Low-temperature specific heat of some quantum mean-field glassy phases

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We investigate analytically the low temperature behavior of the specific heat $C_v(T)$ for a large class of quantum disordered models within mean field approximation. This includes the vibrational modes of a lattice pinned by impurity disorder in the quantum regime, the quantum spherical p-spin-glass and a quantum Heisenberg spin glass. We exhibit a general mechanism, common to all these models, arising from the so-called marginality condition, responsible for the cancellation of the linear and quadratic contributions in T in the specific heat. We thus find for all these models the mean field result $C_v(T) \propto T^3$.

I. INTRODUCTION

While they have been experimentally observed several decades ago,[1] the anomalous low temperature thermodynamical properties of disordered and glassy systems remain a formidable theoretical issue. In particular, measurements of the specific heat $C_v(T)$ in a variety of glasses including structural glasses,[1] disordered crystals,[2] or spin-glasses[3] show a linear behavior $C_v(T) \propto T$ at low temperature. Such a behavior is often explained by the standard two-level systems (TLS) phenomenological argument.[4] Although this TLS argument is very appealing, and successful in many situations, it appears extremely hard to vindicate it from a microscopic description.

The computation of the specific heat of a disordered system starting from a microscopic Hamiltonian is a very complicated task. In this respect, important progress has been achieved by recent developments in mean field methods in quantum spin-glasses[5–9] and related models,[10] allowing for the description of low lying excitations in quantum glasses. However, even in this solvable limit, the analytical computation of $C_v(T)$ is still intricate and the question whether this TLS argument is confirmed or not by Mean Field calculations is still a subject of controversy.[11,12]

In Ref. 8, a quantum extension of the spherical p-spin model was studied. In the marginal spin-glass phase, characterized by a one step Replica Symmetry Breaking (RSB) ansatz together with the marginality condition,[13,14] some indications were found for a linear behavior of the specific heat, although its low T behavior was not extracted analytically. The authors of Ref. 9 have studied a mean field theory of an SU(2) quantum Heisenberg spin-glass. Using a semiclassical expansion in $1/S$, with S the size of the spins, the specific heat was obtained analytically to lowest order in the marginal spin glass phase, also described by a one step RSB solution. At this order, the linear and quadratic terms of the low T expansion of $C_v(T)$ were found to *cancel*, leading to a cubic behavior, $C_v(T) \propto T^3$ (these cancellations were found to occur in the related model of a quantum Ising spin-glass at the lowest order in a similar semiclassical expansion[9]). The expansion to next order appeared to be rather intricate, and it was argued[9] that the accidental cancellation identified to lowest order does not occur to this next order, yielding a linear

contribution to $C_v(T)$. A later numerical solution of the saddle point equation[11] claims instead the absence of this linear contribution and a low T behavior $C_v(T) \propto T^2$.

A class of models for which such mean field methods have been applied with some success, e.g., to compute correlation functions,[10,15] are disordered elastic systems, which cover a wide range of physical situations such as charge density waves,[16] electron glasses,[17,18] and flux lattices,[19–22] for which the quantum regime is relevant. In the elastic limit, where topological defects can be neglected, which is for instance the case in the so-called Bragg glass phase, these systems have been studied, both in the classical[15] and quantum[10] limit, using the Gaussian variational approximation[10,15,23] to the replicated Hamiltonian. In this framework, the specific heat has been studied both in the classical[24] and quantum regime.[12,25] In the quantum limit, of interest here, the specific heat has been computed in a semiclassical expansion in powers of $\hbar$, keeping $\hbar/T$ fixed, similar to the aforementioned $1/S$ expansion.[9] At the leading order, the cancellation of the linear and quadratic terms in $C_v(T)$ was also obtained.[12,25] But surprisingly, the analysis of the next to leading order showed that these cancellations also occur.[25] In view of these results,[9,11,25] it is important to know whether there is a general property, within this mean field approach, leading to the cancellation of the linear term in $C_v(T)$.

In this paper, we identify a general mechanism, common to all these models, relying on the marginality condition, which leads to the cancellation of the linear and quadratic (in T) contribution to $C_v(T)$ at low T. This leads, *independently of any semiclassical expansion nor numerics*, to $C_v(T) \propto T^3$.

The organization of the paper is as follows: In Sec. II, we introduce the different models we will be interested in, and recall the main properties of the saddle point equations. Section III is devoted to the low T expansion itself: we first exhibit the nontrivial low T structure of the variational equations, therefore extending the previous analysis of Refs. 8–10 at finite T, and then turn to the computation of the specific heat. Finally, we draw our conclusions in the last section.

II. MODELS AND MEAN FIELD APPROXIMATIONS

A. Quantum periodic elastic manifold in a random potential (Model I)

We consider a collection of interacting quantum objects of mass M whose position variables are denoted $u_\alpha(R_i, \tau)$. The

equilibrium positions R_i in the absence of any fluctuations form a perfect lattice of spacing a. Interactions result in an elastic tensor $\Phi_{\alpha,\beta}(q)$ which describes the energy associated to small displacements, the phonon degrees of freedom. Impurity disorder is modeled by a τ independent Gaussian random potential $U(x)$ directly coupled to the local density $\rho(x) = \Sigma_i \delta(x - R_i - u(R_i, \tau))$. We will describe systems in the weak disorder regime $a/R_a \ll 1$ where R_a is the translational correlation length, e.g., in a Bragg glass phase where the condition $|u_\alpha(R_i, \tau) - u_\alpha(R_i + a, \tau)| \ll a$ holds, no dislocation being present. The system at equilibrium at temperature $T = 1/\beta$ is described by the partition function $Z = Tre^{-\beta H} = \int Du D\Pi e^{-S/\hbar}$ with the Hamiltonian $H = H_{\text{ph}} + H_{\text{dis}}$:

$$H_{\text{ph}} = \frac{1}{2} \int_q \frac{\Pi(q)^2}{M} + \sum_{\alpha,\beta} u_\alpha(q) \Phi_{\alpha,\beta}(q) u_\beta(-q),$$

$$H_{\text{dis}} = \int d^d x \, U(x) \rho(x, u(x)), \tag{1}$$

and its associated Euclidean quantum action in imaginary time τ

$$-S[\Pi, u] = \int_0^{\beta\hbar} d\tau \int_q i\Pi_\alpha(q, \tau) \partial_\tau u_\alpha(q, \tau) - H, \tag{2}$$

where $u(q, \tau)$ and its conjugated momentum $\Pi(q, \tau)$ satisfy periodic boundary conditions, of period $\beta\hbar$, along the τ axis. One denotes by $\int_q \equiv \int_{\text{BZ}}[d^d q/(2\pi)^d]$ integration on the first Brillouin zone. We will focus here on the case of internal dimension $d \geq 2$. For simplicity we illustrate the calculation on an isotropic system with $\Phi_{\alpha,\beta}(q) = cq^2 \delta_{\alpha\beta}$ and denote disorder correlations

$$\overline{U(x)} = 0, \quad \overline{U(x)U(x')} = \Delta(x - x'). \tag{3}$$

The disorder average is performed using the replica trick $\overline{Z^k} = \int Du e^{-S_{\text{rep}}/\hbar}$ and integrating over Π, after some manipulations,[10] one obtains the following replicated action $S_{\text{rep}} = S_{\text{ph}} + S_{\text{dis}}$ with:

$$S_{\text{ph}} = \int d^d x \, d\tau \frac{c}{2} \sum_a (\nabla_x u_a)^2 + \frac{1}{v^2}(\partial_\tau u_a)^2,$$

$$S_{\text{dis}} = -\frac{1}{2\hbar} \int d^d x \, d\tau \, d\tau' \sum_{ab} R(u_a(x, \tau) - u_b(x, \tau')),$$

$$R(u) = \rho_0^2 \sum_K \Delta_K \cos(K \cdot u). \tag{4}$$

Here $v = \sqrt{c/M}$ is the pure phonon velocity and $\Delta_K = \int d^d x \, e^{iK \cdot x} \Delta(x)$ denote the harmonics of the disorder correlator at the reciprocal lattice vectors K, and $\rho_0 \sim a^{-2}$ the average areal density.

Given the complexity of the replicated action S_{rep}, it has been proposed to study it within the Gaussian Variational Method.[10,15,23] It is implemented by choosing a trial Gaussian action S_0 parametrized by a $k \times k$ matrix in replica space $G_{ab}^{-1}(q, \omega_n)$:

$$S_0 = \frac{1}{2\beta\hbar} \int_q \sum_{a,b} G_{ab}^{-1}(q, \omega_n) u^a(q, \omega_n) u^b(-q, -\omega_n),$$

$$G_{ab}^{-1}(q, \omega_n) = cq^2 \delta_{ab} - \sigma_{ab}, \tag{5}$$

which minimizes the variational free energy $F^{\text{var}} = F_0 + (1/\beta\hbar)\langle S^{\text{rep}} - S_0 \rangle_{S_0}$, where F_0 denotes the free energy computed with S_0. In the limit $k \to 0$, we denote $\widetilde{G}(q, \omega_n) = G_{aa}(q, \omega_n)$ and parametrize $G_{a \neq b}(q, \omega_n)$ by $G(q, u)$, where $0 < u < 1$, which is ω_n independent.[10] Similarly we take $B_{ab}(\tau) = \overline{\langle [u_a(x, \tau) - u_b(x, 0)]^2 \rangle}/m$ with $\widetilde{B}(\tau)$ and $B(u)$ which is τ independent. The best trial Gaussian action (5) is obtained by breaking the replica-symmetry (RSB).[15] A previous analysis[15] revealed indeed the existence of a breakpoint u_c such that $\sigma(u) = \sigma(u_c)$ for $u \geq u_c$. In $d > 2$, where there is a full RSB solution, $\sigma(u)$ is a continuously varying function of u for $u < u_c$. In $d = 2$, for the single cosine model, there is instead a (marginal) one step RSB solution such that $\sigma(u) = 0$ for $u < u_c$.

