions

$$\sum_{\mathbf{n}} \mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}} \int_0^T dt \sin[\tilde{\mathbf{A}}(t)\mathbf{n}] = 0,$$

$$\sum_{\mathbf{n}} \mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}} \int_0^T dt \cos[\tilde{\mathbf{A}}(t)\mathbf{n}] = 0, \tag{16}$$

instead of Eqs. (11) and (15). The parameters of the field in these solutions, which are not simultaneously solutions to Eqs. (11) and (15), are dependent upon the electron dispersion or the overlap integrals $H^{(0)}_{\mathbf{0},\mathbf{n}}$. We will not consider such “model-specific” solutions here, but concentrate on the cases (11) and (15), when localization is a consequence of time-symmetry of the field and spatial symmetry of the system. Within the tight-binding approximation the equations (16) obviously reduce to (11) and (15).

Under the conditions (11) and (15) the diffusion coefficient (10) vanishes, which signifies exceptional localization. In the presence of relaxation the localization, however, is never absolute and is weakly destroyed by higher order terms in $\alpha T \ll 1$, which stem from Eq. (6). The corresponding non-vanishing term is of order $\sim \alpha T$:

$$D_{loc} = \frac{\alpha}{dT}\int \frac{d\mathbf{k}}{V_{BZ}}\int_0^T dt\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)}\int_0^t dt'\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t')}[N_e t' - \rho_{\mathbf{k},\mathbf{k}}(0)t], \tag{17}$$

where for brevity we assumed $\mathbf{R}_{\mathbf{k},\mathbf{k}}(0)=0$.

The PD dc response (9) under the conditions (11) and (15) also vanishes, i.e., it is sensitive to localization. Allowing for slow relaxation we find that localization at short time $\alpha t \ll 1$ is weakly destroyed by terms of the next order αT, stemming from Eq. (8):

$$\langle\langle \mathbf{v}\rangle\rangle_{loc} = -\frac{\alpha}{T}\int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\int_0^T dt\, t\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)} + \frac{\alpha N_e}{T}\int d\mathbf{k} f(\mathbf{k})\int_0^T dt\int_0^t dt'\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)-\tilde{\mathbf{A}}(t')}. \tag{18}$$

Additionally, the second term in Eq. (18) becomes zero under exceptional localization for all fields, possessing shift antisymmetry, Eq. (13).

In conjunction with Eqs. (9) and (18) we would like to note the following. Discussing the average velocity we can adopt either one of the definitions—as an average of the velocity over the period for time t, or as the average displacement during time t, divided by t. In the second case the result will incorporate contributions from all the electron displacements prior to t. For our discussion with the electric current in mind the first definition is more in place. The second one might also be suitable for the discussion of spatial aspects of localization. Both calculations, though, can be performed easily in analytic form. The leading terms of the short-time and long-time expansions in both cases coincide.

The PD ac response is not sensitive to exceptional localization.

Next we consider the low scattering $\alpha T \ll 1$ long time $\alpha t \gg 1$ periodic kinetic regime (PK) [Eq. (7)]. In the long-time limit the influence of initial coherent oscillations is lost completely, and the electron evolution becomes notably different from the purely coherent quantum case [Eqs. (2), (3), and (9)]. The localization/propagation of the electrons can be studied in transparent analytic form for slow relaxation, and coherent effects are most pronounced in this limit.

The leading term of the electron wavepacket velocity at long time $t=mT+\Delta t$, $m \gg (\alpha T)^{-1}$ stems from the second term of Eq. (8):

$$\langle \mathbf{v}(t=mT+\Delta t)\rangle = \frac{N_e}{T}\int d\mathbf{k} f(\mathbf{k})\int_0^T dt'\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t)-\tilde{\mathbf{A}}(t')}, \tag{19}$$

plus higher order terms in αT.

Clearly, the kinetic low-scattering velocity Eq. (19) is different from the coherent one, Eq. (9), which came from the first term of Eq. (8). It is periodic in time with period T, $0 < \Delta t < T$, and in the main term is independent of the scattering rate α, like Eq. (9). Thus the ac response in regime PK is typically nonzero with the leading term $\sim(\alpha T)^0$. After averaging over the period of the field (of any time dependence), the main term of the direct current obtained from Eq. (19), however, vanishes identically. Thus, in the low-scattering long-time regime there is no average drift of the wavepacket, independent of the scattering rate α, in contrast to the dynamic relaxation-free case (9).

As before, the zero value of the averaged velocity (or direct current) derived from Eq. (19) for any field does not necessarily mean localization. Indeed, in the kinetic regime under consideration the diffusion coefficient obtained from Eq. (6),

$$D = \frac{N_e}{\alpha d T^2}\int \frac{d\mathbf{k}}{V_{BZ}}\left[\int_0^T dt\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)}\right]^2, \tag{20}$$

is not only nonzero, but in fact is big, $\sim(\alpha T)^{-1}$. This signifies the fast diffusive spreading of the electron wavepacket for low scattering (large mean free path), and, correspondingly, typical delocalization. We note here also that the kinetic expression (20) cannot reproduce the dynamic zero-scattering case $\alpha T=0$, when the electron propagates ballistically [Eq. (3)]. However, the stated divergence $D\sim(\alpha T)^{-1}$ "feels" that transition for $\alpha T\to 0$ correctly. On a qualitative level, random scattering averages out the velocity, but contributes to the diffusion coefficient.

The time-averaged velocity assumes typically nonzero values in the next order $\sim\alpha T$ in contrast to Eq. (19):

$$\langle\langle \mathbf{v}\rangle\rangle = \frac{\alpha N_e}{T^2}\int d\mathbf{k} f(\mathbf{k})\left(T\int_0^T dt\int_0^t dt' \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)-\tilde{\mathbf{A}}(t')} + \int_0^T dt\int_0^T dt'(t'-t)\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)-\tilde{\mathbf{A}}(t')}\right). \tag{21}$$

Equation (21) is nonzero either in the presence of the dc component of the applied field, or if the left-right symmetry is violated otherwise, like, for example, in a bichromatic field $E(t)=E_1\cos(\omega t)+E_2\cos(2\omega t+\varphi)$.[29]

After the above statement of substantial differences between the dynamic and the kinetic regimes it is interesting to note that the exceptional localization in the low-scattering long-time domain still exists exactly under the same conditions (11) and (15), as in the dynamic case (cf. Ref. 20). However, the manifestations of the effect now are different: whereas in the dynamic case we have vanishing direct current with ac oscillations remaining nonzero, in the kinetic regime we have vanishing leading term of the alternating current $\sim(\alpha T)^0$ for any Δt [Eq. (19)], and a somewhat more complicated situation with the direct current. Namely, the second term in the direct current $\sim\alpha T$ [Eq. (21)], under localization conditions (11) and (15) vanishes identically. For the first one to vanish, however, the additional shift-antisymmetry of the field (13) is required, or it remains typically nonzero otherwise. Thus, localization tends to decrease the direct current by an order in the small parameter $\sim\alpha T$ and does that exactly in the additional presence of shift-antisymmetry of the field, which is rather common. Then the leading nonzero term of the long-time kinetic current becomes the next one $\sim(\alpha T)^2$:

$$\langle\langle \mathbf{v}\rangle\rangle_{loc} = -\frac{\alpha^2 N_e}{T^2}\int d\mathbf{k} f(\mathbf{k})\left(T\int_0^T dt\int_0^t dt'(t-t')\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)-\tilde{\mathbf{A}}(t')} + \int_0^T dtt\int_0^T dt' t'\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)-\tilde{\mathbf{A}}(t')}\right), \tag{22}$$

which corresponds to a weaker delocalization, as compared to Eq. (21). In the absence of shift-antisymmetry, the direct current typically retains its order $\sim\alpha T$, and is provided by the first term in Eq. (21).

