Discrete sliding symmetries, dualities, and self-dualities of quantum orbital compass models and $p+ip$ superconducting arrays

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We study the spin-1/2 two- and three-dimensional orbital compass models relevant to the problem of orbital ordering in transition metal oxides. We show that these systems display self-dualities and gauge-like discrete sliding symmetries. An important and surprising consequence is that these models are dual to (seemingly unrelated) recently studied models of $p+ip$ superconducting arrays. The duality transformations are constructed by means of a path-integral representation in discretized imaginary time and considering its $\mathbb{Z}_2$ spatial reflection symmetries and space-time discrete rotations, we obtain, in a transparent unified geometrical way, several dualities. We also introduce an alternative construction of the duality transformations using operator identities. We discuss the consequences of these dualities for the order parameters and phase transitions of the orbital compass model and its generalizations, and apply these ideas to a number of related systems.

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I. INTRODUCTION

Orbital compass models offer a simple and qualitative description of the ordering of orbital degrees of freedom in a number of complex oxides such as the titanates.[1] The degrees of freedom of these models describe the spatial orientation of the orbital degrees of freedom. Jahn-Teller effects lead to anisotropic orbital compasslike interactions among the orbitals. When combined with the spin degrees of freedom, to which the orbitals are coupled via superexchange in these systems[2] as well as by spin-orbit interactions, they lead to complex phase diagrams with phases that involve both spin and orbital ordering (and disorder) to various degrees. Indeed, these systems offer an interesting laboratory for the study of interesting anisotropic quantum nematic phases, with and without spin order, and are a simple example of electronic liquid crystal phases.[3,4]

Orbital compass models also exhibit unusual and so far not well-studied symmetries which play a big role in their physical properties. In the current paper, we elucidate the discrete "sliding" gauge-like symmetries present in the two- and three-dimensional orbital compass models. In two dimensions, these symmetries involve flipping the orbital degrees of freedom simultaneously along a single row or column of the lattice. These discrete symmetry transformations stand in-between the global symmetries familiar from spin systems and the local symmetries of gauge theories. Although these are not truly gauge symmetries in the sense that they affect the boundary conditions, they are softer than the familiar global symmetries. In fact, for reasons discussed elsewhere[5] these discrete sliding symmetries are alike gauge symmetries in that they cannot be spontaneously broken. A direct consequence of the existence of these discrete sliding symmetries is that their natural order parameters are *nematic*, which are invariant under discrete sliding symmetries. Here we give an explicit construction of the nematic order parameters and potential physical consequences are discussed.

Gauge-like symmetries appear in a number of condensed matter systems. Exact local gauge symmetries are pervasive in the quantum hydrodynamics of incompressible and compressible quantum Hall systems, as a direct expression of their quantum hydrodynamics.[6] Similarly, local gauge symmetries appear naturally in the context of strongly correlated systems such as the *t-J* model, quantum dimer models, and other systems.[7] Of particular interest for the problems discussed in this paper are the *sliding phases* of arrays of Luttinger liquids,[8] quantum Hall smectic (stripe) phases,[9] DNA intercalates in lipid bilayers,[10] as well as in some ring exchange models of frustrated antiferromagnets.[11] The discrete sliding symmetries we discuss here are a discrete, $\mathbb{Z}_2$, version of the *continuous* sliding symmetries of the systems mentioned above. The existence of sliding symmetries has profound effects on their quantum phase transitions, whose behavior only begun to be understood quite recently[12] and still remains largely unexplored.

Among others, discrete sliding symmetries are present in spin,[13–15] orbital,[16–20] and superconducting array systems.[21,22] We further demonstrate that the planar orbital compass model[17,20] and the Xu-Moore model[21,22] of two dimensional $p+ip$ superconducting arrays are, in fact, one and the same system, related by a simple duality transformation. Viewed in that light, the discrete sliding symmetries which the Hamiltonians describing superconducting arrays display are natural. By applying our dualities, we find self-dualities for the three-dimensional orbital compass model and several other systems. These dualities do not rely on operator representations[23,24] nor on standard combinatorial loop/bond counting arguments or summation formulas.[25] Rather, the dualities that we report here appear as simple geometrical reflections between various spin and spatial axis. The dualities investigated in this paper map such trivial geometrical reflection self-dualities in one model onto far less trivial weak-strong coupling self-dualities in other systems. In a formal setting, our dualities correspond to different space-time cuts of a single classical action. Choosing a certain time and spin quantization axis, we find one spatial system while choosing the time axis to lie along another direction in space-time leads to a seemingly very different (yet dual) spatial model. Our purely geometric dualities further extend and complement, from a rather general perspective, the dualities generally derived via techniques such as those in, e.g., Ref. 26.

The plan to the paper is as follows. In Sec. II, we introduce the planar orbital compass model in both its isotropic and anisotropic incarnations. We identify the many gauge-like and single reflection symmetries of this model (the latter reflection symmetry will, as we will later find out, play the role of self-duality). In the aftermath, we construct order parameters invariant under these symmetries.

In Sec. III, we discuss another two dimensional *XY* system that possesses one dimensional gauge-like symmetries. This system has been argued to embody the quintessential physics of a square lattice array of $p+ip$ superconducting grains. As we will show, this model is identical to the planar orbital compass model.

In Sec. IV, we discuss the three-dimensional orbital compass system. This model has been considered to embody the prototypical features of orbital system Hamiltonian and might be directly relevant to the so-called "t_{2g}" systems (such as the vanadates and manganates) in particular. We identify gauge-like symmetries in this system. As in the planar case, we find nematic orders invariant the gauge-like symmetries. This in turn suggests that orbital systems might possess observable nematic orders.

In Sec. V, we employ simple geometrical reflections to derive dualities for extended systems (now residing in three dimensions). The dualities of Xu and Moore[21,22] (derived by Kramers-Wannier loop counting) form a subset of the derived dualities. The central actor in our scheme is a geometrical inversion operator which allows us to set the imaginary time axis along different external space-time directions with similar ideas for choosing the internal spin quantization axis. These operations generate, in turn, many different dual models. With this geometrical understanding of the observed duality in hand, we return to the self-dual point of Refs. 21,22 and make comparisons to other systems.

