Tails of the density of states in a random magnetic field

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We study the tails of the density of states of particles subject to a random magnetic field with nonzero mean with the optimum fluctuation method. Closer to the centers of the Landau levels, the density of states is found to be Gaussian, whereas the energy dependence is nonanalytic near the lower edge of the spectrum.

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The problem of a charged quantum particle constrained to move in a two-dimensional (2D) static random magnetic field (RMF) has attracted considerable theoretical and experimental interest in the past few years. The model plays an important role within the composite fermion picture of the fractional quantum Hall effect.[1] Furthermore, it is supposed to describe states with spin-charge separation in high-T_c superconductors.[2] It is also relevant to the understanding of the properties of a two-dimensional electron gas (2DEG) in lattice-mismatched InAs/InGaAs heterostructures in magnetic fields.[3] In the latter systems the electron gas is nonplanar due to the lattice-mismatched epitaxial growth. When a uniform magnetic field B is applied, the electrons experience an effective inhomogeneous field perpendicular to the nonplanar 2DEG.[3] In addition, a static RMF in 2D inversion layers can be experimentally realized in several ways. One possibility is to use a type-II superconductor with a disordered Abrikosov flux lattice in an external magnetic field as the substrate for the 2DEG.[4] Alternatively, a magnetically active substrate such as a demagnetized ferromagnet with randomly oriented magnetic domains may be used.[5] Recently, static RMFs in 2DEGs were created by applying strong magnetic fields parallel to GaAs Hall bars decorated with randomly patterned magnetic films.[6]

The most fundamental quantity for understanding the electonic properties of a random system is the density of energy levels. The standard method to estimate the density of states (DOS) is to calculate the imaginary part of the trace of the single-particle Green's function by diagrammatic techniques. However, this approach fails in the tails of an energy band where multiple scattering up to infinite order has to be considered in order to take into account correctly the effect of localization of electrons. Also, numerical approaches are bound to fail in the asymptotic tails since here the eigenstates are determined by rare statistical fluctuations of the randomness. Moreover, in the case of RMF, the perturbative approach is also fundamentally problematic since one has to deal with the nondiagonal part of the Green's function, which is not gauge invariant. In addition, the calculation of the Green's function is plagued by infrared divergencies[7–11] that are due to the long-range nature of the correlations of the vector potential, even if the spatial correlations in the RMF are short ranged. It has been suggested that these divergencies are due to the nongauge invariance of the Green's function and therefore unphysical,[9] although, recently, a physical interpretation has been proposed.[11] In order to avoid such difficulties, E. Altshuler *et al.*[9] and Mirlin[12] calculated the DOS of a charge in a RMF using the semiclassical approximation. This is valid when the energy E is much larger than the cyclotron energy $\hbar\omega_c$ corresponding to the mean magnetic field B. Also, it should exceed Γ, the disorder induced width of the Landau levels (LL). A field theoretical approach has been used to determine the DOS in a RMF with zero mean value near the band edge.[8] The tail of the DOS in a system of randomly distributed flux tubes of fixed strength was considered.[13] There are also several numerical studies of the spectrum with different mean and correlation lengths.[14] Recently, mathematically rigorous results have been obtained.[15,16] In particular,[15] upper and lower bounds for the logarithm of the integrated DOS near $E=0$ of some simple Gaussian RMFs with zero mean values have been estimated. For RMFs with nonzero mean values, the limit when E is smaller than $\hbar\omega_c$ and, more generally, the tails of the lower Landau bands have not been considered analytically so far.

It is our purpose to provide nonperturbative results for the DOS in the tails of the lower Landau bands, as broadened by a static RMF. We show that the optimum fluctuation method (OFM),[17,18] being nonperturbative and free from divergencies, can be extended to treat this RMF problem. We consider noninteracting fermions in a RMF with nonzero mean, $B+b(\mathbf{r})$, with $b(\mathbf{r})$ Gaussian distributed and $\langle b(\mathbf{r})\rangle=0$. The Hamiltonian

$$H=\frac{1}{2m_e}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)^2 \tag{1}$$

has a sharp lower edge of the energy spectrum at $E=0$. We concentrate on the energy region near the first Landau band. Since the OFM is especially designed to grasp rare fluctuations, it allows us to calculate the energy and B dependence of the leading terms of the DOS. The correlation function of the Gaussian RMF is assumed as

$$\langle b(\mathbf{r})b(\mathbf{r}')\rangle=\beta(|\mathbf{r}-\mathbf{r}'|) \tag{2}$$

with $\beta(|\mathbf{r}-\mathbf{r}'|)\to 0$ as $|\mathbf{r}-\mathbf{r}'|\to\infty$. The random field is ergodic, i.e., the correlations between different regions decay to zero with increasing distance. We also assume that $\beta(r)$ is characterized by a single scale r_c, the correlation length of the RMF. The probability density for a specific realization