The leading ac response (19) under localization conditions (11) and (15) vanishes. Then the nonzero alternating current is provided in next order $\sim\alpha T$ by Eq. (21) without averaging over t—smaller than without localization:

$$\langle \mathbf{v}(t=mT+\Delta t)\rangle_{loc} = \frac{\alpha N_e}{T}\int d\mathbf{k} f(\mathbf{k})\left(T\int_0^t dt' \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t)-\tilde{\mathbf{A}}(t')} + \int_0^T dt' t' \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t)-\tilde{\mathbf{A}}(t')}\right). \tag{23}$$

An important question is the diffusional spreading of the wavepacket under localization. Interestingly, in the periodic kinetic regime we find localization again, as it was in the dynamic case. In fact, both leading terms of the diffusion coefficient, $\sim(\alpha T)^{-1}$ [Eq. (20)], and the next one, $\sim(\alpha T)^0$, vanish under localization conditions (11) and (15). The resulting diffusion coefficient, derived from Eq. (6), is small $\sim\alpha T$:

$$D_{loc} = \frac{\alpha N_e}{dT}\int \frac{d\mathbf{k}}{V_{BZ}}\left[\int_0^T dt\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)}\int_0^t dt'(t'-t)\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t')} - \frac{1}{T}\left(\int_0^T dt\, t\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(t)}\right)^2\right]. \tag{24}$$

This fact justifies the use of the term localization in the low-scattering periodic kinetic regime.[20] All the quantities of interest (ac and dc response, diffusion coefficient) reveal sensitivity to exceptional localization.

Thus, in the periodic case the electron evolution in the short-time dynamic and in the long-time kinetic regimes are drastically different. However, field-induced coherent localization exists in both regimes under exactly the same conditions upon the parameters of the applied field. The typical effect of localization is to decrease the corresponding quantities by an order in the small parameter $\alpha T\ll 1$. Exceptions are the kinetic diffusion coefficient, when the decrease is two orders in αT, and the kinetic direct current, when additional shift-antisymmetry of the field is required.

B. Commensurate case

Next we pass over to the commensurate case. It takes place when the field parameters satisfy the following equation with some integers μ,μ':

$$\mathbf{E}_0 T = -\frac{\hbar}{e}\mathbf{Q}\frac{\mu}{\mu'} \quad \text{or} \quad \frac{\omega_0}{\omega} = \left|\frac{\mu}{\mu'}\right|. \tag{25}$$

After μ' periods of the external field the quantum evolution of the electronic system takes it back to the same states in $\mathbf{k}$ space, as before. Equivalently, the gain of the wave vector $\Delta\mathbf{k}$ during the period of the field T is commensurate with some vector of the reciprocal lattice $\mathbf{Q}$, or the projection of the vector Bloch frequency $\boldsymbol{\omega}_0=e\mathbf{E}_0/\hbar$ in the constant component of the field $\mathbf{E}_0$ upon $\mathbf{Q}$ is commensurate with the frequency ω of the time-periodic component.

In the commensurate regime the electron wavepacket velocity [see Eq. (5)] at arbitrary time $t=m\mu' T+\Delta t$ (where m is an integer, and $0<\Delta t<\mu' T$) can be calculated as in the periodic case by presenting the integral over time in Eq. (5) as a sum of m integrals over μ' periods each, and summing up the corresponding series:

$$\langle \mathbf{v}(t=m\mu' T+\Delta t)\rangle = e^{-\alpha(m\mu' T+\Delta t)}\int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t)} + \alpha N_e e^{-\alpha\Delta t}\int d\mathbf{k} f(\varepsilon(\mathbf{k})) \times\left(\frac{1-e^{-\alpha m\mu' T}}{1-e^{-\alpha\mu' T}}\int_0^{\mu' T} d(\Delta t')e^{-\alpha(\mu' T-\Delta t')}\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t)-\tilde{\mathbf{A}}(\Delta t')} + \int_0^{\Delta t} d(\Delta t')e^{\alpha\Delta t'}\mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t)-\tilde{\mathbf{A}}(\Delta t')}\right). \tag{26}$$

At short time $\alpha\mu' t \ll 1$ in the main [first in Eq. (26)] term the electron evolves in the commensurate dynamic (CD) regime with average velocity:

$$\langle\langle \mathbf{v}\rangle\rangle = -\frac{2}{\hbar\mu' T}\int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\sum_{\mathbf{n}}\mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}}\frac{\sin\left(\frac{1}{2}\mu\mathbf{Q}\mathbf{n}\right)}{\sin\left(\frac{1}{2}\frac{\mu}{\mu'}\mathbf{Q}\mathbf{n}\right)} \times\int_0^T dt\,\sin\left[\mathbf{k}\mathbf{n}+\tilde{\mathbf{A}}(t)\mathbf{n}+\frac{\mu'-1}{2}\frac{\mu}{\mu'}\mathbf{Q}\mathbf{n}\right]. \tag{27}$$

In this section and in the next one we consider lattices with inversion symmetry. The indefiniteness of the type $\frac{0}{0}$ in Eq. (27) we eliminate by adding an infinitesimal ε to $\frac{1}{2}(\mu/\mu')\mathbf{Q}\mathbf{n}$ and then passing to the limit $\varepsilon\to 0$.

Within the tight-binding approximation (TBA) and under the condition (25), Eq. (27) results in a strict overall localization for any periodic electric field (cf. Refs. 21 and 27). The introduction of overlap beyond the first coordination sphere, however, produces typically weak delocalization due to neighbors $\mathbf{N}$, situated on "resonant" equidistant planes, satisfying

$$\mathbf{Q}\mathbf{N} = 2\pi\mu'\nu \tag{28}$$

with any integer ν and with $\mathbf{Q}$ and μ' the same as in Eq. (25).[27,28] In 1d, for example, these are neighbors at distances multiple of μ'. The overlap with all the other "nonresonant" sites produces oscillations only. Thus, due to neighbors $\mathbf{N}$ the electron is weakly delocalized with wavepacket velocity (27):

$$\langle\langle \mathbf{v}\rangle\rangle = -\frac{2}{\hbar T}\int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\sum_{\mathbf{N}}\mathbf{N}H^{(0)}_{\mathbf{0},\mathbf{N}}\int_0^T dt\,\sin[\mathbf{k}\mathbf{N}+\tilde{\mathbf{A}}(t)\mathbf{N}], \tag{29}$$

which is small due to the small values of $H^{(0)}_{\mathbf{0},\mathbf{N}}$.

Similarly, the leading term of the short-time CD diffusion coefficient (3) $(\alpha\mu' T\ll 1, \alpha t\ll 1)$ is formed by "resonant" neighbors (28) only:

$$D = \frac{2t}{d\hbar^2 T^2}\int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\left(\sum_{\mathbf{N}}\mathbf{N}H^{(0)}_{\mathbf{0},\mathbf{N}}\int_0^T dt\,\sin[\mathbf{k}\mathbf{N}+\tilde{\mathbf{A}}(t)\mathbf{N}]\right)^2 - \frac{2}{d\hbar T}\int d\mathbf{k}\frac{\mathbf{R}_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\sum_{\mathbf{N}}\mathbf{N}H^{(0)}_{\mathbf{0},\mathbf{N}}\int_0^T dt\,\sin[\mathbf{k}\mathbf{N}+\tilde{\mathbf{A}}(t)\mathbf{N}] - \frac{t}{2dN_e}\langle\langle \mathbf{v}\rangle\rangle^2, \tag{30}$$

where $\langle\langle \mathbf{v}\rangle\rangle$ is provided in Eq. (29). Obviously, the short-time diffusion coefficient, like the short-time direct current, is not small in $\alpha\mu' T$, but is small in $H^{(0)}_{\mathbf{0},\mathbf{N}}$, and it grows in time $\sim t$, which corresponds to propagation.

Exceptional induced localization for an arbitrary wavepacket with the account of distant neighbors takes place, if the field satisfies the same Eqs. (11) and (15). In fact, for the direct current (29) and the diffusion coefficient (30) to vanish, it is sufficient for these conditions to hold for "resonant" neighbors $\mathbf{N}$, determined by Eq. (28) only, for sites with nonzero overlap.

Under exceptional localization in the CD regime the electron is weakly delocalized by the next order term $\sim\alpha\mu' T$ of Eq. (26). This term is equivalent to Eq. (18) with the substitution $T\to\mu' T$. With localization conditions (11) and (15) imposed for all sites $\mathbf{n}$, this term is formed by both "resonant" and "nonresonant" neighbors:

$$\langle\langle \mathbf{v}\rangle\rangle_{loc} = \frac{2\alpha}{\hbar T}\int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\sum_{\mathbf{N}}\mathbf{N}H^{(0)}_{\mathbf{0},\mathbf{N}}\cos(\mathbf{k}\mathbf{N}) \times\int_0^T dtt\,\sin[\tilde{\mathbf{A}}(t)\mathbf{N}] - \frac{2\alpha N_e}{\hbar T}\int d\mathbf{k} f(\mathbf{k})\sum_{\mathbf{n}}\mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}}\cos(\mathbf{k}\mathbf{n}) \times\int_0^T dt\int_0^t dt'\,\sin[\tilde{\mathbf{A}}(t)\mathbf{n}-\tilde{\mathbf{A}}(t')\mathbf{n}], \tag{31}$$

where we assumed symmetric initial band filling $\rho_{\mathbf{k},\mathbf{k}}(0)=\rho_{-\mathbf{k},-\mathbf{k}}(0)$. [If we restrict the localization conditions to "resonant" sites only, two more terms in Eq. (31), coming from "nonresonant" contributions, would appear.] The dc response (31) under localization becomes small in the parameter $\alpha\mu' T$, in contrast to Eq. (29).