Using the variational approach, it has been shown in detail[25] that the specific heat is obtained from the T-derivative of internal energy $\overline{\langle H \rangle}$ per unit volume, which, independently of the RSB scheme, can be written in terms of the saddle point solution:

$$\overline{\langle H \rangle} = \frac{1}{\beta} \sum_n \int_q \frac{cq^2 + \Sigma + I(\omega_n)}{cq^2 + \Sigma + M\omega_n^2 + I(\omega_n)} + \frac{1}{\hbar} \int_0^{\beta\hbar} d\tau (F(\widetilde{B}(\tau))$$
$$- F(B)) - \int_0^{w_c} dw(F(B(w)) - F(B)), \tag{6}$$

where $F(X) = \hat{V}(X) - (X/2)\hat{V}'(X)$, $\hat{V}(X) = -\rho_0^2 \Sigma_K \Delta_K \times \exp(-XK^2/2)$, $w_c = \beta u_c$. In (6), the quantities entering this expression are determined by the variational equations:

$$I(\omega_n) = \frac{2}{\hbar} \int_0^{\beta\hbar} d\tau (1 - \cos(\omega_n \tau))(\hat{V}'(\widetilde{B}(\tau)) - \hat{V}'(B)), \tag{7}$$

$$1 = -4\hat{V}''(B)J_2(\Sigma), \quad J_n(x) = \int_q \frac{1}{(cq^2 + x)^n}, \tag{8}$$

with the definitions

$$B = \frac{2}{\beta} \sum_n \int_q \frac{1}{cq^2 + M\omega_n^2 + \Sigma + I(\omega_n)}, \tag{9}$$

$$\widetilde{B}(\tau) = \frac{2}{\beta} \sum_n \int_q \frac{1 - \cos(\omega_n \tau)}{cq^2 + M\omega_n^2 + \Sigma + I(\omega_n)}. \tag{10}$$

The breakpoint w_c, in $d > 2$, is determined by

$$w_c \equiv w_c(\Sigma) = 4 \frac{(J_2(\Sigma))^3}{J_3(\Sigma)} \hat{V}'''(B). \tag{11}$$

We finally quote the following useful relation, valid for a full RSB solution, obtained by combining Eq. (8) and Eq. (11):

$$w_c \delta B + 2\mathcal{J}_2(\Sigma)\delta\Sigma = 0, \tag{12}$$

where δ stands for an infinitesimal variation. An analysis of the stability of the variational solution reveals[23] that Eq. (8) coincides with the vanishing of the transverse eigenvalue (*replicon*) of the Hessian matrix. This property, the so called *marginality condition*, holds automatically in this problem for $d \geq 2$.

As first noticed in Ref. 10 and further investigated in Refs. 12 and 25, the solution of the variational equations can be organized in an expansion in $\hbar$ keeping $\beta\hbar$ fixed. Expanding any quantity Q as $Q = \sum_{n=0}^{\infty} \hbar^n Q_n(\beta\hbar)$, it was shown[10] that this condition describes a *gapless* excitation spectrum characterized by the low frequency behavior of the self-energy (7) $I_0(\omega_n) \propto |\omega_n| + \mathcal{O}(\omega_n^2)$ leading to the analytic continuation $I_0''(\omega) \propto \omega$, where $I(\omega_n \to -i\omega + 0^+) = I'(\omega) + iI''(\omega)$. Thus one expects a power law behavior of the specific heat at low T. In Sec. III, we will compute analytically the low T expansion of the internal energy (6).

B. Quantum spherical *p*-spin glass model (Model II)

We consider a quantum extension of the spherical *p*-spin glass model as studied in Refs. 8 and 26, an interacting system of N continuous spins s_i, $1 < i < N$. This quantum extension consists in considering a continuous spin s_i as an operator associated to a spatial coordinate and introducing its conjugated momentum π_i which satisfies standard commutation relations

$$[s_i, s_j] = [\pi_i, \pi_j] = 0, \quad [\pi_i, s_j] = -i\hbar\delta_{ij}. \tag{13}$$

Following this quantization procedure (see Ref. 7 for alternative quantization rules of multispin Hamiltonians) the quantum *p*-spin glass model is then described by the following Hamiltonian:

$$H[\boldsymbol{\pi}, \mathbf{s}, J] = \frac{\pi^2}{2M} + \sum_{i_1 < \cdots < i_p}^{N} J_{i_1, \ldots, i_p} s_{i_1} \cdots s_{i_p}, \tag{14}$$

where we denote $\pi^2 = \boldsymbol{\pi} \cdot \boldsymbol{\pi}$, with $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_N)$ (similarly for s^2 and $\mathbf{s}$) and impose the spherical constraint

$$\frac{1}{N}\sum_{i=1}^{N} \langle s_i^2 \rangle = 1. \tag{15}$$

In (14), the coupling constants $J_{i1,\ldots,ip}$ are random variables, independently distributed according to a Gaussian distribution of zero mean and variance

$$\overline{J_{i_1,\ldots,i_p}^2} = \frac{\tilde{\mathcal{J}}^2 p!}{2N^{p-1}}. \tag{16}$$

This model (14) is then studied[8] using the formalism of the quantum action in imaginary time (2) together with the use of replicas to implement the average over the disorder (16). After some manipulations, one obtains, in the limit $N \to \infty$ a saddle point equation for the order parameter

$$Q_{ab}(\tau - \tau') = \frac{1}{N}\overline{\langle \mathbf{s}_a(\tau) \cdot \mathbf{s}_b(\tau')\rangle}, \tag{17}$$

where a, b are replica indices. In the limit $k \to 0$, one denotes $Q_{aa}(\tau) = q_d(\tau)$ and parametrizes $Q_{a \neq b}(\tau)$ by $q(u)$ which is τ-independent. Following the authors of Ref. 8, we will work with dimensionless quantities, by redefining the imaginary time $\hat{\tau} = \tilde{J}\tau/\hbar$ and Matsubara frequencies $\hat{\omega}_n = \hbar\omega_n/\tilde{J}$ (in the following we will drop all hats in order to simplify the notations). We also introduce the parameter $\Gamma = \hbar^2/(M\tilde{J})$, which measures the strength of quantum fluctuations. The phase diagram of (14) in the Γ-T plane was found[8] to be characterized by a line $\Gamma_c(T)$ separating a paramagnetic (PM), associated to a diagonal matrix $Q_{ab}(\tau) = q_d(\tau)\delta_{ab}$, from a spin-glass (SG) phase at low T, which we focus on here. The saddle point equations describing this SG phase is solved by a one step RSB ansatz,[8] shown to be exact as in the classical case, such that $q(u) = 0$ for $u < m$ and $q(u) = q_{EA}$ for $u > m$, m being the breakpoint. The internal energy, as a function of the saddle point solution is given by[8]

$$\overline{\langle H \rangle} = \frac{z'}{2} + \frac{p}{4}\int_0^\beta d\tau (q_d^{p-1}(\tau) - q_{EA}^{p-1}) - \frac{p+2}{4}\beta m q_{EA}^p$$
$$- \frac{p+2}{4}\int_0^\beta d\tau (q_d^p(\tau) - q_{EA}^p) \tag{18}$$

with the saddle point equations

$$\tilde{\Sigma}(\omega_n) = \frac{p}{2}\int_0^\beta d\tau(1 - \cos(\omega_n\tau))(q_d^{p-1}(\tau) - q_{EA}^{p-1}), \tag{19}$$

$$z' = \frac{p}{2}\beta m q_{EA}^{p-1}\frac{1 + x_p}{x_p}, \tag{20}$$

and the definitions

$$q_{EA} = 1 - \frac{1}{\beta}\sum_n \frac{1}{\dfrac{\omega_n^2}{\Gamma} + z' + \tilde{\Sigma}(\omega_n)}, \tag{21}$$

$$q_d(\tau) - q_{EA} = \frac{1}{\beta}\sum_n \frac{\cos\omega_n\tau}{\dfrac{\omega_n^2}{\Gamma} + z' + \tilde{\Sigma}(\omega_n)}. \tag{22}$$

The breakpoint is determined by

$$\beta m = x_p \sqrt{\frac{2}{p(x_p + 1)}} q_{EA}^{-p/2}. \tag{23}$$

Combining (20) and (23), one obtains the following useful identity:

$$\frac{1}{z'^2} = \frac{2}{p(1 + x_p)}q_{EA}^{2-p}. \tag{24}$$

Notice that the *p*-spin model is related to the random manifold problem (Model I): the internal energy (18) and the variational equation (19) are indeed formally recovered from Eqs. (6) and (7) by setting $d = 0$ and $\hat{V}(x) = -\frac{1}{2}(1 - (x/2))$.

However here, as it was found in other one step RSB solution,[9,10] one obtains a one parameter family of solutions, indexed by x_p (or equivalently by the breakpoint m). There are then two different ways to determine m.[14] In the statics, m is usually determined by minimizing the free energy: this is the so-called *equilibrium* criterion. The excitation spectrum of the equilibrium SG state is gaped, yielding a specific heat which vanishes *exponentially* at low T. Alternatively, m is determined by imposing the vanishing of the replicon eigenvalue, which leads to the so-called *marginality* condition[8,14]

$$x_p = p - 2. \qquad (25)$$

It has been shown,[8] using a Keldysh mean field approach[26] and performing analytical continuation to imaginary time, that the marginality condition (25) gives indeed the correct solution from the dynamical point of view, i.e., if one considers, in an infinite system, the large waiting time limit—where time translational invariance and equilibrium fluctuation dissipation theorem hold—and finally takes the limit of a vanishing coupling to the thermal bath.

Moreover, this marginal value of x_p was found to be the only one compatible with a *gapless* excitation spectrum.[8] In the $T=0$ limit, it was indeed shown that, in the low frequency limit $\tilde{\Sigma}(\omega_n) \propto |\omega_n| + \mathcal{O}(\omega_n^2)$. Therefore, one expects that the specific heat of the marginally stable SG state vanishes as a power law. In Sec. III, we show how to extract analytically the low T behavior of the marginally stable SG state.