In Sec. VI, we use an operator representation of the duality transformation to rederive the dualities for the orbital compass model which we obtained via geometrical reflections in the previous sections. Section VII is devoted to the conclusions. In the Appendix we discuss the self-duality of "around the cube" models in transverse fields.

II. QUANTUM PLANAR ORBITAL COMPASS MODELS: SYMMETRIES

We start with the planar compass model. The compass models often serve as the simplest caricatures for the physics of 3*d* orbital systems wherein Jahn-Teller interactions as well as magnetic exchange processes are dictated by the orientation of the orbitals at the various lattice sites. In the orbital compass models, the spin variables code for the orbital states. As orbitals extend in real space, all orbital dependent interactions are highly anisotropic-these interactions link the external lattice directions with the internal "spin" (i.e., orbital) orientations. We refer the interested reader to Ref. 27 where the physics of orbital systems and the orbital only models that we investigate is explored in depth.

The planar compass model is defined on the square lattice where at each site $\boldsymbol{r}$ there is a $S=1/2$ operator denoted by $\mathbf{S_r}=(\hbar/2)\sigma_{\mathbf{r}}$. The isotropic planar orbital model Hamiltonian

$$H_{\text{iso}} = -J\sum_{\boldsymbol{r}} (\sigma^x_{\boldsymbol{r}}\sigma^x_{\boldsymbol{r}+\hat{e}_x} + \sigma^z_{\boldsymbol{r}}\sigma^z_{\boldsymbol{r}+e_z}), \tag{1}$$

where the nature of the interaction allows us to set $J>0$.[28]

Unlike the more conventional nearest neighbor spin Hamiltonians which posses a continous global rotational symmetry, the compass model Hamiltonian is not invariant under arbitrary global rotations of all spins. [Physically, the lack of this symmetry is the direct consequence of the coupling between the internal polarization directions (orbital states) and the external lattice directions (as much unlike spins, the orbitals extend in real space).[27]] Instead, this model possesses many nontrivial symmetries corresponding to specific quantized angles of rotation of all spins on given rows/columns and a single additional rather trivial reflection symmetry (which upon mapping will enable us to find a nontrivial weak to strong coupling self-duality in another model). As a consequence of these symmetries, this model harbors an infinite degeneracy of all states and of its ground states in particular. Let us consider the system with open boundary conditions on an $L\times L$ square lattice and let us define an operator on an arbitrary horizontal line (of ordinate z) $\hat{O}_z=\Pi_{x=-L}^{L}\sigma^z_x$ and an operator on an arbitrary vertical line (of horizontal intercept x) $\hat{O}_x=\Pi_{z=-L}^{L}\sigma^x_z$. It is readily verified that for all sites $\vec{r}$ whose z component is $r_z=z$, the product $\hat{O}_z^{-1}\sigma^x_{\vec{r}}\hat{O}_z=-\sigma^x_{\vec{r}}$ while $\hat{O}_z^{-1}\sigma^z_{\vec{r}}\hat{O}_z=\sigma^z_{\vec{r}}$. Similarly, $\hat{O}_x$ inverts the z component of all spins on a vertical line, while leaving σ_x untouched. In the case of symmetric exchange constants for bonds along the x and z axis, as in the compass model under consideration here, (where both exchange constants are equal to J), we further have a single additional $\mathbb{Z}_2$ reflection symmetry ($\sigma_x\to\sigma_z,\sigma_z\to\sigma_x$)—a rotation by π about the symmetric line (the 45 deg line in the xz plane), i.e., $\hat{O}_{\text{Reflection}}=\Pi_{\vec{r}}\exp[i(\pi\sqrt{2}/4)(\sigma^x_{\vec{r}}+\sigma^z_{\vec{r}})]$. For each of these operations, $\hat{O}_\alpha^{-1}H\hat{O}_\alpha=H$. This symmetry, $\hat{O}_{\text{Reflection}}$ is a manifestation of self-duality present in the model—we will explain the origin of this comment later. Putting all of the pieces together, as a consequence of these symmetries, each state is, at least, $\mathcal{O}(2^L)$ degenerate. Formally, these symmetries constitute a gauge-like symmetry which is intermediate between a local gauge symmetry (whose volume scales as the system area) and a global gauge symmetry (whose logarithmic volume is pointlike). These intermediate gauge symmetries suggest that nontrivialities may occur. As it turns out, such large discrete symmetries do not prohibit ordering in classical variants of this model albeit complicating matters significantly.[17,19,20] This ordering tendency may be expected to become fortified by quantum fluctuations ("quantum order out of disorder").[29]

As an aside, we note that the global nematic symmetries (the global rotation of all spins by $n\pi/2$ with $n=0,1,2,3$) are not independent symmetries on top of the gauge-like symmetries discussed above. Rather, there is only one ($\sigma_x \to \sigma_z, \sigma_z \to \sigma_x$)—a rotation by π about the symmetric line (the 45 deg line in the xz plane) additional symmetry supplanting the gauge-like symmetries. To see this, first note that the global inversion operation, $\vec{\sigma} \to -\vec{\sigma}$, is a composite of the row/column inversion symmetries: An inversion of σ_x on all rows followed by an inversion of σ_z on all columns leads to the global inversion operation. Next, note that by fusing the global inversion symmetry with the global $\mathbb{Z}_2$ reflection symmetry, we may produce the four global nematic symmetry operations (rotations by $n\pi/2$). Thus, unlike what is suggested by Ref. 20, the global nematic symmetries do not supplant the gauge-like symmetries and no less important, the quantum system possesses the above reported gauge-like symmetries embodied by the operators $\hat{O}_{x,z}$.