Away from localization there are four terms more to Eq. (31), coming from both "resonant" and "nonresonant" neighbors. In some practical cases without localization this second term $\sim\alpha\mu' T$ of the direct current may be even bigger due to "nonresonant" contributions than the main term $\sim(\alpha\mu' T)^0$ coming from distant "resonant" sites only [Eq. (29)].

The nonvanishing diffusion coefficient D_{loc} is of order $\sim\alpha\mu' T$ and can be obtained from Eq. (17) by substitution $T\to\mu' T$.

In reality it is rather hard to meet the requirements for exceptional localization beyond the tight-binding (or nearest-neighbor) approximation (TBA) rigorously, except for a rectangular step-wise alternating field (see examples below) or finite-radius-overlap approximations.

Within TBA the asymptotics, formed solely by contributions from "resonant" distant neighbors (28), vanish. Then the dynamic direct current in any field without exceptional localization becomes first order in $\alpha\mu' T$, in contrast to Eq. (29):

$$\langle\langle \mathbf{v}\rangle\rangle_{TBA} = -\frac{\alpha}{\hbar}\sin^{-1}\left(\pi\frac{\mu}{\mu'}\right)\sum_{\mathbf{n}(TBA)}\mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}}\int d\mathbf{k}\frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}}\cos(\mathbf{kn})\int_0^T dt\cos\left[\pi\frac{\mu}{\mu'}-\widetilde{\mathbf{A}}(t)\mathbf{n}\right]$$
$$-\frac{\alpha N_e}{\hbar T}\sum_{\mathbf{n}(TBA)}\mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}}\int d\mathbf{k}f(\mathbf{k})\cos(\mathbf{kn})\left[\cot\left(\pi\frac{\mu}{\mu'}\right)\int_0^T dt\int_0^T dt'\cos[\widetilde{\mathbf{A}}(t)\mathbf{n}-\widetilde{\mathbf{A}}(t')\mathbf{n}]\right.$$
$$\left.+2\int_0^T dt\int_0^t dt'\sin[\widetilde{\mathbf{A}}(t)\mathbf{n}-\widetilde{\mathbf{A}}(t')\mathbf{n}]\right]. \tag{32}$$

This can be viewed as overall [i.e., in any field (25)] localization in comparison to the periodic regime and to the commensurate one with long-range overlap.

Under localization conditions (11) and (15) the first two terms in Eq. (32) vanish, but the last one does not. This signifies the considerable decrease (rigorously—elimination) of the effect of exceptional localization within the commensurate tight-binding approximation in comparison to the regimes PD and long-range-overlap CD.

Thus the reduction of the overlap radius to TBA in the CD regime increases overall localization. On this background the direct current loses sensitivity to exceptional localization. The CD alternating current $\sim(\alpha\mu' T)^0$, provided by the first term in Eq. (26), is qualitatively unaffected by neither TBA nor by exceptional localization.

The commensurate kinetic (CK) regime takes place under the condition of Eq. (25) at long time, $\alpha t \gg 1$. The main term of the velocity derived from Eq. (26) at long time $t = m\mu' T + \Delta t$ (with m an integer and $0 < \Delta t < \mu' T$) in the low-scattering limit $\alpha\mu' T \ll 1$ is given by

$$\langle \mathbf{v}(t = m\mu' T + \Delta t)\rangle = -\frac{2N_e}{\hbar T}\sum_{\mathbf{N}}\mathbf{N}H^{(0)}_{\mathbf{0},\mathbf{N}}\int d\mathbf{k}f(\mathbf{k})\cos(\mathbf{kN})\int_0^T dt'\sin[\widetilde{\mathbf{A}}(\Delta t)\mathbf{N}-\widetilde{\mathbf{A}}(t')\mathbf{N}]. \tag{33}$$

Only "resonant" neighbors $\mathbf{N}$ [Eq. (28)] contribute to Eq. (33), like it was in the CD regime [Eq. (29)]. However, this time it is for the alternating current (main term)—Eq. (33) is periodic in time, whereas in the CD regime, considered previously, it was for the direct current. Both cited quantities—the leading kinetic ac, Eq. (33), and dynamic dc, Eq. (29), responses for the commensurate case are thus smaller than those for the periodic regime, as they are provided by overlap with distant **N**th neighbors only. However, these main terms are not small in the relaxation rate $\sim(\alpha\mu' T)^0$. The period of the alternating current in Eq. (33) is $\mu' T$, i.e., the basic frequency in the response is μ' times lower than that of the input field.

The time average of the leading term of the velocity (33) is zero, so that there is no constant drift, independent of the scattering rate α. The nonvanishing direct current is produced in the next order $\sim\alpha\mu' T$ of Eq. (26) [it can also be obtained from Eq. (21) with the substitution $T\to\mu' T$ and some subsequent manipulations]:

$$\langle\langle \mathbf{v}\rangle\rangle = -\frac{2\alpha N_e}{\hbar T}\int d\mathbf{k}f(\mathbf{k})\left(\sum_{\mathbf{n}}\mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}}\cos(\mathbf{kn})\int_0^T dt\int_0^t dt'\sin[\widetilde{\mathbf{A}}(t)\mathbf{n}-\widetilde{\mathbf{A}}(t')\mathbf{n}]\right.$$
$$+\frac{1}{2}\sum_{\mathbf{n}'}\mathbf{n}'H^{(0)}_{\mathbf{0},\mathbf{n}'}\cos(\mathbf{kn}')\cot\left(\pi\frac{\mu\lambda}{\mu'}\right)\int_0^T dt\int_0^T dt'\cos[\widetilde{\mathbf{A}}(t)\mathbf{n}'-\widetilde{\mathbf{A}}(t')\mathbf{n}']$$
$$\left.-\frac{2}{T}\sum_{\mathbf{N}}\mathbf{N}H^{(0)}_{\mathbf{0},\mathbf{N}}\cos(\mathbf{kN})\int_0^T tdt\int_0^T dt'\sin[\widetilde{\mathbf{A}}(t)\mathbf{N}-\widetilde{\mathbf{A}}(t')\mathbf{N}]\right), \tag{34}$$

where by $\mathbf{n}'$ we denote the "nonresonant" neighbors, for which $\mathbf{Qn}' = 2\pi\lambda \neq 2\pi\mu'\nu$. In contrast to the leading ac response [Eq. (33)] it comprises contributions from all the sites, not only "resonant" ones. Thus it is not small in $H^{(0)}_{\mathbf{0},\mathbf{N}}$, and in some cases it may exceed the amplitude of the leading alternating current (33). The leading CK direct current without exceptional localization is of the same order, $\sim\alpha\mu' T$, as the one in PK regime.

The main term of the long-time CK diffusion coefficient ($\alpha\mu' T \ll 1, \alpha t \gg 1$) is formed by "resonant" neighbors (28) only, as in the short-time case (30):

$$D = \frac{2N_e}{\alpha d(\hbar T)^2}\sum_{\mathbf{N}} (\mathbf{N}H^{(0)}_{\mathbf{0},\mathbf{N}})^2\left[\left(\int_0^T dt \cos[\widetilde{\mathbf{A}}(t)\mathbf{N}]\right)^2 + \left(\int_0^T dt \sin[\widetilde{\mathbf{A}}(t)\mathbf{N}]\right)^2\right], \tag{35}$$

and, consequently, diffusion is slow $\sim(H^{(0)}_{\mathbf{0},\mathbf{N}})^2$. However, D is big in $\sim(\alpha\mu' T)^{-1}$, as it was in the periodic case—diffusion tends to be fast due to long mean-free path, but in fact might be slow due to long-range overlap with "resonant" neighbors only. We note that the study of the mean-square displacement, based on Eq. (6) in the long-time domain is valid for high temperatures and/or narrow bands, when $f(\mathbf{k})\to V_{BZ}^{-1}$.