$b(\mathbf{r})$ with correlator (2) is $P[b(\mathbf{r})]=\mathcal{N}\exp\{-S[b(\mathbf{r})]\}$, where $\mathcal{N}$ is the normalization constant and $S[b(\mathbf{r})]$, the action of the RMF, is

$$S[b(\mathbf{r})] = \frac{1}{2}\int b(\mathbf{r})\beta^{-1}(\mathbf{r}-\mathbf{r}')b(\mathbf{r}')d\mathbf{r}d\mathbf{r}'. \qquad (3)$$

The kernel $\beta^{-1}(\mathbf{r}-\mathbf{r}')$ has the property

$$\int d\mathbf{r}'\beta^{-1}(\mathbf{r}-\mathbf{r}')\beta(\mathbf{r}'-\mathbf{r}'') = \delta(\mathbf{r}-\mathbf{r}''). \qquad (4)$$

For a δ-correlated RMF, the action reduces to $S[b(\mathbf{r})] = 1/\beta_0\int b^2(\mathbf{r})d\mathbf{r}$. The configurationally averaged DOS is

$$\rho(E) = \int Db(\mathbf{r})P[b(\mathbf{r})]\rho\{E;[b(\mathbf{r})]\}. \qquad (5)$$

In the low energy tail of the first Landau band states are expected to be localized near strong, exponentially rare fluctuations of $b(\mathbf{r})$. Therefore, the average in Eq. (5) over all configurations, yielding states at energy E is dominated by the most probable realization of $b(\mathbf{r})$. The functional integral in Eq. (5) can be evaluated using the saddle point approximation. Furthermore, only fluctuations in which the lowest level E_0 is equal to E have to be taken into account since a configuration in which E corresponds to an excited level is less probable. Within logarithmic accuracy,[17–22]

$$-\ln\rho(E) \sim \min_{b(\mathbf{r})} S[b(\mathbf{r})]\Big|_{E_0[b(\mathbf{r})]=E}. \qquad (6)$$

Before we present a rigorous treatment, let us first give an intuitive estimate for the DOS closer to the band center, $\Delta E \ll \hbar\omega_c$, where $E=\hbar\omega_c/2-\Delta E$. For short-range RMF, $r_c \ll l_B$, with $l_B=(\hbar c/eB)^{1/2}$ the magnetic length related to B, an optimal configuration near the band center is likely to be a circular magnetic well with depth $\Delta b \ll B$. The corresponding action is $S\approx \pi\Delta b^2 R^2/\beta_0$, where R is the radius of the well. The ground-state energy E_0 of a charged particle in a circular magnetic well of radius R, where the magnetic field is different from the constant magnetic field B, is known.[23] The radii $R_{\rm opt}$ of the optimal wells with lowest action $S_{\rm min}(\Delta E)$, are proportional to l_B and independent on ΔE. Moreover, the depths $\Delta b_{\rm opt} \ll B$ of these wells are proportional to ΔE and do not depend on B,

$$R_{\rm opt} \sim 1.6\, l_B \propto B^{-1/2}, \quad \Delta E \propto \Delta b_{\rm opt}. \qquad (7)$$

The action of the optimum fluctuation with $E_0=E$ is then

$$S(\Delta E) \sim \frac{R_{\rm opt}^2\Delta b_{\rm opt}^2}{\beta_0} \sim \frac{l_B^2\Delta E^2}{\beta_0}. \qquad (8)$$

From Eqs. (6) and (8) we obtain

$$\rho(E) \sim \exp(-\Delta E^2/\Gamma_{\delta,0}^2), \qquad (9)$$

with $\Gamma_{\delta,0}=\alpha\hbar e\beta_0^{1/2}/(m_e c l_B)$, where $\alpha\sim 1.5\cdot 10^{-2}$ is a numerical factor. The variance of the Gaussian is thus proportional to B. Our simple arguments are expected to be valid when the energetic distance from the center of the lowest LL fulfils $\Gamma_{\delta,0} \ll \Delta E \ll \hbar\omega_c$. A completely analogous argument holds for the right tail of the first LL, near the band center. In this case the optimal fluctuations are magnetic circular humps with height $\Delta b \ll B$ and radius $R \propto l_B$ and the leading exponential term of the DOS in the right tail shows the same dependence on the energy shift and B as (9).