The classical (large S) ground state sector of the orbital compass model further possesses an additional continuous $[U(1)]$ symmetry not captured by the discrete $(\mathbb{Z}_2)^{2L+1}$ symmetries $[2^L$ of these associated with horizontal, 2^L associated with vertical discrete spin flip symmetries, and one $\mathbb{Z}_2$ symmetry being the $(\sigma_x \to \sigma_z, \sigma_z \to \sigma_x)$ reflection symmetry$]$ detailed above. This continuous symmetry is made evident by noting that any constant spin field, $\sigma_{\mathbf{r}}=\sigma$, is a ground state. First, we note that $\Sigma_{\alpha=x,z}[\sigma_{\mathbf{r}}^{(\alpha)}]^2$ is constant. Thus, up to an irrelevant constant, the general Hamiltonian of Eq. (1) is

$$H_{\rm iso}^{\rm cl} = \frac{J}{2}\sum_{\mathbf{r},\alpha} [\sigma_{\mathbf{r}}^{(\alpha)} - \sigma_{\mathbf{r}+\hat{e}_\alpha}^{(\alpha)}]^2, \tag{2}$$

which is obviously minimized when the spin field is constant. We emphasize that the continuous symmetries which underscore these ground states are just symmetries of the states and *not* of the Hamiltonian itself. In common parlance, these are *emergent symmetries* specific only to the ground state sector. Therefore, at least in the orbital-only models, we are not in a setting where a Mermin-Wagner argument can be applied.

With an eye toward things to come, let us now introduce and examine the anisotropic planar compass model

$$H = -J_x\sum_{\mathbf{r}} \sigma_{\mathbf{r}}^x\sigma_{\mathbf{r}+\hat{\mathbf{e}}_\mathbf{x}}^x - J_z\sum_{\mathbf{r}} \sigma_{\mathbf{r}}^z\sigma_{\mathbf{r}+\hat{\mathbf{e}}_\mathbf{z}}^z. \tag{3}$$

It is readily verified that this more general Hamiltonian harbors all of the one-dimensional gauge-like symmetries encapsulated by $\hat{O}_{x,z}$. The only symmetry which does not persist for arbitrary J_x, J_z is the reflection symmetry. (Insofar as its underlying physics is concerned, this anisotropic compass model emulates Jahn-Teller distortions on a strained lattice.[27])

The two terms in the anisotropic compass model of Eq. (3), trivially compete. The first term favors ordering of the spins parallel to the x axis while the second favors an ordering of the spins parallel to the z axis. Order becomes more inhibited when the competition between the two terms becomes the strongest ($J_x=\pm J_z$) as it indeed occurs within the compass model of Eq. (1). We note that the gauge like symmetries (encapsulated by the column/row $\hat{O}_{x,z}$ generators) preserve the Hamiltonian also for arbitrary $|J_x| \neq |J_z|$. A natural (smecticlike) order parameter in the orbital compass model monitors the tendency of the spins to order along their preferred directions

$$m = \langle \sigma_{\mathbf{r}}^x\sigma_{\mathbf{r}+\hat{e}_x}^x - \sigma_{\mathbf{r}}^z\sigma_{\mathbf{r}+\hat{e}_z}^z\rangle. \tag{4}$$

[Just as in smectic liquid crystals, having all spins point in the $\hat{e}_\alpha$ direction or in the $(-\hat{e}_\alpha)$ direction is one and the same insofar as the above order parameter is concerned.] This nematiclike order parameter is invariant under all gauge-like symmetries.

Similar to the xy symmetric order parameter above, for the anisotropic planar orbital compass model (say $|J_x|>|J_z|$), we may consider the Ising like nematic-order parameter,

$$m_x = \langle \sigma_{\mathbf{r}}^x\sigma_{\mathbf{r}+\hat{e}_x}^x\rangle, \tag{5}$$

with a similar definition for the system with $|J_z|>|J_x|$. These order parameters are invariant under the gauge-like symmetries of the system.

The classical, large S, rendition of this model, has similar nematiclike order parameters invariant under all gauge-like symmetries.[17,19,20] Here, and in fact for all spins $S>1/2$, the order parameter can be local (not a bond order parameter involving two spins). All quantities $Q_{\alpha\beta}=S^\alpha S^\beta-(1/d)\delta_{\alpha\beta}$, with α,β internal spin indices, and with d the dimension ($d=2$ in the planar higher spin extensions of the orbital compass model) are invariant under all gauge-like symmetries. The order parameter $\langle Q_{11}\rangle$ is anticipated for $|J_x|>|J_z|$ (and $\langle Q_{22}\rangle$ for $|J_z|>|J_x|$). Similar quantities may be introduced for higher dimensional ($d>2$) generalizations of the planar compass model.

III. $p+ip$ SUPERCONDUCTING ARRAYS

A Hamiltonian describing a square lattice of $p+ip$ superconducting grains (e.g., Sr_2RuO_4) was recently suggested[21,22]

$$H = -K\sum_{\square} \sigma^z\sigma^z\sigma^z\sigma^z - h\sum_{\mathbf{r}} \sigma_{\mathbf{r}}^x. \tag{6}$$

Here, the four spin product is the product of all spins residing at the four vertices of a given plaquette $\square$ (not on its bonds as for gauge fields!). As noted by Xu and Moore,[21] the quantity

$$\hat{O}_P = \prod_{\mathbf{r}} \sigma_{\mathbf{r}}^x, \tag{7}$$

with the string product (along "P") extending over all spins in a given row ($r_z=z$) or a given column ($r_x=x$), is conserved. The discrete (gauge-like) sliding symmetry of this model is similar to that of the planar orbital compass model and we will indeed show that these two models are actually dual to each other.

The central derivation in Refs. 21 and 22 was a self-duality of the Hamiltonian in Eq. (6) via a tour de force Wannier Kramers loop counting arguments. The form of this self-duality is somewhat similar (yet still very different) to the beautiful self-dualities of Ref. 31. Similar dualties were discussed in the ring exchange systems of Ref. 11. In these models not only a relation among strong and weak coupling is given by the self-duality but the self-duality further intertwines the various terms [e.g., large h is related to small K in the self-duality of Eq. (6) and vice versa as found by Xu and Moore].

We will shortly establish that the rather complicated looking weak coupling to strong coupling self-duality of Eq. (6) derived by Refs. 21 and 22 immediately follows from a very simple purely geometric ($\mathbb{Z}_2$ reflection) self-duality of the planar orbital compass model. This self-duality may also be related (albeit in a less general fashion) to the trivial geometrical self-duality of the planar orbital compass model via the operator representations of Sec. VI. In the aftermath, the plaquette coefficient K in Eq. (6) may be related to the exchange amplitude J_x of Eq. (3) whereas the transverse magnetic field h of Eq. (6) becomes trivially related to J_z of Eq. (3).