With the distant overlap taken into account, exceptional localization takes place under the same conditions (11) and (15) for all sites $\mathbf{n}$. Then the leading term of the diffusion coefficient (35) vanishes. In fact, it is sufficient for these conditions to hold for "resonant" neighbors (28) only (in the long-range-overlap commensurate regime this is typically the case for the leading terms of expansions that involved single integration over the range $(0,\mu' T)$—cf. Eqs. (29), (30), and (33)). Moreover, under exceptional localization not only one, but two leading terms of D vanish. The resulting diffusion coefficient is of the order of $\sim\alpha T$, as in regime PK, Eq. (24):

$$D_{loc} = -\frac{2\alpha N_e}{\hbar^2 dT^2}\left\{\sum_{\mathbf{N}} (\mathbf{N}H^{(0)}_{\mathbf{0},\mathbf{N}})^2\int_0^T t dt\int_0^T t' dt' \cos[\widetilde{\mathbf{A}}(t')\mathbf{N} - \widetilde{\mathbf{A}}(t)\mathbf{N}] + T\sum_{\mathbf{n}} (\mathbf{n}\overline{H}^{(0)}_{\mathbf{0},\mathbf{n}})^2\int_0^T dt\int_0^t dt'(t-t') \times\cos[\widetilde{\mathbf{A}}(t)\mathbf{n} - \widetilde{\mathbf{A}}(t')\mathbf{n}]\right\}. \tag{36}$$

Contrary to the leading term (35), it comprises contributions from both "resonant" and "nonresonant" sites.

Under localization the leading term of the long-time ac response $\sim(\alpha\mu' T)^0$, Eq. (33), vanishes as well. Conditions (11) and (15) for "resonant" neighbors (28) suffice. The vanishing of the ac response is in analogy to PK regime (19) and in contrast to the CD regime. Under induced localization the ac response $\sim\alpha\mu' T$ is formed by all sites, not only by "resonant" neighbors:

$$\langle \mathbf{v}(t = m\mu' T + \Delta t)\rangle_{loc} = \frac{2\alpha N_e}{\hbar T}\sum_{\mathbf{N}} \mathbf{N}H^{(0)}_{\mathbf{0},\mathbf{N}}\int d\mathbf{k}f(\mathbf{k})\cos(\mathbf{kN}) \times\int_0^T t' dt' \sin[\widetilde{\mathbf{A}}(t')\mathbf{N} - \widetilde{\mathbf{A}}(\Delta t)\mathbf{N}] + \frac{2\alpha N_e}{\hbar}\sum_{\mathbf{n}} \mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}}\int d\mathbf{k}f(\mathbf{k})\cos(\mathbf{kn}) \times\int_0^{\Delta t} dt' \sin[\widetilde{\mathbf{A}}(t')\mathbf{n} - \widetilde{\mathbf{A}}(\Delta t)\mathbf{n}]. \tag{37}$$

In the expression of the direct current (34) under localization the third and the second terms in figure brackets vanish, but the first one does not. Thus, despite the existence of exceptional localization in this case, manifested through the diffusion coefficient, the direct current does not vanish (is qualitatively "insensitive"). This remaining dc response comprises contributions from all the neighbors, both "resonant" and "nonresonant" and thus is not small in $\sim H^{(0)}_{\mathbf{0},\mathbf{N}}$. [If we limit localization conditions to "resonant" neighbors, Eq. (28), only the third term vanishes.]

The question whether the exceptional localization can be observed in practice in the commensurate regime or not depends, as noted above, on the relative magnitude of the expansion terms in $\alpha\mu' T$ and in $H^{(0)}_{\mathbf{0},\mathbf{N}}/H^{(0)}_{\mathbf{0},\mathbf{1}}$.

From the provided consideration we deduce that typically the localization criteria for the leading terms of the current and diffusion coefficient in the long-range-overlap commensurate regime are provided by Eqs. (11) and (15) for "resonant" sites only. The next order terms may be "not sensitive" to exceptional localization.

The situation changes in the tight-binding approximation (TBA). If we reduce the overlap radius to nearest neighbors, then the asymptotics (33) and (35), formed solely by "resonant" distant sites (28), vanish, like it was in the CD case above. The kinetic alternating current without exceptional localization becomes first order $\sim\alpha\mu' T$ in contrast to $\sim(\alpha\mu' T)^0$ in Eq. (33):

$$\langle \mathbf{v}(t = m\mu' T + \Delta t)\rangle_{TBA} = \frac{\alpha N_e}{\hbar}\sum_{\mathbf{n}(TBA)} \mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}}\int d\mathbf{k}f(\mathbf{k})\cos(\mathbf{kn}) \times\left[2\int_0^{\Delta t} dt' \sin[\widetilde{\mathbf{A}}(t')\mathbf{n} - \widetilde{\mathbf{A}}(\Delta t)\mathbf{n}] - \sin^{-1}\left(\pi\frac{\mu}{\mu'}\right)\int_0^T dt' \cos\left(\widetilde{\mathbf{A}}(t')\mathbf{n} - \widetilde{\mathbf{A}}(\Delta t)\mathbf{n} - \pi\frac{\mu}{\mu'}\right)\right], \tag{38}$$

where summation runs over nearest neighbors. Under localization conditions (11) and (15) the second term in Eq. (38) vanishes, but the first one does not. Thus within TBA in our terminology the ac response becomes "not sensitive" to exceptional localization.

The kinetic direct current within TBA retains its order $\sim\alpha\mu' T$, like in the long-range-overlap CK regime. It is provided by the same Eq. (34) with the following modifications: the third term vanishes, summation runs over nearest neighbors, and $\lambda=1$. Under localization conditions (11) and (15) the second term of Eq. (34) vanishes as well, but the first one does not [shift antisymmetry is not compatible with Eq. (25)]. Thus the CK TBA direct current is qualitatively insensitive to exceptional localization, like the long-range-overlap direct current (34) was.

The CK diffusion coefficient decreases under TBA to order $\sim\alpha\mu' T$ in contrast to Eq. (35). The expression of it can be obtained from Eq. (24) by the substitution $T\to\mu' T$ and some subsequent manipulations:

$$D_{TBA} = \frac{\alpha N_e}{2\hbar^2 dT} \sum_{\mathbf{n}(TBA)} (\mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}})^2 \Bigg[T \sin^{-2}\left(\pi \frac{\mu}{\mu'}\right) \int_0^T dt \int_0^T dt' \cos[\tilde{\mathbf{A}}(t)\mathbf{n} - \tilde{\mathbf{A}}(t')\mathbf{n}] + 2\cot\left(\pi\frac{\mu}{\mu'}\right) \int_0^T dt \int_0^T dt'(t-t')\sin[\tilde{\mathbf{A}}(t)\mathbf{n} - \tilde{\mathbf{A}}(t')\mathbf{n}] - 4\int_0^T dt \int_0^t dt'(t-t')\cos[\tilde{\mathbf{A}}(t)\mathbf{n} - \tilde{\mathbf{A}}(t')\mathbf{n}] \Bigg]. \quad (39)$$

Under exceptional localization the first two integrals in Eq. (39) vanish, but the last one does not. Thus, D_{TBA} is qualitatively insensitive to exceptional localization, namely, retains the same order $\sim \alpha\mu' T$. The reduction of the overlap radius to TBA in the commensurate kinetic regime increases the overall localization and, rigorously speaking, eliminates the effect of exceptional localization (11) and (15), at least within the high-temperature approximation for diffusion.

In short, the commensurate regime typically exhibits stronger overall localization than the periodic one—either in smaller overlap, or extra order in $\alpha\mu' T$ within TBA. With long-range overlap, the exceptional localization is manifested qualitatively similar to the periodic case (apart from the factor $H^{(0)}_{\mathbf{0},\mathbf{N}}$), while in the tight-binding approximation it is practically eliminated.

Most formulas for the commensurate regime can be obtained from those for the periodic case with the substitution $T \to \mu' T$ and some subsequent manipulations. In fact, the periodic regime can be viewed as a particular case of the commensurate one with $\mu' = 1$ (all sites are "resonant"). However, the periodic case is much more simple and transparent. Besides, within the commonly used tight-binding approximation these regimes differ drastically. For these reasons we considered them separately.