C. Quantum SU(N) spin-glass (Model III)

We consider the Heisenberg quantum spin-glass, defined by the following Hamiltonian:[6,9]

$$H = \frac{1}{N\mathcal{N}} \sum_{ij} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \qquad (26)$$

where the original spin symmetry group SU(2) is extended to SU(N) (Ref. 6) and the large N limit is taken. These $\mathcal{N}$ spins occupy the sites of a fully connected lattice. In (26), the coupling J_{ij} are random variables, independently distributed according to a Gaussian distribution of zero mean and variance

$$\overline{J_{ij}^2} = J^2. \qquad (27)$$

Using the imaginary time path-integral formalism (we will set $\hbar=1$ from the beginning for this model), together with replicas to implement the average over the disorder (27), the model is mapped, in the infinite range limit, onto a self-consistent single site problem described by the action[5]

$$S_{\text{eff}} = S_B - \frac{J^2}{2N} \int_0^\beta d\tau d\tau' \, Q_{ab}(\tau - \tau') \mathbf{S}_a(\tau) \cdot \mathbf{S}_b(\tau'), \qquad (28)$$

where S_B is the Berry phase imposing the spin commutation relations,[6] together with the self-consistent equation:

$$Q_{ab}(\tau - \tau') = \frac{1}{N^2} \langle \mathbf{S}_a(\tau) \cdot \mathbf{S}_b(\tau') \rangle_{S_{\text{eff}}}, \qquad (29)$$

where $\langle \cdots \rangle_{S_{\text{eff}}}$ stands for an average computed with the action S_{eff}. Using a bosonic representation of the spin operator S in terms of Schwinger bosons, $S_{\alpha\beta} = b_\alpha^\dagger b_\beta - S\delta_{\alpha\beta}$ with the constraint $\Sigma_\alpha b_\alpha^\dagger b_\alpha = SN$, this model (26) can be described analytically in the limit $N \to \infty$ which then constitutes a mean field theory of the fully connected model (26) where the spins have the symmetry SU(2). In this limit, the original self-consistent equation (29) reduces to an equation for the boson Green's function $\mathcal{G}^{ab}(\tau) \equiv -\langle Tb^a(\tau)b^{\dagger b}(0) \rangle$. In the limit $k \to 0$, one parametrizes $\mathcal{G}^{aa}(\tau)$ by $\tilde{\mathcal{G}}(\tau) - \tilde{g}$, such that $\lim_{\tau \to \infty} \tilde{\mathcal{G}}(\tau) = 0$ at $T=0$ and, $\mathcal{G}^{a \neq b}(\tau)$ by $-g(u)$, which is τ-independent. The phase diagram of (26) in the $T-S$ plane has been established in the large N limit.[9] A line $S_c(T)$ separates a paramagnetic phase, described by a diagonal matrix in replica space $\mathcal{G}^{ab}(\tau) = \delta_{ab}\tilde{\mathcal{G}}(\tau)$, and where several crossovers were found to occur in the quantum regime,[9] from a spin glass phase, which we focus on here. In this SG phase, the saddle point equations are solved by a one step RSB ansatz, such that $g(u) = 0$ for $u < x$ and $g(u) = g$ for $u > x$, x being the breakpoint and $\tilde{g} = g$. The starting point of our computation of the specific heat is the expression for the internal energy per unit volume[9]

$$\overline{\langle H \rangle} = -\frac{J^2}{2} \int_0^\beta d\tau (\tilde{\mathcal{G}}(\tau) - g)^2 (\tilde{\mathcal{G}}(-\tau) - g)^2 - \frac{J^2}{2} \beta(x-1)g^4 \qquad (30)$$

in terms of the saddle point solution

$$\hat{\Sigma}(i\nu_n) = J^2 \int_0^\beta d\tau (e^{-i\nu_n \tau} - 1)((\tilde{\mathcal{G}}(\tau) - g)^2 (\tilde{\mathcal{G}}(-\tau) - g) + g^3), \qquad (31)$$

$$\beta x = \frac{1}{Jg^2} \left(\frac{1}{\Theta} - \Theta \right), \qquad (32)$$

with the definitions

$$g = S + \tilde{\mathcal{G}}(\tau = 0^-),$$

$$\tilde{\mathcal{G}}(i\nu_n) = \frac{1}{i\nu_n - \dfrac{Jg}{\Theta} - \hat{\Sigma}(i\nu_n)}, \qquad (33)$$

where ν_n is a bosonic Matsubara frequency. Similarly to the spherical p-spin model (19)–(23), one obtains a one-parameter family of solutions, parametrized by Θ, or equivalently by the breakpoint x. Here also, if one chooses the *equilibrium* criterion, the excitation spectrum is gapped. Instead, if one imposes the vanishing of the replicon eigenvalue, one obtains the *marginality* condition[9]

$$\Theta = \Theta_R = \frac{1}{\sqrt{3}}. \tag{34}$$

Using an expansion in $1/S$—similar to the semiclassical expansion for the elastic manifold[10]—it has been shown explicitly[9] that the marginality condition (34) is the only one compatible with a gapless excitation spectrum, such that $\hat{\Sigma}''(\omega) \propto \omega + \mathcal{O}(\omega^2)$, where $\hat{\Sigma}(\nu_n \to -i\omega + 0^+) = \hat{\Sigma}(\omega) + i\hat{\Sigma}''(\omega)$. Although the connexion between this Matsubara formalism using the marginality condition (34) and the true Hamiltonian dynamics in real time at the mean field level has not yet been established for the present case, the study of similar one step RSB solutions for which this connection has been done,[8,27] suggests that (34) gives indeed the correct solution from the dynamical point of view. On physical grounds, in the present case of the ordered phase of a quantum SG with continuous symmetry, this choice seems also natural as one indeed expects a gapless excitation spectrum.[9]

Although the specific heat in the equilibrium SG state vanishes exponentially at low T, we focus here on the low T behavior of $C_v(T)$ in the marginal state (34), which is instead expected to vanish as a power law.[9]

III. LOW TEMPERATURE ANALYSIS

In this section, we compute the low temperature expansion of the internal energy for the different models presented before, from which we directly obtain the specific heat $C_v(T)$. To do so, we start by deriving some general identities, which form the background of our analysis. The finite temperature behavior of the internal energy requires the low temperature expansion of the saddle point equations, which we then present in detail. For that purpose, we will use the notation, for any quantity of interest Q, $Q = \Sigma_{n=0}^{\infty} T^n Q^{(n)}$. We finally turn to the behavior of $C_v(T)$ in the last paragraph of this section.

A. General properties

We first focus on the two first problems evocated here (1) and (14), which show, formally, a strong similarity. In particular, at variance with Model III (26), these two systems exhibit a two point Green's function (10) and (22) which is invariant under the transformation $\omega_n \to -\omega_n$. We focus on the low T behavior of integrals over imaginary time τ which enter both the variational equation and the computation of the internal energy. We will use the notations of the disordered elastic Hamiltonian (7) and (11), the transposition to the p-spin model being straightforward.

We first suppose that Σ and $I(\omega_n)$, as a function of ω_n are independent of T, i.e., $\Sigma^{(n)} = 0$, $\forall n > 0$ and similarly for $I(\omega_n)$ with $I(\omega_n) \sim |\omega_n| + \mathcal{O}(\omega_n^2)$. Then, as shown in Appendix A, one has the low temperature expansion, for any function $\mathcal{H}(X)$ that can be expanded as a power serie around 0, $\mathcal{H}(X) = \Sigma_{k=0}^{\infty} a_k X^k$,

$$\int_0^{\beta\hbar} d\tau \cos(\omega_n\tau)(\mathcal{H}(\widetilde{B}(\tau)) - \mathcal{H}(B)) = C^{\mathrm{st}} + \mathcal{O}(T^4), \tag{35}$$

where here, and in the following, C^{st} stands for a (generic) quantity independent of T (it may however depend on the Matsubara frequency). Indeed, the gapless structure of the spectral function suggests that only even powers of T should enter this expansion (35). But as shown in detail in Appendix A, one can check explicitly that the term $\propto T^2$ *vanishes*. This property (35) is rather independent of the structure of the variational equations, only requiring $I''(\omega) \propto \omega$ at low frequency.

For the other model discussed here (26), one could not show a so general statement (35) involving any function $\mathcal{H}$, due to the absence of the symmetry $i\nu_n \to -i\nu_n$ of the Green's function $\widetilde{\mathcal{G}}(i\nu_n)$ (33), which renders the calculations more subtle. Nevertheless, with the same assumptions as above that the complete self-energy $Jg/\theta + \hat{\Sigma}(i\nu_n)$ (33) is independent of T, with $\hat{\Sigma}(i\nu_n) \propto |\nu_n| + \mathcal{O}(\nu_n^2)$, for the particular integrals involved here, one can show (see Appendix B for more detail),

$$\int_0^{\beta} d\tau((\widetilde{\mathcal{G}}(\tau) - g)^2(\widetilde{\mathcal{G}}(-\tau) - g)^2 - g^4) = C^{\mathrm{st}} + \mathcal{O}(T^4),$$

$$\int_0^{\beta} d\tau e^{-i\nu_n\tau}((\widetilde{\mathcal{G}}(\tau) - g)^2(\widetilde{\mathcal{G}}(-\tau) - g) + g^3) = C^{\mathrm{st}} + \mathcal{O}(T^4). \tag{36}$$

These properties are very useful tools to investigate the low T behaviors of physical quantities in these models.

B. Variational equations

We now turn to the low T expansion of the saddle point equations for the three different models.