IV. SYMMETRIES OF THE THREE-DIMENSIONAL ORBITAL COMPASS MODEL

The canonical prototype of all orbital-spin[2] and orbital-orbital interactions is the orbital compass model.[30] The model is defined on the cubic lattice where at each site $\vec{r}$ there is an $S=1/2$ operator denoted by $\vec{S}_{\vec{r}}=(\hbar/2)\vec{\sigma}_{\vec{r}}$. The orbital model Hamiltonian

$$H = J\sum_{\vec{r}} (\sigma^x_{\vec{r}}\sigma^x_{\vec{r}+\hat{e}_x} + \sigma^y_{\vec{r}}\sigma^y_{\vec{r}+\hat{e}_y} + \sigma^z_{\vec{r}}\sigma^z_{\vec{r}+e_z}). \tag{8}$$

Let us define an operator on an arbitrary xy plane P (of intercept z) $\hat{O}_{P;z}=\Pi_{\vec{r}\in P}\sigma^z_{\vec{r}}$ with similar definitions for $\hat{O}_{P;x}$ and $\hat{O}_{P;y}$. These operators may be recast as rotations by π about an axis orthogonal to the plane. For all sites $\vec{r}$ in the xy plane P whose z component is $r_z=z$, up to a multiplicative phase factor, the operator $\hat{O}_{P;z}=\exp[i(\pi/2)\sigma^z_P/\hbar]$ with $\sigma^z_P=\Sigma_{\vec{r}\in P}\sigma^z_{\vec{r}}$. The products $\hat{O}^{-1}_{P;z}\sigma^{P;x,y}_{\vec{r}}\hat{O}_{P;z}=-\sigma^{x,y}_{\vec{r}}$ while $\hat{O}^{-1}_{P;z}\sigma^z_{\vec{r}}\hat{O}_{P;z}=\sigma^z_{\vec{r}}$. Similarly, $\hat{O}_{P;x}$ inverts the y and z component of all spins on the yz plane of intercept x while leaving σ_x untouched. These "string" operators spanning the entire plane commute with the Hamiltonian, $[H,\hat{O}_{P;\alpha}]=0$. The classical orbital compass model has an exact $[\mathbb{Z}_2]^{3L^2}$ symmetry (along each chain parallel to the cubic α (x, y, or z) axis, we may reflect the α spin component, $S_\alpha\to -S_\alpha$, while keeping all other spin components unchanged, $S_{\beta\neq\alpha}\to S_{\beta\neq\alpha}$). The quantum orbital compass model has a lower exact $[\mathbb{Z}_2]^{3L}$ gauge-like symmetry (forming a subset of the larger $[\mathbb{Z}_2]^{3L^2}$ symmetry present for classical spins). As alluded to above, the gauge like $[\mathbb{Z}_2]^{3L}$ symmetries of this quantum $S=1/2$ case (as well as all representations), become evident once we rotate, with no change ensuing in the Hamiltonian, all spins in a plane orthogonal to the cubic lattice direction α by π about the internal S_α quantization axis.

As before, let us now introduce and examine the anisotropic orbital compass model

$$H = -J_x\sum_{\mathbf{r}} \sigma^x_{\mathbf{r}}\sigma^x_{\mathbf{r}+\hat{\mathbf{e}}_\mathbf{x}} - J_y\sum_{\mathbf{r}} \sigma^y_{\mathbf{r}}\sigma^y_{\mathbf{r}+\hat{\mathbf{e}}_\mathbf{y}} - J_z\sum_{\mathbf{r}} \sigma^z_{\mathbf{r}}\sigma^z_{\mathbf{r}+\hat{\mathbf{e}}_\mathbf{z}}. \tag{9}$$

(The isotropic orbital compass model corresponds to $J_{x,y,z}=-J$.) The anisotropic orbital compass model possesses all of the gauge-like symmetries of the isotropic orbital compass model (planar rotations in the quantum model and more numerous single line inversions in the classical case). Further, if at least any two of the three exchange constants $\{J_\alpha\}$ are identical the system possess a reflection symmetry.

As in the planar orbital compass model, nematiclike order parameters may be constructed for both the isotropic and anisotropic systems. Thus, we naturally predict the existence of observable nematic orbital orders in t_{2g} systems.

In what follows, we will also investigate a related system governed by the Hamiltonian

$$H = -J_x\sum_{\mathbf{r}} \sigma^x_{\mathbf{r}}\sigma^x_{\mathbf{r}+\hat{\mathbf{e}}_\mathbf{x}} - J_y\sum_{\mathbf{r}} \sigma^z_{\mathbf{r}}\sigma^z_{\mathbf{r}+\hat{\mathbf{e}}_\mathbf{y}} - J_z\sum_{\mathbf{r}} \sigma^z_{\mathbf{r}}\sigma^z_{\mathbf{r}+\hat{\mathbf{e}}_\mathbf{z}}. \tag{10}$$

Note the similarity between the XY model of Eq. (10) and the orbital compass model of Eq. (9). In the limit $J_z=0$, Eq. (10) trivially degenerates into the strained planar orbital compass model of Eq. (3).

We will construct new "plaquette models" (in which the spins reside on the lattice sites not on bonds) dual to Eq. (10) which possess a self-duality and gauge-like symmetry, naturally extending the results of Refs. 21 and 22.

V. DUALITIES AND SELF-DUALITIES FOUND BY PLANAR REFLECTIONS

We now transform the zero temperature quantum problem of Eq. (1) onto a classical problem in $(d+1)$ dimensions. To this end, we work in a basis quantized along $\sigma^z(=\pm 1)$. We now consider the basis spanned by two spins $(\sigma^z,\sigma^{z\prime})$ at the same spatial site $\mathbf{r}$ yet at two consecutive imaginary times τ and $(\tau+\Delta\tau)$. The transfer matrices corresponding to $\alpha e^{\bar{h}\sigma^x}$ (stemming, in the imaginary time formalism from a propagator $e^{-H\Delta\tau}$ such as $e^{h\sigma^x\Delta\tau}$ with $\bar{h}\equiv h\Delta\tau$) and $e^{\bar{J}\sigma^z\sigma^{z\prime}}$ (or, with space time coordinates explicitly instated, $e^{\bar{J}\sigma^z_{\mathbf{r},\tau}\sigma^z_{\mathbf{r},\tau+\Delta\tau}}$) are the same provided that $\tanh\bar{h}=e^{-2\bar{J}}$ (or equivalently $\sinh 2\bar{h}\sinh 2\bar{J}=1$) and $\alpha=(2\sinh\bar{J})^{1/2}$. Similarly, the nonvanishing eigenvalues of the transfer matrices

$$\exp[K_x\sigma^z_{i,\tau}\sigma^z_{i+1,\tau}\sigma^z_{i,\tau+\Delta\tau}\sigma^z_{i+1,\tau}] \tag{11}$$

and

$$\exp[\bar{J}_x\sigma^x_{i,\tau}\sigma^x_{i+1,\tau}] \tag{12}$$

are equivalent once $\sinh 2K_x\sinh 2\bar{J}_x=1$.