With long-range RMFs, the analysis of the DOS near the band center is simpler: The localization radius of a typical state of the order of l_B is much shorter than the radius of an optimal potential well. The correlation length r_c, and the energy E of such a state is, in leading order, equal to the first LL energy in the total field $B-\Delta b$. The energy shift is thus proportional to Δb, and the RMF acts exactly like a random electrostatic potential. Since for long-range RMFs the radius of the well is the largest length scale, the probability distribution (3) can be approximated as

$$P[\Delta b] \sim \exp[-\Delta b^2/\beta(0)]. \qquad (10)$$

Hence,

$$-\ln\rho(E) \sim \frac{\Delta E^2}{\Gamma_0^2}, \qquad (11)$$

with $\Gamma_0=\hbar e\beta(0)^{1/2}/m_e c$, and the exponent of the DOS does not depend on B, as long as the inequality $l_B \ll r_c$ is fulfilled. Equation (11) is valid if $\Gamma_0 \ll \Delta E \ll \hbar\omega_c$. Similar considerations are expected to yield a Gaussian DOS also in the tails of higher Landau bands, in the regions $\Gamma_n \ll |E-(n+1/2)\hbar\omega_c| \ll \hbar\omega_c$, where Γ_n is the width of the nth LL. In these regions, the DOS resembles the one of independent charged particles in a Gaussian electrostatic potential.[22,24]

Due to the sharp band edge at $E=0$ the DOS is expected to approach zero more rapidly for energies $E \ll \hbar\omega_c$. For short-range RMF, states with arbitrarily small energies can be obtained when they are localized in regions of area $\mathcal{A}$, inside which $b\approx 0$ and outside which $b\approx B$. The action of these fluctuations is

$$S \sim \mathcal{A}B^2/\beta_0, \qquad (12)$$

and the ground-state energy scales like the one in a potential well, $E\sim\hbar^2/2m_e\mathcal{A}$. Thus, $1/\mathcal{A}\propto E$ and the DOS becomes a nonanalytic function of E,

$$\rho(E) \sim \exp(-K_0 B^2/\beta_0 E), \qquad (13)$$

with $K_0\sim\pi\hbar^2/2m_e$. The above picture is analogous to the argument used by Lifshitz to estimate the tail of the DOS of a particle subject to a Poissonian random potential generated by short-range, repulsive impurities in zero magnetic field.[18,20,25] The argument holds for a Poissonian distribution of magnetic fluxes, too.[13]

Intuitively, for large correlation length $r_c \gg l_B$, there is a right neighborhood of $E=0$ such that, each corresponding optimum well has a radius much larger than the correlation length $R \gg r_c$. Equivalently, $E \ll \hbar^2/(m_e r_c^2)$. The larger the correlation length, the closer to the band edge the energies of the optimal states must be in order to fulfil this inequality. Defining

$$\beta_0 \equiv \int \beta(\xi) d^2\xi \sim \beta(0) r_c^2, \tag{14}$$

the action of $b(\mathbf{r})$ is still given by Eq. (12) and the DOS by Eq. (13). Hence, for $E \to 0$, the DOS becomes independent of the correlation length r_c.

In order to obtain exact expressions for the DOS, we now derive the variational equations which determine the shape of the optimal fluctuations and wave functions.[19,21] According to Eq. (6), we must search for the maximum of the probability distribution Eq. (3) under the constraint $E_0 = E$. For the weaker constraint

$$\det\{H[b(\mathbf{r})] - E\} = 0, \tag{15}$$

or, equivalently, $\underline{E_n} = E$ for some energy level E_n, the optimum fluctuation $\overline{b}(\mathbf{r})$ of the RMF must satisfy

$$\int \beta^{-1}(\mathbf{s}-\mathbf{s}')\overline{b}(\mathbf{s}')d\mathbf{s}' + \mu \left. \frac{\delta \det\{H[b(\mathbf{r})] - E\}}{\delta b(\mathbf{s})} \right|_{b=\overline{b}} = 0, \tag{16}$$