1. Model I

Most of the properties presented here, and their extension to the spin-glass models (14) and (26), have been suggested by the expansion in powers of $\hbar$, keeping $\beta\hbar$ fixed. We will shortly remind here the main features of the semiclassical expansion of the variational equations (7) and (8). We use the notation, for any quantity $Q = \Sigma_{k=0}^{\infty} \hbar^k Q_k(\beta\hbar)$. Although at lowest order, the complete solution of the variational equations[10] shows that Σ_0 and $I_0(\omega_n)$ are independent of T, Σ_1 and $I_1(\omega_n)$ become T dependent at the next order.[25] At low temperature the following structure was *explicitly* obtained

$$\Sigma_1 = \Sigma_1^{(0)} + \left(\frac{T}{\hbar}\right)^2 \Sigma_1^{(2)} + O((T/\hbar)^4), \tag{37}$$

$$I_1(\omega_n) = I_1^{(0)}(\omega_n) + \left(\frac{T}{\hbar}\right)^2 I_1^{(2)}(\omega_n) + O((T/\hbar)^4),$$

$$I_1^{(2)}(\omega_n) = -(1 - \delta_{n,0})\Sigma_1^{(2)}, \tag{38}$$

such that the finite temperature corrections of $\Sigma_1 + I_1(\omega_n)$ are $\propto T^4$ for $\omega_n \neq 0$. It was also noticed that the peculiar term (38), once inserted in B (9), generates odd powers of T, the lowest one being T^3, in the low T expansion at higher order in the expansion in $\hbar$. We extend here these properties (37) and (38) independently on the semiclassical approximation. We show indeed that the following expansion, up to order $\mathcal{O}(T^4)$:

$$\Sigma = \Sigma^{(0)} + T^2\Sigma^{(2)} + T^3\Sigma^{(3)},$$

$$I(\omega_n) = I^{(0)}(\omega_n) - (1 - \delta_{n,0})(T^2\Sigma^{(2)} + T^3\Sigma^{(3)}), \tag{39}$$

with $I^{(0)}(\omega_n) \propto |\omega_n| + \mathcal{O}(\omega_n^2)$, is a consistent solution of the variational equations (7) and (8). To do so, we compute the low T expansion of the r.h.s. of the equation for $I(\omega_n)$ given the forms (39). We thus need an extension of the general property (35), when Σ and $I(\omega_n)$ have the form given in Eq. (39). As shown in Appendix A, the low T behavior of such integrals over τ (35) are in that case given by

$$\int_0^{\beta\hbar} d\tau \cos(\omega_n\tau)(\mathcal{H}(\tilde{B}(\tau)) - \mathcal{H}(B))$$

$$= C^{\text{st}} + 2\hbar\delta_{n,0}(T^2\Sigma^{(2)} + T^3\Sigma^{(3)})J_2(\Sigma^{(0)})\mathcal{H}'(B^{(0)}) + \mathcal{O}(T^4). \tag{40}$$

If one applies this general formula (40) with $\mathcal{H} = 2\hat{V}'(B)$ to the r.h.s. of Eq. (7), one obtains up to order $\mathcal{O}(T^4)$,

$$\frac{2}{\hbar}\int_0^{\beta\hbar} d\tau(1 - \cos(\omega_n\tau))(\hat{V}'(\tilde{B}(\tau)) - \hat{V}'(B))$$

$$= C^{\text{st}} + 4(1 - \delta_{n,0})(T^2\Sigma^{(2)} + T^3\Sigma^{(3)})J_2(\Sigma^{(0)})$$

$$\times J_2(\Sigma^{(0)})\hat{V}''(B^{(0)}) \tag{41}$$

$$= C^{\text{st}} - (1 - \delta_{n,0})(T^2\Sigma^{(2)} + T^3\Sigma^{(3)}), \tag{42}$$

where, in the last line, we have used the marginality condition (8) at $T = 0$. This relation (41) thus shows explicitly the consistency of the low T expansion (39) proposed for the exact solution of the variational equations. Importantly, although the general property shown above (40) holds independently of the saddle point equation, a solution such as (39) is consistent *provided the marginality condition holds*.

2. Model II

Inspired by the previous analysis, we show that a consistent solution of the variational equations for the quantum p-spin model (19) and (20) is given, up to order $\mathcal{O}(T^4)$ by

$$z' = z'^{(0)} + T^2z'^{(2)} + T^3z'^{(3)},$$

$$\tilde{\Sigma}(\omega_n) = \tilde{\Sigma}^{(0)}(\omega_n) - (1 - \delta_{n,0})(T^2z'^{(2)} + T^3z'^{(3)}), \tag{43}$$

with $\tilde{\Sigma}^{(0)}(\omega_n) \propto |\omega_n| + \mathcal{O}(\omega_n^2)$. This is shown by using the general low T expansion, an extension of (40) to the present case (43),

$$\int_0^{\beta\hbar} d\tau \cos(\omega_n\tau)(\mathcal{H}(q_d(\tau)) - \mathcal{H}(q_{\text{EA}}))$$

$$= C^{\text{st}} - \delta_{n,0}(T^2z'^{(2)} + T^3z'^{(3)})\frac{\mathcal{H}'(q_{\text{EA}}^{(0)})}{(z'^{(0)})^2} + \mathcal{O}(T^4). \tag{44}$$

Thus applying (44) with $\mathcal{H}(X) = (p/2)X^{p-1}$ yields the low T expansion of the r.h.s. of the variational equation (19) up to order $\mathcal{O}(T^4)$,

$$\frac{p}{2}\int_0^\beta d\tau(1 - \cos(\omega_n\tau))(q_d^{p-1}(\tau) - q_{\text{EA}}^{p-1})$$

$$= C^{\text{st}} - \frac{p(p-1)}{2}(1 - \delta_{n,0})(T^2z'^{(2)} + T^3z'^{(3)})\frac{(q_{\text{EA}}^{(0)})^{p-2}}{(z'^{(0)})^2}$$

$$= C^{\text{st}} - \frac{p-1}{1 + x_p}(1 - \delta_{n,0})(T^2z'^{(2)} + T^3z'^{(3)}), \tag{45}$$

where we have used, in the last line, the identity (24). Thus, one sees on (45) that the expression given in Eq. (43) is a consistent solution of (19) provided $x_p = p - 2$, i.e., the solution is marginally stable (25).

3. Model III

For this model, we show that, similarly to the two previous ones, a consistent solution of the variational equations (31) is given, up to order $\mathcal{O}(T^4)$ by

$$g = g^{(0)} + T^2g^{(2)} + T^3g^{(3)},$$

$$\hat{\Sigma}(i\nu_n) = \hat{\Sigma}^{(0)}(i\nu_n) + \frac{J}{\Theta}(\delta_{n,0} - 1)(T^2g^{(2)} + T^3g^{(3)}), \tag{46}$$

with $\hat{\Sigma}^{(0)}(i\nu_n) \propto |\nu_n| + \mathcal{O}(\nu_n^2)$. To show the consistency of this solution (46), we perform the low temperature expansion of the r.h.s. of the equation for $\hat{\Sigma}(i\nu_n)$ (31) given (46). One obtains up to order $\mathcal{O}(T^4)$,

$$J^2\int_0^\beta d\tau(e^{-i\nu_n\tau} - 1)[(\tilde{\mathcal{G}}(\tau) - g)^2(\tilde{\mathcal{G}}(-\tau) - g) + g^3]$$

$$= C^{\text{st}} + (\delta_{n,0} - 1)3\Theta J(T^2g^{(2)} + T^3g^{(3)}). \tag{47}$$

Again, this expansion (47) shows that the structure exhibited in (46) is a consistent solution of the variational equation (31) provided $3\Theta = 1/\Theta$, i.e., $\Theta = \Theta_R = 1/\sqrt{3}$, the marginality condition (34).

C. Specific heat: Low temperature expansion

We now turn to the computation of the low temperature behavior of the specific heat.

1. Model I

Our starting point is the expression for the variational internal energy $\langle H \rangle$ given in (6). We first analyze the low temperature behavior of the first term in (6), namely the sum

over Matsubara frequencies. Importantly, we notice that, in this sum, the contribution of the mode $\omega_n=0$ is independent of the peculiar structure of $I(\omega_n)$ exhibited in (38): this allows to avoid ambiguities coming from the analytic continuation of such a term $\propto(1-\delta_{n,0})$ in (38). We can thus safely transform this discrete sum in an integral:

$$\frac{1}{\beta}\sum_n \int_q \frac{cq^2+\Sigma+I(\omega_n)}{cq^2+\Sigma+M\omega_n^2+I(\omega_n)}$$

$$= \int_{-\infty}^{+\infty} \frac{d\omega}{\pi}\hbar\omega\rho_{\text{DOS}}(\omega)f_B(\omega), \qquad (48)$$

$$\rho_{\text{DOS}}(\omega) = \int_q \frac{-\frac{c}{v^2}\omega I''(\omega)}{\left(cq^2-\frac{c}{v^2}\omega^2+\Sigma+\hat{I}'(\omega)^2+(I''(\omega))^2\right)}, \qquad (49)$$

where $\rho_{\text{DOS}}(\omega)$ is the density of states. Using that $\rho_{\text{DOS}}(\omega) = \rho_{\text{DOS}}^{(0)}(\omega)+T^4\rho_{\text{DOS}}^{(4)}(\omega)$ (39), together with the low frequency behavior $\rho_{\text{DOS}}^{(0)}\propto\omega^2$, one obtains straightforwardly

$$\frac{1}{\beta}\sum_n \int_q \frac{cq^2+\Sigma+\hat{I}(\omega_n)}{cq^2+\Sigma+M\omega_n^2+\hat{I}(\omega_n)} = C^{\text{st}}+\mathcal{O}(T^4). \qquad (50)$$

One performs the low T expansion of the second term in (6), the integral over τ, using the property (40) which holds for the solution of the variational equations we have found (39):

$$\int_0^{\beta\hbar} d\tau(F(\tilde{B}(\tau))-F(B))$$

$$= C^{\text{st}}+2(T^2\Sigma^{(2)}+T^3\Sigma^{(3)})J_2(\Sigma^{(0)})F'(B^{(0)})+\mathcal{O}(T^4). \qquad (51)$$

One finally tackles the analysis of the last term in (6), the integral over w, by using the property that $B(w)$ is independent of T for $w<w_c$,[25]

$$\int_0^{w_c} dw(F(B(w))-F(B))$$

$$= C^{\text{st}}+w_c^{(0)}(T^2B^{(2)}+T^3B^{(3)})F'(B^{(0)})+\mathcal{O}(T^4). \qquad (52)$$

This property (52) is then obvious for a full RSB solution where $B(w_c^-)=B$ and was shown[25] to hold also for the marginal one step RSB solution in $d=2$.