C. Incommensurate case

The incommensurate regime takes place when Eqs. (7) and (25) do not hold, or, in other words, when there is no reciprocal lattice vector $\mathbf{Q}$, parallel to $\mathbf{E}_0$, or the magnitudes of all $\mathbf{Q}$ and $\omega_0 T$ are incommensurate. Then the electron wavepacket velocity at arbitrary time $t = mT + \Delta t$ (where m is an integer, and $0 < \Delta t < T$) can be calculated in analogy to the previous cases by presenting the integral over time in Eq. (5) as a sum of m integrals over the field period each, and summing up the corresponding series:

$$\begin{aligned}\langle \mathbf{v}(t = mT + \Delta t)\rangle &= e^{-\alpha t} \int d\mathbf{k} \frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}} \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t) - m\boldsymbol{\omega}_0 T} + \alpha N_e e^{-\alpha \Delta t} \int d\mathbf{k} f(\mathbf{k}) \Bigg([1 - 2e^{\alpha T}\cos(\boldsymbol{\omega}_0 T\mathbf{n}) + e^{2\alpha T}]^{-1} \\ &\times \int_0^T d(\Delta t') e^{\alpha \Delta t'} [e^{-\alpha m T} \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t) - \tilde{\mathbf{A}}(\Delta t') - m\boldsymbol{\omega}_0 T} - e^{-\alpha(m-1)T} \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t) - \tilde{\mathbf{A}}(\Delta t') - (m+1)\boldsymbol{\omega}_0 T} \\ &+ e^{\alpha T} \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t) - \tilde{\mathbf{A}}(\Delta t') - \boldsymbol{\omega}_0 T} - \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t) - \tilde{\mathbf{A}}(\Delta t')}] + \int_0^{\Delta t} d(\Delta t') e^{\alpha \Delta t'} \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t) - \tilde{\mathbf{A}}(\Delta t')} \Bigg),\end{aligned} \quad (40)$$

where we made use of the symmetry of $\rho_{\mathbf{k},\mathbf{k}}(0)$ and $f(\mathbf{k})$ in $\pm\mathbf{k}$. Equation (40) is valid for lattices with inversion symmetry.

In the low-scattering $\alpha T \ll 1$, short-time $\alpha t \ll 1$ incommensurate dynamic regime (ID) the leading term of the velocity is

$$\langle \mathbf{v}(t = mT + \Delta t)\rangle = \int d\mathbf{k} \frac{\rho_{\mathbf{k},\mathbf{k}}(0)}{V_{BZ}} \mathbf{v}_{\mathbf{k}+\tilde{\mathbf{A}}(\Delta t) - m\boldsymbol{\omega}_0 T}. \quad (41)$$

Obviously, the short-time ac response (41) is nonzero, like it was in both periodic and commensurate dynamic regimes. The calculation of the dc response requires some comments. Contrary to the regimes PD and CD, the velocity Eq. (41) is not periodic in time, even in the "quasistationary" limit of slow relaxation. Hence there is no natural time period of evolution for averaging to calculate the "dc" response—in contrast to T in the periodic and $\mu' T$ in the commensurate case. To obtain the average short-time response we use the short-time expansion (41) and calculate the average over an infinite time range $(0, mT)$, $m \to \infty$. The resulting leading term of the dc response $\sim(\alpha T)^0$ becomes zero always, as $m \to \infty$, in contrast to regimes PD [typical propagation in order $\sim(\alpha T)^0$] and long-range-overlap CD [typically weak delocalization due to distant "resonant" neighbors $\sim(\alpha T)^0$], but similar to the tight-binding CD case.

The nonvanishing incommensurate dynamic direct current appears in next order $\sim\alpha T$, like it was in the CD TBA regime:

$$\langle\langle\mathbf{v}\rangle\rangle = \frac{\alpha N_e}{\hbar T}\sum_{\mathbf{n}} \mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}}\int d\mathbf{k}f(\mathbf{k})\cos(\mathbf{kn}) \times\left[\sin^{-1}\left(\frac{1}{2}\boldsymbol{\omega}_{\mathbf{0}}T\mathbf{n}\right)\int_0^T dt\int_0^T dt'\cos\left(\tilde{\mathbf{A}}(t)\mathbf{n}-\tilde{\mathbf{A}}(t')\mathbf{n} - \frac{1}{2}\boldsymbol{\omega}_{\mathbf{0}}T\mathbf{n}\right) - 2\int_0^T dt\int_0^t dt'\sin[\tilde{\mathbf{A}}(t)\mathbf{n}-\tilde{\mathbf{A}}(t')\mathbf{n}]\right]. \tag{42}$$

Under localization conditions (11) and (15) the first term in figure brackets of Eq. (42) vanishes, but the second one does not, in similarity with the CD TBA direct current (32). Shift antisymmetry (13) is incompatible with the incommensurability condition.

The leading term of the short-time diffusion coefficient $\sim(\alpha T)^0$ vanishes exactly. Thus, in the ID regime the electrons are more localized (overall localization in any field, satisfying incommensurability condition) than in PD and CD with long-range overlap, when the electrons typically drift in order $\sim(\alpha T)^0$. However, the ID behavior is qualitatively similar to overall localization in CD TBA. Within the relaxation-free formulation Eq. (3) the mean-square displacement has no components, growing with time (both in short-time and long-time limits).

In the incommensurate kinetic regime (IK) for low scattering $\alpha T \ll 1$ the main term of the ac velocity in the long-time limit $\alpha t \gg 1$ is of the order αT:

$$\langle\mathbf{v}(t=mT+\Delta t)\rangle = -\frac{\alpha N_e}{\hbar}\sum_{\mathbf{n}} \mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}}\int d\mathbf{k}f(\mathbf{k})\cos(\mathbf{kn}) \times\left\{2\int_0^{\Delta t} dt'\sin[\tilde{\mathbf{A}}(\Delta t)\mathbf{n}-\tilde{\mathbf{A}}(t')\mathbf{n}] - \int_0^T dt'\left[\sin[\tilde{\mathbf{A}}(\Delta t)\mathbf{n}-\tilde{\mathbf{A}}(t')\mathbf{n}] + \cos[\tilde{\mathbf{A}}(\Delta t)\mathbf{n} - \tilde{\mathbf{A}}(t')\mathbf{n}]\cot\left(\frac{1}{2}\boldsymbol{\omega}_{\mathbf{0}}T\mathbf{n}\right)\right]\right\}, \tag{43}$$

which is smaller than the corresponding typical responses $\sim(\alpha T)^0$ in PK and long-range CK regimes, and qualitatively similar to CK TBA Eq. (38). Obviously, the absence of resonance between the field and the electron evolution in the band decreases the oscillations.

In contrast to the incommensurate dynamic case above, the kinetic long-time velocity (43) is time periodic with period T and reveals no dependence on m. The external field, together with relaxation, imposes its periodicity on electron evolution, though coherent effects are still there. Under localization conditions (11) and (15) the second integral in figure brackets of the ac response, Eq. (43), vanishes, but the first one does not. Thus the IK ac response does not become zero under exceptional localization, in contrast to the corresponding results for regimes PK (19) and long-range CK (33), and in analogy to CK TBA, Eq. (38).

The incommensurate kinetic direct current is of the same order $\sim\alpha T$, as the ID dc response, and as the dc response in regimes PK and CK:

$$\langle\langle\mathbf{v}\rangle\rangle = -\frac{\alpha N_e}{\hbar T}\sum_{\mathbf{n}} \mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}}\int d\mathbf{k}f(\mathbf{k})\cos(\mathbf{kn}) \times\left[2\int_0^T dt\int_0^t dt'\sin[\tilde{\mathbf{A}}(t)\mathbf{n}-\tilde{\mathbf{A}}(t')\mathbf{n}] - \cot\left(\frac{1}{2}\boldsymbol{\omega}_{\mathbf{0}}T\mathbf{n}\right)\int_0^T dt\int_0^T dt'\cos[\tilde{\mathbf{A}}(t)\mathbf{n}-\tilde{\mathbf{A}}(t')\mathbf{n}]\right]. \tag{44}$$

Under conditions (11) and (15) the second integral in the expression of the direct current Eq. (44) vanishes, while the first one does not. Thus the IK direct current, as well as the ac response, is qualitatively not sensitive to exceptional localization.