In the standard imaginary time mapping of quantum systems to classical actions, we identify $\bar{J}_\alpha=J_\alpha\Delta\tau$ with the aforementioned $\Delta\tau$ the lattice spacing along the imaginary time direction.

The generalized classical Euclidean action corresponding to Eq. (1) is

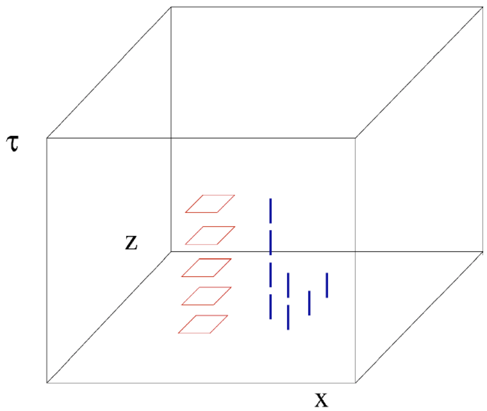

FIG. 1. The classical Euclidean action corresponding to the Hamiltonian of Eq. (6) at zero temperature in a basis quantized along the σ^z direction. The transverse field leads to bonds parallel to the imaginary time axis while the four plaquette interactions become replicated along the imaginary time axis. Taking an equal time slice of this system we find the four spin term of Eq. (6) and the on-site magnetic field term. If we interchange τ with z, we find the anisotropic planar orbital compass model of Eq. (3) in the basis quantized along the σ^x direction.

$$S = -K_x \sum_{\square \in (x\tau)\ \text{plane}} \sigma^z_{\mathbf{r},\tau}\sigma^z_{\mathbf{r},\tau+\Delta\tau}\sigma^z_{\mathbf{r}+\hat{\mathbf{e}}_x,\tau}\sigma^z_{\mathbf{r}+\hat{\mathbf{e}}_x,\tau+\Delta\tau} - (\Delta\tau)J_z\sum_{\mathbf{r}} \sigma^z_{\mathbf{r}}\sigma^z_{\mathbf{r}+e_z}. \tag{13}$$

A schematic of this action in Euclidean space time is shown in Fig. 1. If we relabel the axes and replace the spatial index x with the temporal index τ, we will immediately find the classical action corresponding to the the Hamiltonian of Eq. (6) depicting $p+ip$ superconducting grains in a square grid. This trivially suggests that the anisotropic planar orbital compass system [Eq. (3)] and the Xu-Moore Hamiltonian [Eq. (6)] are dual to each other. In Sec. VI, we sketch a detailed derivation of this duality by the operator dualities of Refs. 23 and 24. This classical action follows from the equivalence of the transfer matrices corresponding to Eqs. (11) and (12) or, alternatively, from the equivalence of Eq. (3) to (6) (which will be proved in detail by operator representations in Sec. VI) and the relation between the transfer matrices corresponding to $e^{\bar{h}\sigma^x}$ and $e^{\bar{J}\sigma^z\sigma^{z'}}$.

We find that the classical action corresponding to the model of Eq. (10) is

$$S = \Bigg[-\tanh^{-1}(e^{-2J_x\Delta\tau}) \sum_{\square \in x\tau\ \text{plane}} \sigma\sigma\sigma\sigma - \Delta\tau J_z \times \sum_{z\ \text{direction}} \sigma\sigma - \tanh^{-1}(e^{-2J_y\Delta\tau}) \sum_{\square \in y\tau\ \text{plane}} \sigma\sigma\sigma\sigma \Bigg]. \tag{14}$$

Here and elsewhere, $\sigma = \pm 1$ are c numbers and we omit the (z) polarization superscripts.

We now extend the duality of self-duality of Eq. (6) to the three-dimensional arena. First note that by interchanging the imaginary time coordinate τ with the spatial z coordinate, we find that

$$H = -\left(K_{xz} \sum_{\square \in xz} \sigma^z\sigma^z\sigma^z\sigma^z + K_{yz} \sum_{\square \in yz} \sigma^z\sigma^z\sigma^z\sigma^z + h\sum_{\mathbf{r}} \sigma^x_{\mathbf{r}} \right) \tag{15}$$

is dual to the system given by the Hamiltonian of Eq. (10). Let us now derive self-dualities of this extended three dimensional system (en passant, effortlessly proving the central result of Refs. 21 and 22).

Expressing the action corresponding to the Hamiltonian of Eq. (10) in a spin eigenbasis of σ^x and inverting the spatial z and x coordinates of any site $\mathbf{r}$ (a $\mathbb{Z}_2$ operation), we obtain

$$S_{\text{dual}} = \Bigg[-\tanh^{-1}(e^{-2J_z\Delta\tau}) \sum_{\square \in x\tau\ \text{plane}} \sigma\sigma\sigma\sigma - \Delta\tau J_x \times \sum_{z\ \text{direction}} \sigma\sigma - \tanh^{-1}(e^{-2J_y\Delta\tau}) \sum_{\square \in y\tau\ \text{plane}} \sigma\sigma\sigma\sigma \Bigg]. \tag{16}$$

Looking at Eqs. (14) and (16), we see that if

$$S = \left(A \sum_{\square \in x\tau\ \text{plane}} \sigma\sigma\sigma\sigma + B \sum_{z\ \text{direction}} \sigma\sigma + C \sum_{\square \in y\tau\ \text{plane}} \sigma\sigma\sigma\sigma \right) \tag{17}$$

then

$$S_{\text{dual}} = \left(\tilde{A} \sum_{\square \in x\tau\ \text{plane}} \sigma\sigma\sigma\sigma + \tilde{B} \sum_{z\ \text{direction}} \sigma\sigma + \tilde{C} \sum_{\square \in y\tau\ \text{plane}} \sigma\sigma\sigma\sigma \right). \tag{18}$$