Using the consequence of the marginality condition (12), one thus sees clearly that the quadratic and cubic contributions to this low T behavior *cancel* between Eq. (51) and Eq. (52). And this yields

$$\overline{\langle H\rangle} = C^{\text{st}}+\mathcal{O}(T^4). \qquad (53)$$

This results in the low temperature behavior of the specific heat:

$$C_v(T)\propto T^3+\mathcal{O}(T^4). \qquad (54)$$

Although the coefficient of this cubic term is very hard to extract by the method presented here, it has been explicitly computed at the lowest order in the aforementioned semiclassical expansion and it was found to be nonzero.[12]

2. Model II

We analyze the low T behavior of the internal energy within the variational method (18) of the quantum p-spin model. We treat separately the two lines of (18). Using (44), one obtains

$$\frac{p}{4}\int_0^\beta d\tau(q_d^{p-1}(\tau)-q_{\text{EA}}^{p-1})$$

$$= C^{\text{st}}-(T^2z'^{(2)}+T^3z'^{(3)})\frac{p(p-1)}{4}\frac{(q_{\text{EA}}^{(0)})^{p-2}}{(z'^{(0)})^2}+\mathcal{O}(T^4). \qquad (55)$$

Using this expansion and the identity (24) one obtains that the quadratic and cubic terms in the low T expansion of the first line in (18) cancel provided $x_p=p-2$ (25):

$$\frac{z'}{2}+\frac{p}{4}\int_0^\beta d\tau(q_d^{p-1}(\tau)-q_{\text{EA}}^{p-1}) = C^{\text{st}}+\mathcal{O}(T^4). \qquad (56)$$

We now focus on the second line of (18). Using (44), one has immediately

$$-\frac{p+2}{4}\int_0^\beta d\tau(q_d^p(\tau)-q_{\text{EA}}^p)$$

$$= C^{\text{st}}+(T^2z'^{(2)}+T^3z'^{(3)})\frac{p(p+2)}{4}\frac{(q_{\text{EA}}^{(0)})^{p-1}}{(z'^{(0)})^2}+\mathcal{O}(T^4). \qquad (57)$$

Finally combining (23) with (24), one obtains the expansion of the remaining term in (18) up to order $\mathcal{O}(T^4)$:

$$-\frac{p+2}{4}\beta mq_{\text{EA}}^p = -\frac{p(p+2)}{4}\frac{z'^{(2)}T^2+z'^{(3)}T^3}{(z'^{(0)})^2}(q_{\text{EA}}^{(0)})^{p-1}. \qquad (58)$$

Collecting the different contributions to $\overline{\langle H\rangle}$ (55), (57), and (58), one obtains

$$\overline{\langle H\rangle} = C^{\text{st}}+\mathcal{O}(T^4) \qquad (59)$$

which leads to the low temperature behavior of the specific heat

$$C_v(T)\propto T^3+\mathcal{O}(T^4). \qquad (60)$$

3. Model III

The low T expansion of the internal energy of the Heisenberg spin-glass model (30) is performed using the relation derived in Appendix B which, given the solution of the variational equations we have found, yield

$$-\frac{J^2}{2}\int_0^\beta d\tau[(\tilde{\mathcal{G}}(\tau)-g)^2(\tilde{\mathcal{G}}(-\tau)-g)^2-g^4]$$

$$= C^{\mathrm{st}} + 2J\Theta g^{(0)}(g^{(2)}T^2 + g^{(3)}T^3) + \mathcal{O}(T^4). \quad (61)$$

The expansion of the last term in (30) is straightforwardly computed using the relation (32)

$$-\frac{J^2}{2}\beta x g^4 = C^{\mathrm{st}} - J g^{(0)}\left(\frac{1}{\Theta}-\Theta\right)(g^{(2)}T^2 + g^{(3)}T^3) + \mathcal{O}(T^4).$$

$$(62)$$

Thus combining Eq. (61) and Eq. (62), one sees that the quadratic and cubic terms come with a prefactor $3\Theta - 1/\Theta$, which thus vanishes for the marginally stable solution, corresponding to $\Theta = \Theta_R = 1/\sqrt{3}$. This yields the low temperature behavior of the specific heat

$$C_v(T) \propto T^3 + \mathcal{O}(T^4). \quad (63)$$

For this model too, the amplitude of the cubic term has been computed in a $1/S$, semiclassical, expansion, and found to be nonzero.[9]

IV. CONCLUSION

To sum up, we have computed the low temperature specific heat of a rather wide class of quantum disordered systems with continuous degrees of freedom, using a mean field approximation, including disordered elastic systems in $d \geq 2$, the spherical p-spin-glass and the Heisenberg spin-glass. For all these models, we have obtained that the Mean Field approximation yields the low T behavior $C_v(T) \propto T^3$ (54), (60), and (63). For the quantum p-spin model, our result is at variance with the linear behavior claimed from the previous analysis in Ref. 8. There, the variational equations were solved in the $T=0$ limit. This solution[8] at $T=0$, which is recovered by the present analysis, exhibits the low frequency behavior of the self-energy $\tilde{\Sigma}^{(0)}(\omega_n) \propto |\omega_n|$ (43), which in turns leads to the low temperature behavior of the Edwards-Anderson parameter $q_{\mathrm{EA}} - q_{\mathrm{EA}}^{(0)} = \mathcal{O}(T^2)$. One would thus conclude,[8] from this low T behavior together with the expression for the internal energy (18), that $\overline{\langle H \rangle} = C^{\mathrm{st}} + \mathcal{O}(T^2)$ at low T. However, as we have shown here, a thorough computation of the coefficient of this quadratic contribution to $\overline{\langle H \rangle}$ (which was not performed in Ref. 8) shows that it actually vanishes. In addition, we have shown that the cubic contribution to $\overline{\langle H \rangle}$ also vanishes (59). For the Heisenberg spin-glass model, the cancellation of the linear term in $C_v(T)$ obtained here is in agreement with the numerical solution of the saddle point equation of Ref. 11. And the nontrivial structure of the saddle point solution elucidated here (46), which plays a crucial role in the cancellation of the T^2 term in $C_v(T)$, could help to clarify numerically the status of the quadratic contribution to $C_v(T)$ obtained in Ref. 11.

This mean field result $C_v(T) \propto T^3$ is at variance with the linear behavior commonly expected from two-level systems.[4] As we have shown, the cancellation of the linear and quadratic contributions to $C_v(T)$ strongly relies upon the margin-

ality condition. And it is worthwhile to notice that the physical picture associated to this marginal stability criterion, which enforces the (quantum) dynamics along the flat directions of the free energy landscape, seems qualitatively different from the argument stemming from TLS.[4]

For the case of manifolds, our result, within the Gaussian Variational Approximation also applies to nonperiodic elastic structures, e.g., domain walls, when they can be solved by a full RSB ansatz (or its limiting case of a marginal one step RSB). And although this mean field approach is always an approximation for the periodic case, it becomes exact for the nonperiodic one, in the limit where the number of components of the displacement field becomes infinite, as it is for the spherical p-spin-glass model (14), in the limit $N \to \infty$, or for the Heisenberg spin-glass model (26) when both $N, \mathcal{N} \to \infty$. An outstanding question remains to know whether and how this result is modified away from mean field, which clearly deserves further numerical and analytical investigations.

APPENDIX A: LOW TEMPERATURE EXPANSION FOR MODEL I AND MODEL II: DETAILED CALCULATIONS

In this appendix, we show some general properties of the low temperature expansion of multiple sums over Matsubara frequencies. We will use here the notations of the elastic manifold (7)–(11) (the extension to the quantum p-spin model being straightforward).

1. A first stage with multiple Matsubara sums

To begin with, we restrict ourselves to the case where Σ and $I(\omega_n)$ as a function of ω_n do not depend on temperature T, for all ω_n including the mode $\omega_n=0$. We also assume the low frequency behavior $I(\omega_n) \sim |\omega_n| + \mathcal{O}(\omega_n^2)$ (7). Here we are interested in the low temperature expansion of the following quantity (35):

$$\int_0^{\beta\hbar} d\tau(\mathcal{H}(\tilde{B}(\tau)) - \mathcal{H}(B)). \quad (A1)$$

As we will see, the first nonvanishing finite temperature correction is *a priori* of order T^2: we show that this contribution in fact cancels for any (smooth enough) function $\mathcal{H}(X)$. To show this cancellation, we show the following property, for any integer m:

$$\int_0^{\beta\hbar} d\tau(\tilde{B}(\tau)^m - B^m) = C^{\mathrm{st}} + \mathcal{O}(T^4). \quad (A2)$$