where μ is a Lagrange multiplier. Using

$$\det\{H[b(\mathbf{r}) - E\} = \exp(\mathrm{tr}\ln\{H[b(\mathbf{r})] - E\}), \tag{17}$$

and assuming that the ground-state energy $E_0[\overline{b}(\mathbf{r})]$ is equal to E, we find

$$\int \beta^{-1}(\mathbf{s}-\mathbf{s}')\overline{b}(\mathbf{s}')d\mathbf{s}' + \mu'(E) \left. \frac{\delta E_0[b(\mathbf{r})]}{\delta b(\mathbf{s})} \right|_{b=\overline{b}} = 0 \tag{18}$$

with $\mu'(E) \equiv \mu \Pi_{n=1}^{\infty}(E_n - E)$. In the Coulomb gauge, we can write the Hamiltonian as a function of $b(\mathbf{r})$ and calculate $\delta E_0 / \delta b(\mathbf{s})$. The variational Eq. (18) yields

$$\overline{b}(\mathbf{s}) = -\mu'(E)\frac{e}{c}\int d^2\mathbf{s}'\beta(\mathbf{s}-\mathbf{s}')\int d^2\mathbf{r}\,\mathbf{j}_0 \cdot \mathbf{a}_\Phi(\mathbf{r}-\mathbf{s}'), \tag{19}$$

where

$$\mathbf{a}_\Phi(\mathbf{r}-\mathbf{s}) = \frac{1}{2\pi}\frac{\tilde{z} \times (\mathbf{r}-\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^2} \tag{20}$$

and $\mathbf{j}_0 = \Psi_0^* \Pi \Psi_0 / 2m_e$ is the ground-state current, Π is the kinetic momentum of the particle. Equation (19), together with

$$\frac{1}{2m_e}\Pi^2\Psi_0 = E\Psi_0 \tag{21}$$

determines the optimal magnetic field $b(\mathbf{r})$ and ground-state wave-function $\Psi_0(\mathbf{r})$.

We have solved Eqs. (19) and (21) iteratively both in the case $r_c \ll l_B$ and in the case $r_c \gg l_B$ for RMFs with Gaussian correlators

$$\beta(\mathbf{r}-\mathbf{r}') = \frac{\beta'}{2\pi r_c^2}\exp\left(-\frac{|\mathbf{r}-\mathbf{r}'|^2}{2r_c^2}\right). \tag{22}$$

Since the RMF distribution function is rotationally invariant, circular symmetry is assumed.

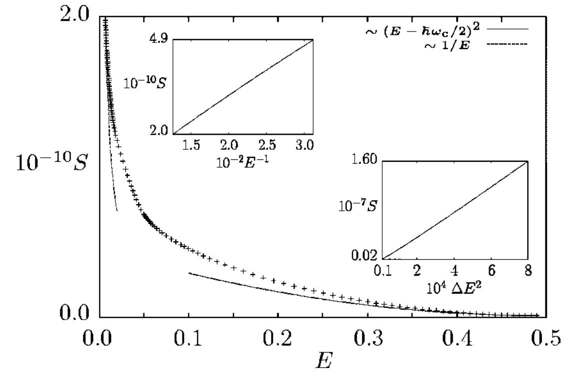

FIG. 1. Behavior of the effective action $S(E) = -\ln\rho(E)$ in a short-range RMF, $r_c = 0.1 l_B$. Energies are in units of $\hbar\omega_c$ and $\Gamma_0 \sim 10^{-4}\hbar\omega_c$. Left and right insets: energy dependence of S near the band edge and near the band center, respectively.

For short-range RMFs the result for the DOS $\rho(E)$ is shown in Fig. 1 as a function of energy E, where $S(E) = -\ln\rho(E)$. At the band edge we find indeed that the action of the optimum fluctuation shows the characteristic behavior as a function of the energy E, $S \sim E^{-1}$ while closer to the band center it changes to $S \propto \Delta E^2$. This reflects the physical origins of the corresponding typical wave functions, and is consistent with the qualitative arguments, given above.

Figure 2 shows the optimal fluctuations near the band center and the band edge for short-range fluctuations. Near the center, the typical fluctuations are shallow wells, compared to B, with relatively steep walls (the change in b is about 0.3 Δb on length scales of l_B, where Δb is the depth of the well). In the case of long-range fields, near the band center, the radius of the ground state is much smaller than the size of the well.