The leading term $\sim(\alpha T)^0$ of the long-time IK diffusion coefficient vanishes exactly in any field. Nonvanishing is the next term $\sim\alpha T$:

$$D = \frac{\alpha N_e}{2d\hbar^2 T}\sum_{\mathbf{n}} \mathbf{n}H^{(0)}_{\mathbf{0},\mathbf{n}} \times\left[T\sin^{-2}\left(\frac{1}{2}\boldsymbol{\omega}_{\mathbf{0}}T\mathbf{n}\right)\int_0^T dt\int_0^T dt'\cos[\tilde{\mathbf{A}}(t)\mathbf{n}-\tilde{\mathbf{A}}(t')\mathbf{n}] - 2\cot\left(\frac{1}{2}\boldsymbol{\omega}_{\mathbf{0}}T\mathbf{n}\right)\int_0^T dt\int_0^T dt'(t-t')\sin[\tilde{\mathbf{A}}(t)\mathbf{n}-\tilde{\mathbf{A}}(t')\mathbf{n}] - 4\int_0^T dt\int_0^t dt'(t-t')\cos[\tilde{\mathbf{A}}(t)\mathbf{n}-\tilde{\mathbf{A}}(t')\mathbf{n}]\right], \tag{45}$$

which is much smaller than the corresponding result in regimes PK $\sim(\alpha T)^{-1}$, and long-range CK $\sim(\alpha T)^{-1}$, but is qualitatively analogous to CK TBA $\sim\alpha T$. Under localization conditions (11) and (15) the first two terms in figure brackets of Eq. (45) vanish, but the last one does not. Thus, Eqs. (43)–(45) suggest that there is no rigorous exceptional localization in regime IK—contrary to regimes PK, Eq. (20), and long-range CK, Eq. (35), but in analogy to CK TBA, Eq. (39)—at least in the approximation of uniform band filling, Eq. (6).

In short, the incommensurate regime typically exhibits stronger overall localization, on a qualitative level similar to the commensurate tight-binding case, with no rigorous exceptional localization above this background.

Apart from "pure" regimes, "mixed" cases are possible, when along different axes different types of evolution take place.

D. Examples

Below we provide some particular examples, for simplicity in one dimension, which illustrate the mentioned regimes.

(1) First we consider the ac harmonic field with phase shift φ: $E(t)=E\sin(\omega t+\varphi)$, which always belongs to the periodic case. In the low-scattering $\alpha T \ll 1$, short-time $\alpha t \ll 1$ dynamic regime (PD) the electron is typically delocalized (cf. Ref. 19):

$$\langle\langle v\rangle\rangle = \frac{2}{\hbar}\int dk \frac{\rho_{k,k}(0)}{V_{BZ}}\sum_{n>0} nH_{0,n}^{(0)}\cos(kn)J_0(\varepsilon n)\sin(\varepsilon n\cos\varphi),$$

$$\varepsilon = \frac{eE}{\hbar\omega}. \tag{46}$$

Here J_0 is the Bessel function. The direct current vanishes for $\varphi=\pi/2$ (cosine field) and for $\varepsilon=\pi\nu/\cos\varphi$, where ν is an arbitrary integer. However, typically the average velocity (46) is nonzero, when Eqs. (11) and (15) are not fulfilled—due to the initial-field-value effect in the absence of relaxation.[29] Exceptional localization is possible only in the tight-binding approximation (TBA) under the condition $J_0(\varepsilon)=0$.[19]

In the low-scattering $\alpha T \ll 1$, long-time $\alpha t \gg 1$ periodic kinetic regime (PK) the leading term $\sim(\alpha T)^0$ of the ac response is typically nonzero:

$$\langle v(t)\rangle = -\frac{2N_e}{\hbar}\int dk\, f(k)\sum_{n>0} nH_{0,n}^{(0)}\cos(kn)J_0(\varepsilon n) \times\sin[\varepsilon n\cos(\omega t+\varphi)]. \tag{47}$$

The dc response is identically zero—both the average of Eq. (47) and the typically nonvanishing next term (21). There is no constant kinetic drift in the absence of dc field or in the absence of left-right symmetry violation (like in a bichromatic field[29]). The kinetic diffusion coefficient is

$$D = \frac{2N_e}{\alpha d\hbar^2}\sum_{n>0}[nH_{0,n}^{(0)}J_0(\varepsilon n)]^2. \tag{48}$$

The long-time diffusion coefficient (48) is independent of phase φ; averaged kinetic (in contrast to dynamic) quantities are time invariant. Exceptional localization is possible within TBA only,[20] $J_0(\varepsilon)=0$. Then the ac velocity (47) vanishes in the main term $\sim(\alpha T)^0$, along with the diffusion coefficient (48) $\sim(\alpha T)^{-1}$. Under localization the nonvanishing term of the diffusion coefficient, Eq. (24), summed over the band filling k, is positive and independent of φ as well.

(2) Next let us consider the alternating step-wise rectangular field with a dc component $E_0\pm E$ and period T.[19] First we address the periodic regime $\omega_0 T=2\pi\mu$. In the low-scattering $\alpha T\ll 1$, short-time $\alpha t\ll 1$ periodic dynamic case the average velocity is

$$\langle\langle v\rangle\rangle = -\frac{4\Delta\omega}{\hbar T(\omega_0^2-\Delta\omega^2)}\int dk\frac{\rho_{k,k}(0)}{V_{BZ}}\sum_n H_{0,n}^{(0)} \times\cos(kn)\left[1-(-1)^{\mu n}\cos\left(\frac{\Delta\omega Tn}{2}\right)\right], \tag{49}$$

where $\Delta\omega=eE\hbar^{-1}$. Exceptional localization conditions (11) and (15) with the account of all sites (including distant neighbors) are fulfilled, if the field parameters satisfy either one of the following conditions for η, where $\Delta\omega T=2\pi\eta$:

$$\text{for } \mu-\text{even}, \eta-\text{any even integer, except } \eta=\mu, \text{or}$$

$$\text{for } \mu-\text{odd}, \eta-\text{any odd integer, except } \eta=\mu. \tag{50}$$

For $\eta=\mu$ the direct current (49) vanishes as well, but the diffusion coefficient does not.

In the periodic kinetic long-time low-scattering regime $(\alpha t\gg 1, \alpha T\ll 1)$ the leading term $\sim(\alpha T)^{-1}$ of the diffusion coefficient is

$$D = \frac{16N_e\Delta\omega^2}{\alpha d\hbar^2T^2(\omega_0^2-\Delta\omega^2)^2}\sum_n (H_{0,n}^{(0)})^2 \times\left[1-(-1)^{\mu n}\cos\left(\frac{\Delta\omega Tn}{2}\right)\right]. \tag{51}$$

Under exceptional localization, Eq. (50), it vanishes entirely. In contrast to that, in the leading term of the average velocity $\sim\alpha T$,

$$\langle\langle v\rangle\rangle = \frac{4\pi\alpha\mu N_e}{\hbar T(\omega_0^2-\Delta\omega^2)}\int dkf(k)\sum_n H_{0,n}^{(0)}\cos(kn) \times\left\{1-\left[1-(-1)^{\mu n}\cos\left(\frac{\Delta\omega Tn}{2}\right)\right] \times\frac{32\Delta\omega^2}{(\omega_0^2-\Delta\omega^2)^2n^2T^2}\right\}, \tag{52}$$

under the condition (50) only the second term in figure brackets vanishes. The dc response (52), however, does become zero with the additional shift-antisymmetry (13), which in this case is equivalent to the requirement $\mu=0$ (absence of the dc component E_0 in the applied field).

Next we consider the commensurate regime, $\omega_0\mu' T=2\pi\mu$. The dynamic direct current resembles Eq. (49), with the difference that it is formed by "resonant" neighbors (28) only [note also the absence of μ' in the exponent of (-1)]:

$$\langle\langle v\rangle\rangle = -\frac{4\Delta\omega}{\hbar T(\omega_0^2-\Delta\omega^2)}\int dk\frac{\rho_{k,k}(0)}{V_{BZ}}\sum_\nu H_{0,\mu'\nu}^{(0)}\cos(k\mu'\nu) \times\left[1-(-1)^{\mu\nu}\cos\left(\frac{\Delta\omega T\mu'\nu}{2}\right)\right]. \tag{53}$$

Exceptional localization with the account of distant neighbors takes place under the condition (50) with $\Delta\omega\mu' T = 2\pi\eta$.