We introduce the notation

$$K(\omega_n) = J_1(M\omega_n^2 + \Sigma + I(\omega_n)). \qquad (A3)$$

Inserting the definitions of B (9) and $\tilde{B}(\tau)$ (10) in (A2), one obtains

$$\int_0^{\beta\hbar} d\tau (\tilde{B}(\tau)^m - B^m)$$

$$= \frac{(2\hbar)^m}{(\beta\hbar)^m} \sum_{n_1,\dots,n_m} \sum_{k=1}^m (-1)^k C_m^k$$

$$\times \int_0^{\beta\hbar} d\tau \cos(\omega_{n_1}\tau) \cdots \cos(\omega_{n_k}\tau)$$

$$\times K(\omega_{n_1}) \cdots K(\omega_{n_k}) K(\omega_{n_{k+1}}) \cdots K(\omega_{n_m}), \quad (A4)$$

where $C_m^k = m!/((m-k)!k!)$. Performing the integral over τ, using the property of parity $K(\omega_n) = K(-\omega_n)$, this expression can be written as

$$\int_0^{\beta\hbar} d\tau (\tilde{B}(\tau)^m - B^m)$$

$$= -2\hbar J_1(\Sigma) m B^{m-1} + \sum_{k=2}^m (-1)^k C_m^k$$

$$\times \frac{(2\hbar)^k}{(\beta\hbar)^{k-1}} \sum_{n_1,\dots,n_{k-1}} K(\omega_{n_1}) \cdots K(\omega_{n_{k-1}})$$

$$\times K(\omega_{n_1} + \cdots + \omega_{n_{k-1}}) B^{m-k}. \qquad (A5)$$

We now focus on the low temperature expansion of the multiple sum over Matsubara frequencies. In that purpose, we use the spectral representation of the Green's function:

$$\frac{1}{cq^2 + M\omega_n^2 + \Sigma + I(\omega_n)} = -\int_{-\infty}^{\infty} \frac{d\omega}{\pi} A(q,\omega) \frac{1}{i\omega_n - \omega},$$

$$A(q,\omega) = \frac{I''(\omega)}{(cq^2 - M\omega^2 + \Sigma + I'(\omega))^2 + (I''(\omega))^2}, \quad (A6)$$

where the spectral function $A(q,\omega)$ is the imaginary part of the retarded function and we remind $I(\omega_n \to -i\omega + 0^+) = I'(\omega) + iI''(\omega)$. All the temperature dependence is then contained in the different Bose factors:

$$\frac{1}{(\beta\hbar)^{k-1}} \sum_{n_1,\dots,n_{k-1}} K(\omega_{n_1}) \cdots K(\omega_{n_{k-1}}) K(\omega_{n_1} + \cdots + \omega_{n_{k-1}})$$

$$= \frac{(-1)^k}{(\pi)^k} \int d\epsilon_1 \cdots d\epsilon_k A(\epsilon_1) \cdots A(\epsilon_k) \qquad (A7)$$

$$\times \frac{1}{(\beta\hbar)^{k-1}} \sum_{n_1,\dots,n_{k-1}} \frac{1}{i\omega_{n_1} - \epsilon_1} \cdots \frac{1}{i\omega_{n_1} + \cdots + i\omega_{n_{k-1}} - \epsilon_k}$$

$$= \frac{1}{\pi^k} \int d\epsilon_1 \cdots d\epsilon_k A(\epsilon_1) \cdots A(\epsilon_k)(f_B(\epsilon_1) - f_B(\epsilon_k)) \quad (A8)$$

$$\times (f_B(\epsilon_2) - f_B(\epsilon_k - \epsilon_1))(f_B(\epsilon_3) - f_B(\epsilon_k - \epsilon_1 - \epsilon_2)) \cdots$$

$$\frac{(f_B(\epsilon_{k-1}) - f_B(\epsilon_k - \epsilon_1 - \epsilon_2 - \cdots - \epsilon_{k-2}))}{\epsilon_k - (\epsilon_1 + \cdots + \epsilon_{k-1})}, \qquad (A9)$$

where $\mathcal{A}(\omega) = \int_q A(q,\omega)$ and $f_B(\epsilon)$ is the Bose factor. This expression (A7) has a very interesting structure which allows us to extract simply the term of order T^2. Indeed, considering the low temperature expansion of the following term (which is analogous of a Sommerfeld expansion in the fermionic case) for any function $\mathcal{H}(x)$ with $\mathcal{H}(\mu) = 0$

$$\int d\epsilon \mathcal{H}(\epsilon) f_B(\epsilon - \mu) = -\int_{-\infty}^{\mu} d\epsilon \mathcal{H}(\epsilon) + \left(\frac{T}{\hbar}\right)^2 \frac{\pi^2}{3} H'(\mu) + \mathcal{O}(T^4) \qquad (A10)$$

notice of course that the assumption $\mathcal{H}(\mu = 0)$ is of course crucial here (A10). This expansion (A10) allows us to write formally the Bose factor, when inserted in an integral over frequency ϵ:

$$f_B(\epsilon - \mu) \equiv -\theta(-\epsilon + \mu) + \left(\frac{T}{\hbar}\right)^2 \frac{\pi^2}{3} \delta(\epsilon - \mu)\partial_\epsilon + \mathcal{O}(T^4), \qquad (A11)$$

where $\theta(x)$ is the step function $[\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ if $x < 0]$, and the notation ∂_ϵ stands for a derivative acting on the function which enters multiplicatively with $f_B(\epsilon)$ the integral over ϵ. This form (A11) is very suitable to extract the coefficient of the term of order $\mathcal{O}(T^2)$ in (A7). Indeed, using (A11), one discovers in (A7), that only the terms where the Bose factors have *only one frequency* in their argument do contribute to order $\mathcal{O}(T^2)$. If we expand, for instance $f_B(\epsilon_k - \epsilon_1 - \epsilon_2)$ in (A7) one obtains, for $k \geqslant 3$

$$\frac{1}{(\beta\hbar)^2} \frac{\pi^2}{3} \frac{1}{\pi^k} \int d\epsilon_1 \cdots d\epsilon_k \delta(-\epsilon_k + \epsilon_1 + \epsilon_2)\partial_{\epsilon_k}$$

$$\times \left[\mathcal{A}(\epsilon_1) \cdots \mathcal{A}(\epsilon_k)(-\theta(-\epsilon_1) + \theta(-\epsilon_k)) \right.$$

$$\times (-\theta(-\epsilon_2) + \theta(-\epsilon_k + \epsilon_1)) \cdots$$

$$\left. \frac{(-\theta(-\epsilon_{k-1}) + \theta(-\epsilon_k + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-2}))}{\epsilon_k - (\epsilon_1 + \cdots + \epsilon_{k-1})} \right]. \qquad (A12)$$

Because of the function $\delta(\epsilon_k - \epsilon_1 - \epsilon_2)$, one must take the derivative of the terms $\partial_{\epsilon_k}(-\theta(-\epsilon_2) + \theta(-\epsilon_k + \epsilon_1)) = -\delta(-\epsilon_k + \epsilon_1)$ (which is *a priori* the only nonvanishing term), which yields

$$\frac{1}{(\beta\hbar)^2} \frac{\pi^2}{3} \frac{1}{\pi^k} \int d\epsilon_1 \cdots d\epsilon_k \delta(-\epsilon_k + \epsilon_1 + \epsilon_2) \delta(-\epsilon_k + \epsilon_1)$$

$$\times \mathcal{A}(\epsilon_1) \cdots \mathcal{A}(\epsilon_k)(-\theta(-\epsilon_1) + \theta(-\epsilon_k)) \cdots$$

$$\frac{(-\theta(-\epsilon_{k-1}) + \theta(-\epsilon_k + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-2}))}{\epsilon_k - (\epsilon_1 + \cdots + \epsilon_{k-1})} = 0. \qquad (A13)$$

We can easily generalize this mechanism to Bose factors

having more than 3 frequencies in their arguments. The case of Bose factors having 2 frequencies in their argument has to be treated separately. This one, i.e., $f_B(\epsilon_k - \epsilon_1)$ in (A7) yields

$$
\frac{1}{(\beta\hbar)^2} \frac{\pi^2}{3} \frac{1}{\pi^k} \int d\epsilon_1 \cdots d\epsilon_k \delta(-\epsilon_k + \epsilon_1) \partial_{\epsilon_k} \Bigg[\mathcal{A}(\epsilon_1) \cdots \mathcal{A}(\epsilon_k)
$$
$$
\times (-\theta(-\epsilon_1) + \theta(-\epsilon_k))
$$
$$
\times (-\theta(-\epsilon_3) + \theta(-\epsilon_k + \epsilon_1 + \epsilon_2)) \cdots
$$
$$
\frac{(-\theta(-\epsilon_{k-1}) + \theta(-\epsilon_k + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-2}))}{\epsilon_k - (\epsilon_1 + \cdots + \epsilon_{k-1})} \Bigg]. \quad (A14)
$$

Here again, one has to take the derivative $\partial_{\epsilon_k}(-\theta(-\epsilon_1) + \theta(-\epsilon_k)) = -\delta(-\epsilon_k)$, which yields

$$
\frac{1}{(\beta\hbar)^2} \frac{\pi^2}{3} \frac{1}{\pi^k} \int d\epsilon_1 \cdots d\epsilon_k \delta(-\epsilon_k + \epsilon_1) \delta(-\epsilon_k) \Bigg[\mathcal{A}(\epsilon_1) \cdots \mathcal{A}(\epsilon_k)
$$
$$
\times (-\theta(-\epsilon_3) + \theta(-\epsilon_k + \epsilon_1 + \epsilon_2)) \cdots
$$
$$
\frac{(-\theta(-\epsilon_{k-1}) + \theta(-\epsilon_k + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-2}))}{\epsilon_k - (\epsilon_1 + \cdots + \epsilon_{k-1})} \Bigg] = 0,
$$
$$
(A15)
$$

where we have used $\mathcal{A}(0) = 0$. We conclude from this analysis that only the Bose factors having only one frequency in their argument, such as $f_B(\epsilon_1), f_B(\epsilon_2), \ldots$ do contribute to the coefficient of order T^2 in (A7). Notice that they all give the same contribution, thus the low temperature expansion of (A7) can be written for $k \geq 3$