Here, $A = -\tanh^{-1}(e^{-2\bar{J}_x})$, $B = -\bar{J}_z$, and $C = -\tanh^{-1}(e^{-2\bar{J}_y})$. Similarly, $\tilde{A} = -\tanh^{-1}(e^{-2\bar{J}_z})$, $\tilde{B} = -\bar{J}_x$, and $\tilde{C} = C$. Eliminating $\bar{J}_{x,z}$, we find that

$$\sinh 2\tilde{A} \sinh 2B = 1,$$

$$\sinh 2A \sinh 2\tilde{B} = 1,$$

$$\tilde{C} = C. \tag{19}$$

Taken together, these relations imply that

$$\sinh 2A \sinh 2B = 1 \tag{20}$$

is a self-dual line for any value of C.

This extends the dualities of Refs. 21 and 22 in a very natural fashion to higher dimensions. Moreover, note in this formalism the dualities just "fall into our lap"—no involved calculations nor loop counting were necessary. The duality is a trivial geometrical reflection.

VI. OPERATOR DUALITY TRANSFORMATIONS OF THE QUANTUM PLANAR ORBITAL COMPASS MODELS ONTO QUANTUM ISING PLAQUETTE MODELS

A duality between the planar orbital compass model [Eq. (3)] and the Xu-Moore model of Eq. (6) is suggested by the

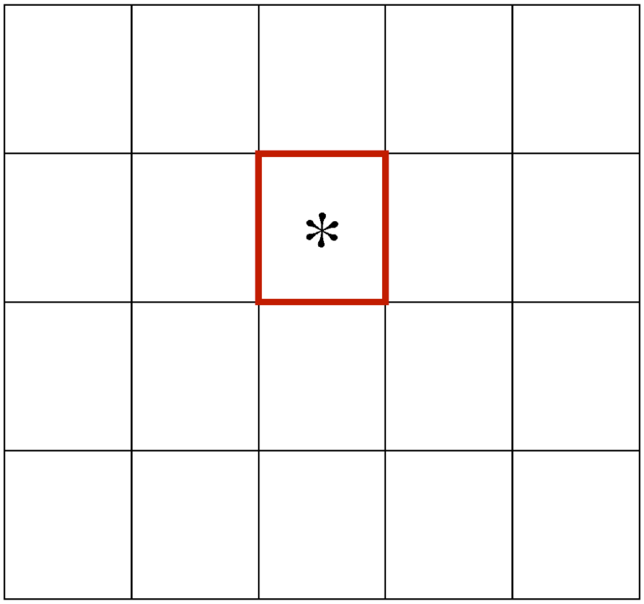

FIG. 2. The lattice and a dual lattice site (marked by an asterisk "*" at a plaquette center). Here we illustrate the representation of σ^z as on a given dual plaquette site as the product of τ^x operators placed on all four bonds composing the plaquette of the original lattice.

cubic point group operation interchanging x with τ in Fig. 1. We now prove this duality at all temperatures. To this end, we invoke a simple operator duality transformation followed by a summation over the horizontal bonds (which amounts to a standard gauge fix) in the model that results. The upshot of the up and coming discussion is that the quantum planar compass model of Eq. (1) can be mapped onto the Hamiltonian of Eq. (6) precisely at its zero temperature self-dual point.

The salient feature of the Pauli matrices σ^x and σ^z is that they anticommute at a common site while commuting everywhere else. It is readily verified[23,24] that these relations are preserved by the canonical duality relations on the dual lattice

$$\sigma^z_{\mathbf{r}} = \tau^x \tau^x \tau^x \tau^x \tag{21}$$

with the plaquette product of τ^x on the right hand-side corresponding to the four spins surrounding the dual lattice site $\mathbf{r}^*$ (the center of the plaquette as shown in Fig. 2 below), and

$$\sigma^x_{\mathbf{r}} = \prod_{x \leq x^*} \tau^z_{x,x+\hat{e}_z}, \tag{22}$$

the product of τ^z placed along vertical bonds (linking x and $x+\hat{e}_z$) along a horizontal line—see Fig. 3. The series of transformations below leading to Eq. (24) may be vividly followed in Figs. 4 and 5.

Inserting Eqs. (21) and (22) into Eq. (1), we obtain

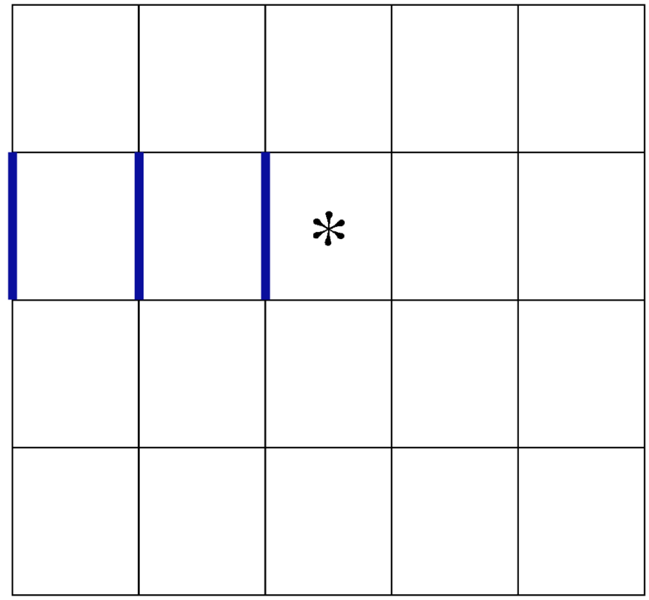

FIG. 3. A graphical representation of σ^x the string product of τ^z on all vertical bonds from the boundary up to the dual lattice site.