In experiments some kind of random electrostatic potential is always present. Let us assume that the RMF and the random potential (RP) are independent random quantities and that they are both long ranged. For a weak RP, $W(0)^{1/2} \ll \hbar e \beta(0)^{1/2}/(mc)$ (where $W(\mathbf{r}) = \langle V(0)V(\mathbf{r})\rangle$), the RMF is dominant in the tail at positive energies except for a narrow region close to $E = 0$. At $E \ll \hbar\omega_c$ the action of an optimal

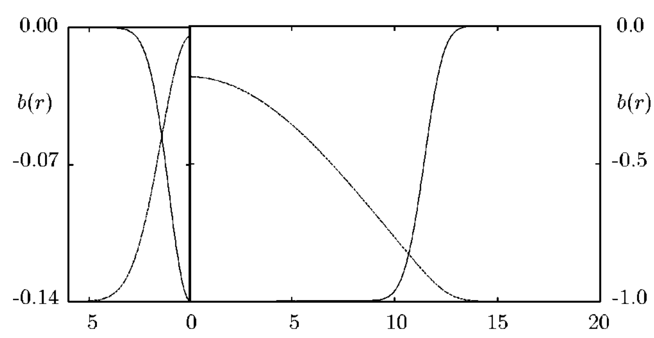

FIG. 2. Optimum magnetic wells $b(r)$ (solid line) and the corresponding ground-state wave functions $\Psi(r)$ (dashed line) for correlation length $r_c = 0.1 l_B$. Left: closer to the center of the band, energy shift $\Delta E = 3.6 \cdot 10^{-2}\hbar\omega_c$. Right: near the band edge; ground-state energy $E = 1.8 \cdot 10^{-2}\hbar\omega_c$. r is in units of l_B and $b(r)$ in units of B.

well of the RP is $S_{\mathrm{RP}} \sim (E-\hbar\omega_c/2)^2/2W(0) \sim \hbar^2\omega_c^2/4W(0)$, whereas $S_{\mathrm{RMF}} \sim \hbar^2\omega_c^2/(2mEr_c^2\delta\omega_c^2)$, where $\delta\omega_c = e\beta(0)^{1/2}/mc$ and r_c is the correlation length of the RMF. The RMF will dominate on the RP if $S_{\mathrm{RMF}} < S_{\mathrm{RP}}$; therefore, the exponent of the DOS will be proportional to $1/E$ if $1 \ll R^2/r_c^2 \ll \hbar^2\delta\omega_c^2/W(0)$, with $R^2 \sim \hbar^2/mE$. Hence, the results presented in this paper break down at energy $E_c \sim W(0)/mr_c^2\delta\omega_c^2$. For large, negative energies $E \to -\infty$, the RMF becomes irrelevant and the DOS is purely classical, $-\ln \rho(E) = E^2/2W(0)$.

In conclusion, we have determined the density of states of a charged particle in a spatially correlated, randomly varying magnetic field with nonvanishing average. The latter provides Landau levels which are broadened into Landau bands by the RMF, well separated for sufficiently small disorder. In the regions of the tails of the Landau bands which are accessible neither by perturbative multiple scattering expansions nor by numerical calculations, we have found that the average DOS is determined by the *typical* configuration of the magnetic field. This is reflected in the energy dependence of the effective action $S(E)$ and the fact that the latter is proportional to the logarithm of the DOS $\rho(E)$, which is found to be asymptotically singular at the lower edge of the energy spectrum and becomes quadratic as a function of the energy closer toward the center of the band. In order to determine the preexponential factor of the DOS, one integrates over the fluctuations of the magnetic field around the saddle point configuration satisfying Eq. (19).[21,22,26] As a final remark, we want to discuss briefly the relevance of our work to the CF description of the fractional quantum Hall effect. Within this model, electrons are replaced by fermions experiencing a fictitious magnetic field proportional to the particle density in addition to the external one. In the presence of a random impurity potential, at the mean field level, the particle density is spatially inhomogeneous due to screening and the fictitious magnetic field has thus a spatially stochastic component.[1] However, since the random impurity potential and the RMF are not independent random quantities, our theory cannot be straightforwardly applied to the CF model. This issue is subject to future work and will be published elsewhere.

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