$$
\frac{1}{(\beta\hbar)^{k-1}} \sum_{n_1, \ldots, n_{k-1}} K(\omega_{n_1}) \cdots K(\omega_{n_{k-1}}) K(\omega_{n_1} + \cdots + \omega_{n_{k-1}})
$$
$$
= \mathcal{J}_k + \frac{k}{(\beta\hbar)^2} \frac{\pi}{3} \mathcal{A}'(0) \mathcal{J}_{k-1} + O(\beta^{-4}),
$$

$$
\mathcal{J}_k = \frac{1}{\pi^k} \int d\epsilon_1 \cdots d\epsilon_k \mathcal{A}(\epsilon_1) \cdots \mathcal{A}(\epsilon_k)(-\theta(-\epsilon_1) + \theta(-\epsilon_k))
$$
$$
\times (-\theta(-\epsilon_2) + \theta(-\epsilon_k + \epsilon_1))
$$
$$
\times (-\theta(-\epsilon_3) + \theta(-\epsilon_k + \epsilon_1 + \epsilon_2)) \cdots
$$
$$
\frac{(-\theta(-\epsilon_{k-1}) + \theta(-\epsilon_k + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{k-2}))}{\epsilon_k - (\epsilon_1 + \cdots + \epsilon_{k-1})}, \quad (A16)
$$

where we have used $\mathcal{A}(-\epsilon) = -\mathcal{A}(\epsilon)$ to treat the term $f_B(\epsilon_k)$ in (A7). For the particular case $k = 2$, one has

$$
\frac{1}{\beta\hbar} \sum_{n_1} K(\omega_{n_1})^2 = \mathcal{J}_2 + \frac{2}{(\beta\hbar)^2} \frac{\mathcal{A}'(0)}{3} \int d\epsilon \frac{\mathcal{A}(\epsilon)}{\epsilon} + \mathcal{O}(T^4)
$$
$$
= \mathcal{J}_2 + \frac{2}{(\beta\hbar)^2} K(\Sigma) \frac{\pi}{3} \mathcal{A}'(0) + O(\beta^{-4}),
$$
$$
(A17)
$$

where we have used $\int d\epsilon \mathcal{A}(\epsilon)/\epsilon = \pi J_1(\Sigma)$. Finally, we need the low T expansion of B (9):

$$
B = B^{(0)} + 2\hbar \left(\frac{T}{\hbar}\right)^2 \frac{\pi}{3} \mathcal{A}'(0) + \mathcal{O}(T^4). \quad (A18)
$$

Using these behaviors, one obtains

$$
\int_0^{\beta\hbar} d\tau (\widetilde{B}(\tau)^m - B^m)
$$
$$
= C^{\text{st}} + \left(\frac{T}{\hbar}\right)^2 \Bigg(-(2\hbar)^2 J_1(\Sigma) \mathcal{A}'(0) \frac{\pi}{3} m(m-1)(B^{(0)})^{m-2}
$$
$$
+ (2\hbar)^2 2 C_m^2 J_1(\Sigma) \mathcal{A}'(0) \frac{\pi}{3} (B^{(0)})^{m-2} + (2\hbar)^3 C_m^2
$$
$$
\times \mathcal{J}_2 (m-2) \mathcal{A}'(0) \frac{\pi}{3} (B^{(0)})^{m-3} + \sum_{k=3}^{m} (2\hbar)^k (-1)^k
$$
$$
\times C_m^k k \frac{\pi}{3} \mathcal{A}'(0) \mathcal{J}_{k-1} (B^{(0)})^{m-k} + \sum_{k=3}^{m-1} (2\hbar)^{k+1} (-1)^k
$$
$$
\times C_m^k (m-k) \frac{\pi}{3} \mathcal{A}'(0) \mathcal{J}_k (B^{(0)})^{m-k-1} \Bigg) + \mathcal{O}(T^4). \quad (A19)
$$

First, we notice that the two first terms $\propto T^2$ just cancel and moreover

$$
\sum_{k=3}^{m} (2\hbar)^k (-1)^k C_m^k \Bigg(k \frac{\pi}{3} \mathcal{A}'(0) \mathcal{J}_{k-1} (B^{(0)})^{m-k} + \sum_{k=3}^{m-1} (2\hbar)^k
$$
$$
\times (-1)^k C_m^k (m-k)(2\hbar) \frac{\pi}{3} \mathcal{A}'(0) \mathcal{J}_k (B^{(0)})^{m-k-1} \Bigg)
$$
$$
= \frac{\pi}{3} \mathcal{A}'(0) \Bigg(-(2\hbar)^3 3 C_m^3 \mathcal{J}_2 (B^{(0)})^{m-3} + \sum_{k=4}^{m} (-1)^k (2\hbar)^k
$$
$$
\times \mathcal{J}_{k-1} (B^{(0)})^{m-k} (k C_m^k - (m-k+1) C_m^{k-1}) \Bigg)
$$
$$
= -(2\hbar)^3 \frac{m(m-1)(m-2)}{2} \frac{\pi}{3} \mathcal{A}'(0) \mathcal{J}_2 (B^{(0)})^{m-3}. \quad (A20)
$$

This identity (A20) combined with (A19) yields finally to the announced property

$$
\int_0^{\beta\hbar} d\tau (\widetilde{B}(\tau)^m - B^m) = C^{\text{st}} + \mathcal{O}(T^4) \quad (A21)
$$

as announced in the text (35).

2. Handling the peculiar term $\propto (1 - \delta_{n,0})$

We now consider the extension of this property (A21) to the case where $\Sigma + I(\omega_n)$ now depend on T, and are of the form (39). We want to follow the same steps as previously (A21), and use the spectral representation of the Green function. Therefore, we need in principle to know the analytical continuation of the term $\propto (1 - \delta_{n,0})$ in (39). In order to avoid this ambiguity, we start by isolating the term $\propto (1 - \delta_{n,0})$ in (39):

$$\Sigma + I(\omega_n) = \Sigma + C(1 - \delta_{n,0}) + \tilde{I}(\omega_n),$$

$$\tilde{I}(\omega_n) \sim |\omega_n| + \mathcal{O}(\omega_n^2),$$

$$\Sigma = \Sigma^{(0)} + T^2\Sigma^{(2)} + T^3\Sigma^{(3)} + \mathcal{O}(T^4),$$

$$C = -T^2\Sigma^{(2)} - T^3\Sigma^{(3)} + \mathcal{O}(T^4),$$

$$\tilde{I}(\omega_n) = \tilde{I}^0(\omega_n) + \mathcal{O}(T^4), \tag{A22}$$

where $\tilde{I}(\omega_n)$ is defined such that there is no more ambiguity concerning its analytical continuation. We want to study the T dependence of integrals over imaginary time such as (A5) when the solution of the variational equations are of the form (39). In order to extract the quadratic and cubic terms in this expansion, we follow the previous analysis, except that, here, the mode $\omega_n=0$ has to be treated separately (39). Notice however that this ambiguity does not exist for the computation of $\tilde{B}(\tau)$ (10). We analyze the low T behavior of (A5) in the following way:

$$\int_0^{\beta\hbar} d\tau(\tilde{B}(\tau)^m - B^m) = \frac{1}{\hbar}\int_0^{\beta\hbar} d\tau(\tilde{B}(\tau)^m - \hat{B}^m) + \beta\hbar(\hat{B}^m - B^m), \tag{A23}$$

$$\hat{B} = \frac{2}{\beta}\sum_n J_1(\Sigma + C + M\omega_n^2 + \tilde{I}(\omega_n)). \tag{A24}$$

Using the previous property (A21), one has simply

$$\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau(\tilde{B}(\tau)^m - \hat{B}^m) = C^{\text{st}} + \mathcal{O}(T^4) \tag{A25}$$

and the leading corrections at finite temperature are then given by the second term in (A23),

$$\beta(\hat{B}^m - B^m) = -\beta\hbar\sum_{k=1}^m C_k^m\left(\frac{2}{\beta}\int_q \frac{C}{(cq^2+\Sigma)(cq^2+\Sigma+C)}\right)^k \hat{B}^{m-k}$$

$$= 2\hbar(T^2\Sigma^{(2)} + T^3\Sigma^{(3)})J_2(\Sigma^{(0)})$$

$$\times m(B^{(0)})^{m-1} + \mathcal{O}(T^4) \tag{A26}$$

since $B^{(0)}=\hat{B}^{(0)}$, and more generally, one can write, for any ω_n and any function $\mathcal{H}(X)$ the generalization of (A21) to this case (39)

$$\int_0^{\beta\hbar} d\tau \cos(\omega_n\tau)(\mathcal{H}(\tilde{B}(\tau)) - \mathcal{H}(B))$$

$$= C^{\text{st}} + 2\hbar\,\delta_{n,0}(T^2\Sigma^{(2)} + T^3\Sigma^{(3)})$$

$$\times J_2(\Sigma^{(0)})\mathcal{H}'(B^{(0)}) + \mathcal{O}(T^4) \tag{A27}$$

as quoted in the text (40).