$$\begin{aligned} H_{\mathrm{iso}} = -J\sum_{\boldsymbol{r}^*}\Bigg(& \tau^z_{x^*+\hat{e}_x, x^*+\hat{e}_x+\hat{e}_z} \\ & + \sum_{\boldsymbol{r}^*} \tau^x_{\mathbf{r}^*,\mathbf{r}^*+\hat{\mathbf{e}}_\mathbf{z}} \tau^x_{\mathbf{r}^*+2\hat{\mathbf{e}}_\mathbf{z},\mathbf{r}^*+2\hat{\mathbf{e}}_\mathbf{z}} \tau^x_{\mathbf{r}^*+2\hat{\mathbf{e}}_\mathbf{z},\mathbf{r}^*+2\hat{\mathbf{e}}_\mathbf{z}-\hat{\mathbf{e}}_\mathbf{x}} \tau^x_{\mathbf{r}^*+\hat{\mathbf{e}}_\mathbf{z},\mathbf{r}^*+\hat{\mathbf{e}}_z-\hat{\mathbf{e}}_x} \\ & \times \tau^x_{\mathbf{r}^*+\hat{\mathbf{e}}_\mathbf{z}-\hat{\mathbf{e}}_x,\mathbf{r}^*-\hat{\mathbf{e}}_\mathbf{x}} \tau^x_{\mathbf{r}^*-\hat{\mathbf{e}}_\mathbf{x},\mathbf{r}^*} \Bigg). \end{aligned} \tag{23}$$

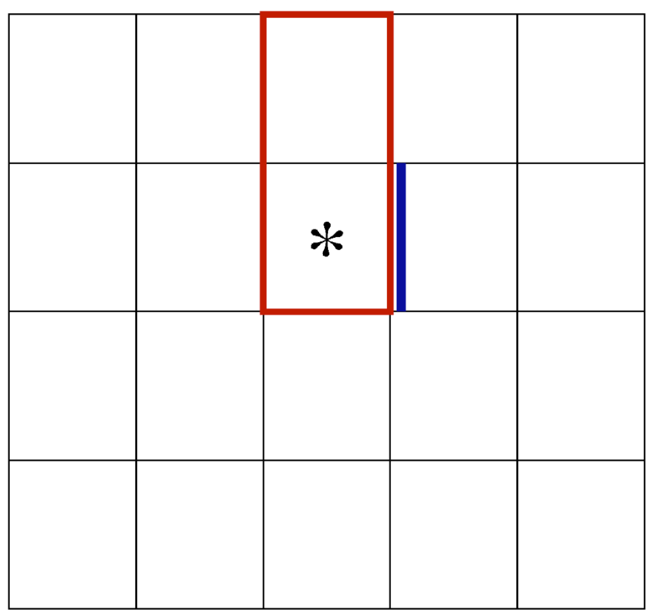

FIG. 4. The product of $J_x \sigma^x_{\mathbf{r}} \sigma^x_{\mathbf{r}+\hat{\mathbf{e}}_\mathbf{x}}$ becomes $J_x \tau^z$ on a single vertical bond on the right-hand side of the dual plaquette site corresponding to $\mathbf{r}$. The product $J_z \sigma^z_{\mathbf{r}} \sigma^z_{\mathbf{r}+\hat{\mathbf{e}}_\mathbf{z}}$ becomes in the dual representation the product of all τ^x operators forming the outer shell of a vertical domino multiplied by J_z. Putting all of the pieces together, the Hamiltonian becomes the sum of J_z multiplying a domino shell of τ^x on bonds augmented by J_x multiplying a single vertical bond on which τ^z is placed.

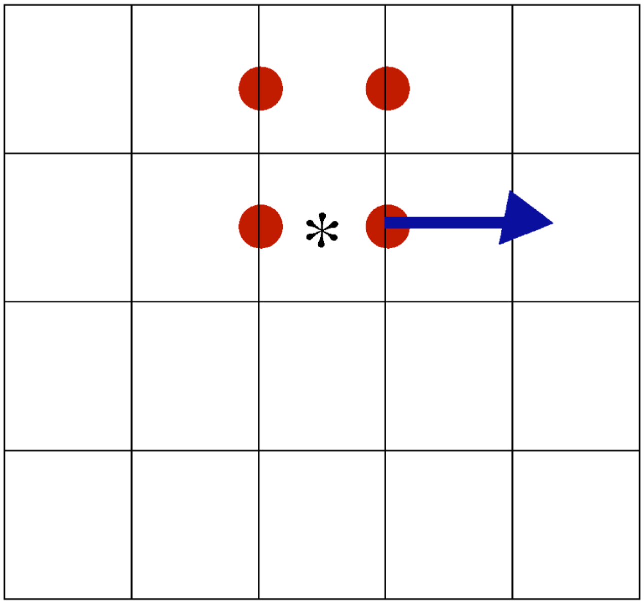

FIG. 5. Choosing a gauge in which $\tau^x=1$ on all horizontal bonds, identifying the centers of the vertical bonds as sites, we find that the Hamiltonian corresponds to the product of four σ^x operators on the vertices of a plaquette $[Ks^x_{\mathbf{r}^*}s^x_{\mathbf{r}^*+\hat{\mathbf{e}}_\mathbf{z}}s^x_{\mathbf{r}^*+\hat{\mathbf{e}}_\mathbf{z}-\hat{\mathbf{e}}_\mathbf{x}}s^x_{\mathbf{r}^*-\hat{\mathbf{e}}_\mathbf{x}}]$ augmented by an external transverse field giving rise to $hs^z_{\mathbf{r}^*}$. Here, $h=J_x$ and $K=J_y$. Thus, the planar orbital compass model is dual to the superconducting array system of Refs. 21 and 22.

The first term corresponds to an external transverse magnetic field of strength J along the z axis acting on all vertical bonds while the second term encapsulates the product of six bonds forming the outer shell of two plaquettes pasted together along the z axis. The bond common to the two plaquettes evaporated due to the relation $\tau_x^2=1$. The net result of Eq. (23) is shown in Fig. 4.