The commensurate kinetic ($\alpha t \gg 1, \alpha T \ll 1$) diffusion coefficient in the leading term $\sim(\alpha T)^{-1}$

$$D = \frac{16 N_e \Delta\omega^2}{\alpha d \hbar^2 T^2 (\omega_0^2 - \Delta\omega^2)^2} \sum_\nu (H^{(0)}_{0,\mu'\nu})^2 \times \left[1 - (-1)^{\mu\nu} \cos\left(\frac{\Delta\omega T \mu' \nu}{2} \right) \right], \qquad (54)$$

is also formed by "resonant" neighbors only; $N = \mu'\nu$, ν are all positive integers. It vanishes under exceptional localization Eq. (50) with $\Delta\omega\mu' T = 2\pi\eta$ as well. The CK direct current $\sim \alpha\mu' T$ under exceptional localization does not vanish.

Within the tight-binding approximation the CK diffusion coefficient is of higher order $\sim \alpha\mu' T$ and is qualitatively insensitive to exceptional localization. The same is true for the CK TBA direct current. We do not provide the corresponding rather lengthy expressions, which can be deduced from Eqs. (39) and (34). Similar conclusions are also valid for the incommensurate kinetic regime, Eqs. (44) and (45).

Thus, the alternating stepwise rectangular field provides a rare example, when exceptional localization is rigorously possible with the account of long-range overlap.

IV. CONCLUSIONS

In summary, we have considered analytically the field-induced localization and response of electrons in a one-band model conductor and the effect of relaxation in the nearly coherent regime. All the provided expansions are exact within the adopted constant-relaxation-time approximation, Eqs. (5) and (6). By discussing localization/delocalization we always imply the leading term of the corresponding expansion in slow relaxation αT. Besides, there are always higher-order weakly delocalized terms.

For low scattering at short time the pure coherent (dynamic) quantum evolution is reproduced. In the dynamic case the overall localization increases in passing from the periodic through commensurate to the incommensurate regime [typical dc response decreases from $\sim(\alpha T)^0$ to $\sim \alpha T$, diffusion coefficient from $\sim t(\alpha T)^0$ to $\sim \alpha T$, ac response retains its order $\sim(\alpha T)^0$]. Under exceptional localization the responses, in passing from the periodic through the commensurate to the incommensurate regime, retain their order. However, on the background of the increasing overall localization, the exceptional localization becomes less pronounced—from the periodic regime (dc response $\sim \alpha T$ instead of $\sim(\alpha T)^0$) to the incommensurate one (dc response $\sim \alpha T$ irrespective of exceptional localization). Localization in the dynamic regime is manifested by oscillatory evolution with no shift and no diffusive dispersion of the electron wavepacket on average, in contrast to typical coherent propagation in the periodic and in the long-range-overlap commensurate regimes. Dynamic ac response under localization, however, exists unaffected.

The low-scattering long-time kinetic regimes differ considerably from the dynamic ones. The overall localization increases in passing from the periodic through commensurate to the incommensurate regime, though in a manner different from the dynamic case [typical dc response retains its order $\sim \alpha T$, diffusion coefficient decreases from $\sim(\alpha T)^{-1}$ to $\sim \alpha T$, ac response decreases from $\sim(\alpha T)^0$ to $\sim \alpha T$]. Under exceptional localization the kinetic responses typically retain their order in passing from the periodic through commensurate to the incommensurate regime (except for the additional reduction of the dc current by one order in αT with the additional requirement of shift-antisymmetry in the periodic case). Thus, upon the background of the increasing overall localization, the exceptional localization becomes manifested less (qualitatively vanishes) in passing from the periodic [dc response $\sim(\alpha T)^2$ instead of $\sim \alpha T$, ac response $\sim \alpha T$ instead of $\sim(\alpha T)^0$, diffusion coefficient $\sim \alpha T$ instead of $\sim(\alpha T)^{-1}$] through commensurate to the incommensurate regime (dc response $\sim \alpha T$, ac response $\sim \alpha T$ and diffusion coefficient $\sim \alpha T$—all irrespective of exceptional localization). We note that localization did not affect qualitatively the ac response in the dynamic case but does in the kinetic.

In the commensurate regime with the account of long-range-overlap between the sites, some of the responses (provided by the leading terms of the expansions of velocity and mean-square displacement) are formed not by all sites, but by "resonant" neighbors N [Eq. (28)] solely. These responses are different in the dynamic case (direct current and diffusion coefficient) and in the kinetic one (alternating current and diffusion coefficient). The reduction of the overlap radius down to the tight-binding approximation (TBA) decreases these asymptotics at least by one order in $\alpha\mu' T$. In general, the commensurate regime with long-range overlap has qualitatively much in common with the periodic one. The commensurate tight-binding regime, on the contrary, resembles the incommensurate one:

- In the periodic and long-range-commensurate dynamic regimes the exceptional localization decreases the diffusion coefficient from big (corresponding to propagation), $\sim t(\alpha T)^0$, to small, $\sim \alpha T$, and the dc response from $\sim(\alpha T)^0$ to $\sim \alpha T$, as compared to the delocalized case. The ac dynamic response remains unaffected. In the periodic and long-range-commensurate kinetic regimes the exceptional localization decreases the diffusion coefficient by two orders from big, $\sim(\alpha T)^{-1}$, to small, $\sim \alpha T$, and the ac response by one order in αT, as compared to the delocalized case. The dc kinetic response typically remains qualitatively unaffected.
- In the tight-binding-commensurate and incommensurate regimes, both dynamic and kinetic, the exceptional localization does not change the order of either the diffusion coefficient, dc or ac.

The periodic regime, as compared to the long-range commensurate one, however, has one additional aspect: in the former case the field can additionally possess shift antisymmetry, and then the PK direct current vanishes under exceptional localization, while to the latter regime that does not apply.

Thus, in general, the field-induced localization "survives" the introduction of slow relaxation, though in a modified form. Two distinct types of localization might be considered: overall localization based solely on commensurability of field frequency and Bloch frequency, and more subtle and rare exceptional localization, which requires particular values of amplitudes, frequencies, and phases of field components to satisfy two simple integral equations or some symmetry properties. Both the overall and exceptional localization are manifested differently through the ac and dc responses and diffusion in the dynamic (short-time or relaxation-free) and kinetic (long-time) regimes, as discussed above.

The stated qualitative differences between the dynamic and kinetic regimes signify that the straightforward substitution of the quantum kinetic problem (with scattering/relaxation included) by a coherent relaxation-free formulation is inadequate, except for the very short-time regime. Because of that, the long-time kinetic results cannot even reproduce the long-time coherent (or dynamic) case as a limit for $\alpha T=0$, and vice versa. However, some indirect analogies between these cases are still present—between the relaxation-free dc response and the kinetic ac response, for example.

From the point of view of mathematics, the equations and their solutions in the dynamic and kinetic regimes differ considerably. However, the qualitative aspects of field-induced localization/delocalization in both regimes have much in common, while the conditions for localization exactly coincide for slow relaxation.

The mathematical reasons for the cited differences between the dynamic and the kinetic regimes are the exponential damping of the kinetic velocity and the additional integral over time in Eqs. (5) and (6), which accounts for the implicit averaging over scattering events. The reason for the similarity lies, however, in the same integral over time period. In fact, similar averages appear in the calculations of average dynamic velocity (dc response at short-time) and momentary values of kinetic velocity (ac response at long time), the latter averaged over scattering during the period of the field. This is exactly the reason for the interrelation of these two effects (dynamic and kinetic), which otherwise could be entirely different. Another necessary ingredient for that similarity is the slow-relaxation limit. Not only did it allow the simple complete analytical study of the problem. It also enabled the expansion of damping exponentials, which otherwise would enter the kinetic localization criteria, rendering these criteria, as well as the resulting formulas different from the dynamic (or pure coherent) case.

The condition for the observation of either dynamic or kinetic results is determined by the time of observation: at short time, comparable to interscattering time the first one takes place, while for long times, exceeding the inverse relaxation rate, the second one should be observed.

Obviously, the response in the periodic, commensurate, and incommensurate regimes is considerably different. In reality, however, due to the fluctuations of the field frequency the response should be produced by some weighted average of these routes with frequencies in the (close) vicinity. Then the typically delocalized regimes (periodic in the first place) should provide the major escape channel (delocalization). In such a case the effect of exceptional localization can be used to control response and electron localization by the field parameters.

The situation will be different for such a choice of basic frequency, when there are no periodic-regime frequencies in the vicinity. Even more to it, if the nearest-neighbor overlap (or TBA) applies, then there will be no delocalized contribution from the commensurate regime as well. In such a case the response will be governed entirely by the incommensurate results, with little sensitivity of the response to exceptional localization (neither parameter vanishes).