Next, we choose the longitudinal gauge wherein all horizontal bonds have $\tau^z=1$. This can be achieved via explicit gauge transformations or by simply noting that in the representation with horizontal bonds with $\tau^z=1$, the duality relations of Eqs. (21) and (22) become identical to the duality relations in one-dimensional spin chains (performed independently for each horizontal row) which trivially satisfy the commutation of spin variables of different sites, the anticommutation of the x and z components of the spin on the same site and the square of each spin operator. In this longitudinal gauge where $\tau^x_{\mathbf{r}^*\mathbf{r}^*+\hat{e}_x}=1$, the Hamiltonian now involves only vertically oriented bonds (parallel to the z axis). Defining spins $s^\alpha_{\mathbf{r}^*}=\tau^\alpha_{\mathbf{r}^*,\mathbf{r}^*+\hat{\mathbf{e}}_\mathbf{z}}$,

$$H_{\mathrm{iso}} = -K\sum_{\mathbf{r}^*} s^x_{\mathbf{r}^*}s^x_{\mathbf{r}^*+\hat{\mathbf{e}}_\mathbf{z}}s^x_{\mathbf{r}^*+\hat{\mathbf{e}}_\mathbf{z}-\hat{\mathbf{e}}_\mathbf{x}}s^x_{\mathbf{r}^*-\hat{\mathbf{e}}_\mathbf{x}} - h\sum_{\mathbf{r}^*} s^z_{\mathbf{r}^*}, \tag{24}$$

with the new parameters $h=K$ being equal to the former J of Eq. (23). Thus the isotropic planar compass orbital model lies precisely on the zero temperature self-dual line $h=k$ of Eq. (24). This result is shown in Fig. 5.

VII. CONCLUSIONS

In conclusion, we investigated several systems displaying discrete (gauge-like) sliding symmetries and illustrated that two such systems are dual to each other. The enhanced discrete sliding symmetries in these systems go hand in hand with a dimensional reduction that occurs in several limiting cases of these systems [e.g., the system of Eq. (6) in the limit of $h=0$ is none other than $1+1$-dimensional version of the one-dimensional Ising model]. We found nematic-order parameters invariant under these symmetries. This suggests the specter of detectable orbital nematic orders in t_{2g} orbital systems.

The superconducting array model of Refs. 21 and 22 is dual to the planar orbital compass model and as such has a finite temperature transition for a large S incarnation at its self-dual point.

We find that dual models may be derived by flipping the spatial and imaginary time axis (and/or quantization axis). In an upcoming work, we will elaborate on this approach to dual models as different cuts of a higher dimensional theory.[32]

The nature of the quantum phase transitions in these systems remains an open problem. A straightforward examination of the Xu-Moore model shows that the finite temperature transition from a high temperature disordered phase to a low temperature phase in which the product of Ising spins on pairs of sites belonging to different sublattices orders. This classical transition is continuous, and in the universality class of the two-dimensional (2D) classical Ising model. Although the nature of the $T=0$ transition is not established at the present time, there are suggestions that it may actually be a continuous quantum phase transition. The presence of the discrete sliding symmetry suggests that this is an unconventional quantum phase transition whose universal behavior is worth understanding.

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APPENDIX: SELF-DUALITY OF AROUND THE CUBE MODELS IN THE PRESENCE OF TRANSVERSE FIELDS

In this appendix, we explicitly generalize the self-duality that we obtained earlier for plaquette models with a transverse field to cubic models with eight spin interactions augmented by a transverse field. Such a duality was alluded to in Refs. 21 and 22.

Later, we derive this duality by going back and forth from various quantum systems to corresponding $(d+1)$-dimensional classical actions when different spin quantization and spatial lattice directions are chosen.

To prove the self-duality of such cubic systems, first consider the Hamiltonian

$$H = -K \sum_{\square \in xy \text{ plane}} \sigma^y\sigma^y\sigma^y\sigma^y - J_z \sum_{\text{bonds along } z \text{ axis}} \sigma^z\sigma^z. \tag{A1}$$

If we write down the classical action in a spin basis quantized along the σ^z axis, we find

$$S_1 = -K_1 \sum_{\text{cubes in } xy\tau} \sigma\sigma\sigma\sigma\sigma\sigma\sigma\sigma - J_{z1} \sum_{\text{bonds along } z \text{ axis}} \sigma\sigma. \tag{A2}$$

Here $\sinh 2K_1 \sinh 2\overline{K}=1$ with $\overline{K}\equiv K\Delta\tau$, and $J_{z1}=J_z\Delta\tau$. Alternatively, if we write down the classical action corresponding to Eq. (A1) in a spin basis polarized along σ^y, we find

$$S_2 = -K_2 \sum_{\square \in xy \text{ plane}} \sigma\sigma\sigma\sigma - J_{z2} \sum_{\square \in z\tau \text{ plane}} \sigma\sigma\sigma\sigma. \tag{A3}$$

Here, $K_2=K\Delta\tau$ and $\sinh 2J_{z2} \sinh 2\overline{J_z}=1$ with $\overline{J_z}=J_z\Delta\tau$. Thus, we find that the classical actions S_1 and S_2 are dual to each other. The classical action S_1 also corresponds to the Hamiltonian

$$H_{\text{cube}} = -K_* \sum_{\text{cubes in } xyz} \sigma^z\sigma^z\sigma^z\sigma^z\sigma^z\sigma^z\sigma^z\sigma^z - h_{z*}\sum_{\mathbf{r}} \sigma^x_{\mathbf{r}}, \tag{A4}$$

when written in a spin basis quantized along σ^z. Thus, H_{cube} may be represented by the classical action S_2. Putting all of the pieces together we find that

$$\sinh 2K_*\Delta\tau \sinh 2K_2 = 1,$$

$$\sinh 2h_{z*}\Delta\tau \sinh 2J_{z2} = 1. \tag{A5}$$

Interchanging, in the action $S_2, K\to J, x\to z, y\to\tau$, we obtain an action ($S_3$) whose partition function is identically the same. By the same steps outlined above, the classical action S_3 corresponds (via duality transformations) to the Hamiltonian

$$\tilde{H}_{\text{cube}} = -\tilde{K}_* \sum_{\text{cubes in } xyz} \sigma^z\sigma^z\sigma^z\sigma^z\sigma^z\sigma^z\sigma^z\sigma^z - \tilde{h}_{z*}\sum_{\mathbf{r}} \sigma^x_{\mathbf{r}}. \tag{A6}$$

This establishes the duality between H_{cube} and $\tilde{H}_{\text{cube}}$,

$$\tilde{K}_* = h_{z*},$$

$$\tilde{h}_{z*} = K_*. \tag{A7}$$

Fusing these relations together, we find that $h_*=K_{z*}$ constitutes a self-dual line of H_{cube}.

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