Another point is that the delocalized regimes are pretty rare, and in fact have an infinitesimal weight as compared to the incommensurate one. If the signal is assumed to scan some frequency range, it may be considered to populate the neighboring frequencies equally, and that greatly diminishes the weight of the periodic and commensurate contributions, depending on the rate of interfrequency transitions. That also works for the prevailing influence of the incommensurate regime.

Yet another point is that the frequency-mixing processes discussed above will take place in the intermediate time range, and thus should bear some more resemblance to the dynamic short-time case. That would mean the faster spreading due to propagation in the delocalized regimes and their increased contribution.

The situation will be different—much more simple and transparent—in the case when the sample is electrostatically shielded. Then the dc component is exactly zero and the pure periodic regime of evolution takes place, with only the ac input component fluctuating. Then the provided results for the periodic regime can be applied directly, with all the varied opportunities for coherent control of localization and response.

We believe the stated mechanisms can be used for the construction of sensors and novel devices for information processing, based on superlattices in the low-temperature nearly coherent regime. The necessary conditions might be achievable, for example, on the outer surface of spaceships in dark.

We will expand this study in future publications by introducing frequency fluctuations and different diagonal and nondiagonal relaxation rates, and by lifting the limitation to high temperatures (or uniform equilibrium band filling) for the long-time kinetic diffusion coefficient. The latter approximation is not only logically unsatisfactory for the nearly coherent regime, but is expected to affect the corresponding long-time asymptotics.

The localization/delocalization properties (dc response, ac response, diffusion coefficient) for all the regimes at low scattering $\alpha T \ll 1$ are summarized in Table I.

Due to the fast decrease of overlap integrals $H^{(0)}_{\mathbf{0},\mathbf{n}}$ with $\mathbf{n}$, the effect of exceptional localization in superlattices might be qualitatively well observable in many cases when it holds for nearest-neighbor sites only.

Finally let us address the question of observability of the considered effects in experiments on GaAs/GaAlAs semiconductor superlattices. The relaxation time α^{-1} we assume $\sim$1 ps (Ref. 10 at $T=10$ K, Ref. 26). To meet the slow-scattering condition $\alpha T \ll 1$, the laser frequency should be high enough, $\nu=\omega/2\pi=1/T \gg \alpha=1$ THz, in our case. On the other hand, for the one-band approximation to be valid, the laser photon energy should be smaller than the bandgap Δ,

TABLE I. Systematization of regimes of intraband evolution for electrons in a one-band model in a time-periodic electric field. By * we mark formulas with modifications, cited in the text. By $^\diamond$ we mark equations that require additional shift-antisymmetry for exceptional localization. By † we mark quantities qualitatively insensitive to exceptional localization and by ‡—totally insensitive ones.

Dynamic (short time, $\alpha t \ll 1$)	Kinetic (long time, $\alpha t \gg 1$)
Periodic, $\omega_0/\omega=\mu$ (typically delocalization)	
dc not small $\sim(\alpha T)^0$, Eq. (9)	dc small $\sim \alpha T$, Eq. (21)
ac always not small $\sim(\alpha T)^0$, Eqs. (2) and (8) *	ac not small $\sim(\alpha T)^0$, Eq. (19)
Diffusion coefficient big $D\sim t(\alpha T)^0$, Eq. (10)	Diffusion coefficient big $\sim(\alpha T)^{-1}$, Eq. (20)
Exceptional localization	
dc small $\sim\alpha T$, Eq. (18)	dc smaller $\sim(\alpha T)^2$ $^\diamond$, Eq. (22)
ac always not small $\sim(\alpha T)^0$, Eqs. (2) and (8) *‡	ac small $\sim\alpha T$, Eq. (23)
Diffusion coefficient small $\sim\alpha T$, Eq. (17)	Diffusion coefficient small $\sim\alpha T$, Eq. (24)
Commensurate, $\omega_0/\omega=\mu/\mu'$ (typically weak delocalization, TBA—localization)	
dc not small $\sim(\alpha\mu' T)^0$, but small in $H_{\mathbf{0},\mathbf{N}}$, Eq. (29) [$\sim\alpha\mu' T$ within TBA, Eq. (32)]	dc small $\sim\alpha\mu' T$, Eq. (34) [$\sim\alpha\mu' T$ within TBA, Eq. (34) *]
ac always not small $\sim(\alpha\mu' T)^0$, Eq. (26) * [same within TBA]	ac not small $\sim(\alpha\mu' T)^0$, but small $\sim H_{\mathbf{0},\mathbf{N}}$ Eq. (33) [$\sim\alpha\mu' T$ within TBA, Eq. (38)]
Diffusion coefficient big $D\sim t(\alpha\mu' T)^0$, but small in $H_{\mathbf{0},\mathbf{N}}$ Eq. (30) [$\sim\alpha\mu' T$ within TBA]	Diffusion coefficient big $\sim(\alpha\mu' T)^{-1}$, but small in $H_{\mathbf{0},\mathbf{N}}$, Eq. (35) [$\sim\alpha\mu' T$ within TBA, Eq. (39)]
Exceptional localization	
dc small $\sim\alpha\mu' T$, Eq. (31) [same order $\sim\alpha\mu' T$ within TBA, Eq. (32) *]	dc small $\sim\alpha\mu' T$, Eq. (34) *† [$\sim\alpha\mu' T$ within TBA, Eq. (34) *†]
ac always not small $\sim(\alpha\mu' T)^0$, Eq. (26) *‡ [same within TBA ‡]	ac small $\sim\alpha\mu' T$, Eq. (37) [$\sim\alpha\mu' T$ within TBA, Eq. (38) *†]
Diffusion coefficient small $\sim\alpha\mu' T$, Eq. (17) *	Diffusion coefficient small $\sim\alpha\mu' T$, Eq. (36) [$\sim\alpha\mu' T$ within TBA, Eq. (39) *†]
Incommensurate, $\omega_0/\omega\neq\mu/\mu'$ (localization)	
dc small $\sim\alpha T$, Eq. (42)	dc small $\sim\alpha T$, Eq. (44)
ac always not small $\sim(\alpha T)^0$, Eq. (41)	ac small $\sim\alpha T$, Eq. (43)
Diffusion coefficient small $\sim\alpha T$	Diffusion coefficient small $\sim\alpha T$, Eq. (45)
Exceptional localization—all parameters qualitatively insensitive	

$\nu \ll \Delta/h$. For the bandgap to be bigger, the superlattice GaAs wells should be narrow enough and the spacer GaAlAs layers should be thick enough. For example, in Ref. 50 they are 5.5 and 32.5 nm, respectively, so that the bandgap is 152 meV. For the interband transitions to be negligible, we assume the upper limiting frequency ~10 THz, corresponding to photon energy ~40 meV. Thus there should be a rather narrow range for the laser frequency inbetween 10^{12} and 10^{13} Hz, where our results for the coherent control of induced localization through intraband evolution should be valid. The manufacture of bigger-bandgap superlattices with longer relaxation time (lower-temperature measurements) should increase the range of its applicability.

In optical superlattices the relaxation processes are much slower, so that the conditions for the observation of the considered effects are more favorable. For example, the apparent deviation from the constant velocity at short time in the inset of Fig. 2 of Ref. 44 obviously is due to the stated difference between the short-time and long-time response.

The study of the dynamic (short-time) regime in semiconductor superlattices requires subpicosecond measurements. For all-optical response it is accessible, though such measurements of transient currents pose more problems.[2,5,6,13,45] In contrast to that, optical lattices require the time scale of only ~1–10 μs (Refs. 43 and 44), which should be quite accessible. In any case, the provided theoretical consideration of the short-time dynamic regime we hope will serve for the clarification of the nontrivial time evolution of the response and coherent control in superlattices.

We believe that the field-induced localization effects, exercised through intraband evolution, can be observed at low temperatures in the electromagnetic response of high-quality semiconductor superlattices, quantum wires, and dot arrays in the mid-THz range, along with optical lattices. We hope that the mechanisms discussed here could serve for the construction of novel sensors and ultrafast information-processing devices.

ACKNOWLEDGMENTS

One of the authors (K.P.) acknowledges financial support from NSERC (Canada), DAAD (Germany), and RFBR (Russia). K.P. also acknowledges helpful discussions with Professor A.A. Ovchinnikov and Dr. D.V. Dmitriev.

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