APPENDIX B: HEISENBERG SPIN GLASS: LOW TEMPERATURE EXPANSION

1. General properties

We present here the detailed analysis of the low temperature behavior of integrals over imaginary time of the form (30)

$$\mathcal{I} = \int_0^\beta d\tau[(\tilde{\mathcal{G}}(\tau) - g)^2(\tilde{\mathcal{G}}(-\tau) - g)^2 - g^4]. \tag{B1}$$

We first consider the case where the Green's function $\tilde{\mathcal{G}}(i\nu_n)$, as a function of $i\nu_n$ is independent of T. Similarly to the analysis performed in Appendix A, we develop the integrand in (B1) and perform the integral over τ: we are then left with multiple sums over Matsubara frequencies. Depending on the number of Green's functions they involve, the integral over τ generates 4 different types of terms:

$$\mathcal{I} = \sum_{i=1}^4 \mathcal{I}_i,$$

$$\mathcal{I}_1 = \frac{1}{\beta^3}\sum_{i\nu_1, i\nu_2, i\nu_3} \tilde{\mathcal{G}}(i\nu_1)\tilde{\mathcal{G}}(i\nu_2)\tilde{\mathcal{G}}(i\nu_3)\tilde{\mathcal{G}}(i\nu_1 + i\nu_2 - i\nu_3),$$

$$\mathcal{I}_2 = -2g\frac{1}{\beta^2}\sum_{i\nu_1, i\nu_2} [\tilde{\mathcal{G}}(i\nu_1)\tilde{\mathcal{G}}(i\nu_2)\tilde{\mathcal{G}}(i\nu_1 + i\nu_2)$$

$$+ \tilde{\mathcal{G}}(i\nu_1)\tilde{\mathcal{G}}(i\nu_2)\tilde{\mathcal{G}}(i\nu_1 - i\nu_2)],$$

$$\mathcal{I}_3 = 2g^2\frac{1}{\beta}\sum_{i\nu_1} [2\tilde{\mathcal{G}}(i\nu_1)^2 + \tilde{\mathcal{G}}(i\nu_1)\tilde{\mathcal{G}}(-i\nu_1)],$$

$$\mathcal{I}_4 = -4g^3\tilde{\mathcal{G}}(i\nu_n = 0). \tag{B2}$$

We follow the standard analysis and use the spectral representation of the Green's function to handle these terms

$$\tilde{\mathcal{G}}(i\nu_n) = -\int_{-\infty}^\infty \frac{d\omega}{\pi}\rho(\omega)\frac{1}{i\nu_n - \omega}, \tag{B3}$$

$$\rho(\omega) = -\text{Im}\,\tilde{\mathcal{G}}(i\nu_n \to \omega + i0^+). \tag{B4}$$

Performing then, the sums over the Matsubara frequencies, we are left with the same kind of structure as found in the previous models (A7),

$$\mathcal{I}_1 = \frac{1}{\pi^4}\int d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 \prod_{i=1}^4 \rho(\epsilon_i)(f_B(\epsilon_1) - f_B(\epsilon_4))(f_B(\epsilon_2)$$

$$- f_B(\epsilon_4 - \epsilon_1))\frac{(f_B(\epsilon_3) - f_B(\epsilon_1 + \epsilon_2 - \epsilon_4))}{\epsilon_4 + \epsilon_3 + \epsilon_2 - \epsilon_1}. \tag{B5}$$

Under this form, we analyze straightforwardly the low temperature behavior, using the property demonstrated previously that only Bose factors with *one* frequency in their argument do contribute to this sum. Using the expansion (A10), we obtain up to order $\mathcal{O}(T^4)$,

$$
\mathcal{I}_1 = \frac{1}{\pi^4} \int d\epsilon_1 d\epsilon_2 d\epsilon_3 d\epsilon_4 \prod_{i=1}^{4} \rho(\epsilon_i)
$$

$$
\times (-\theta(-\epsilon_1) + \theta(-\epsilon_4))(-\theta(-\epsilon_2) + \theta(-\epsilon_4 + \epsilon_1))
$$

$$
\times \frac{(-\theta(-\epsilon_3) + \theta(-\epsilon_1 - \epsilon_2 + \epsilon_4))}{\epsilon_4 + \epsilon_3 + \epsilon_2 - \epsilon_1}
$$

$$
+ T^2 \frac{4}{3\pi^2} \rho'(0) \int d\epsilon_1 d\epsilon_2 d\epsilon_3 \rho(\epsilon_1) \rho(\epsilon_2) \rho(\epsilon_3)
$$

$$
\times (-\theta(-\epsilon_1) + \theta(-\epsilon_3)) \frac{(-\theta(-\epsilon_2) + \theta(-\epsilon_1 + \epsilon_3))}{\epsilon_3 + \epsilon_2 - \epsilon_1},
$$

$$
\tag{B6}
$$

where we have used the relation [which is valid although $\rho(\epsilon)$ is not an odd function]:

$$
\int d\epsilon_1 d\epsilon_2 d\epsilon_3 \rho(\epsilon_1) \rho(\epsilon_2) (\rho(\epsilon_3) + \rho(-\epsilon_3)) (\theta(-\epsilon_1) - \theta(-\epsilon_3))
$$

$$
\times \frac{(-\theta(-\epsilon_2) + \theta(-\epsilon_1 + \epsilon_3))}{\epsilon_3 + \epsilon_2 - \epsilon_1} = 0. \tag{B7}
$$

Indeed, one shows this relation (B7) by noticing that the step functions reduce the interval of integration to $\epsilon_3 > 0$, $\epsilon_2 > 0$, $\epsilon_1 < 0$ and $\epsilon_3 < 0$, $\epsilon_2 < 0$, $\epsilon_1 > 0$. A simple permutation of the integration variables then lead to the relation (B7).

The analysis of $\mathcal{I}_2$ requires the low T expansion of g,

$$
g = S + \int \frac{d\epsilon}{\pi} \rho(\epsilon) f_B(\epsilon) = g^{(0)} + T^2 \frac{\pi}{3} \rho'(0) + \mathcal{O}(T^4),
$$

$$
\tag{B8}
$$

$\mathcal{I}_2$ can be written as

$$
\mathcal{I}_2 = 2g \frac{1}{\pi^3} \int d\epsilon_1 d\epsilon_2 d\epsilon_3 \rho(\epsilon_1) \rho(\epsilon_2) \rho(\epsilon_3)
$$

$$
\times \left[\frac{(f_B(\epsilon_1) - f_B(\epsilon_3)(f_B(\epsilon_2) - f_B(\epsilon_3 - \epsilon_1)))}{\epsilon_1 + \epsilon_2 - \epsilon_3} \right.
$$

$$
\left. + \frac{(f_B(\epsilon_1) - f_B(\epsilon_3)(f_B(\epsilon_2) - f_B(\epsilon_1 - \epsilon_3)))}{\epsilon_1 - \epsilon_2 - \epsilon_3} \right] \tag{B9}
$$

from which we obtain the low T expansion up to order $\mathcal{O}(T^4)$,

$$
\mathcal{I}_2 = C^{\mathrm{st}} - T^2 \frac{4}{3\pi^2} \rho'(0) \int d\epsilon_1 d\epsilon_2 d\epsilon_3 \rho(\epsilon_1) \rho(\epsilon_2) \rho(\epsilon_3)
$$

$$
\times (-\theta(-\epsilon_1) + \theta(-\epsilon_3)) \frac{(-\theta(-\epsilon_2) + \theta(-\epsilon_1 + \epsilon_3))}{\epsilon_3 + \epsilon_2 - \epsilon_1}
$$

$$
+ \frac{2g^{(0)}}{3\pi} \frac{\rho'(0)}{\beta^2} \int d\epsilon_1 d\epsilon_2 \rho(\epsilon_1) \rho(\epsilon_2)
$$

$$
\times \left[4 \frac{\theta(-\epsilon_1) - \theta(-\epsilon_2)}{\epsilon_2 - \epsilon_1} - 2 \frac{\theta(\epsilon_1) - \theta(-\epsilon_2)}{\epsilon_1 + \epsilon_2} \right]. \tag{B10}
$$

One obtains in a similar way the expansion of $\mathcal{I}_3$, writing

$$
\mathcal{I}_3 = \frac{g^2}{\pi^2} \int d\epsilon_1 d\epsilon_2 \rho(\epsilon_1) \rho(\epsilon_2)
$$

$$
\times \left[4 \frac{f_B(\epsilon_2) - f_B(\epsilon_1)}{\epsilon_1 - \epsilon_2} + 2 \frac{f_B(\epsilon_1) - f_B(-\epsilon_2)}{\epsilon_1 + \epsilon_2} \right] \tag{B11}
$$

from which we obtain the low T expansion:

$$
\mathcal{I}_3 = C^{\mathrm{st}} + T^2 \frac{2g^{(0)}}{3\pi} \rho'(0) \int d\epsilon_1 d\epsilon_2 \rho(\epsilon_1) \rho(\epsilon_2) \tag{B12}
$$

$$
\left[-4 \frac{\theta(-\epsilon_1) - \theta(-\epsilon_2)}{\epsilon_2 - \epsilon_1} + 2 \frac{\theta(\epsilon_1) - \theta(-\epsilon_2)}{\epsilon_1 + \epsilon_2} \right]
$$

$$
+ T^2 4\pi [g^{(0)}]^2 \rho'(0) \widetilde{\mathcal{G}}(i\nu_n = 0), \tag{B13}
$$

where we have used $\int d\epsilon \rho(\epsilon)/\epsilon = \pi \widetilde{\mathcal{G}}(i\nu_n = 0)$. As $\widetilde{\mathcal{G}}(i\nu_n)$ does not depend on T, one obtains the expansion of $\mathcal{I}_4$ up to order

$$
\mathcal{I}_4 = C^{\mathrm{st}} - T^2 4\pi [g^{(0)}]^2 \rho'(0) \widetilde{\mathcal{G}}(i\nu_n = 0) + \mathcal{O}(T^4). \tag{B14}
$$

Finally, collecting the quadratic contributions in (B6), (B10), (B12), and (B14) yields

$$
\mathcal{I} = \int_0^\beta d\tau [(\widetilde{\mathcal{G}}(\tau) - g)^2 (\widetilde{\mathcal{G}}(-\tau) - g)^2 - g^4] = C^{\mathrm{st}} + \mathcal{O}(T^4)
$$

$$
\tag{B15}
$$

as announced in the text (36).

Using the same method exposed previously, we study the T dependence of the integral over τ which enters the variational equation of this model (31),

$$
\mathfrak{I} = \int_0^\beta d\tau e^{-i\nu_n \tau} [(\widetilde{\mathcal{G}}(\tau) - g)^2 (\widetilde{\mathcal{G}}(-\tau) - g) + g^3]. \tag{B16}
$$

As previously, after we have performed the integrals over τ in (B16), we have to handle exactly the same integrals as in $\mathcal{I}_2$, $\mathcal{I}_3$, $\mathcal{I}_4$. The mechanism of cancellation of the quadratic term is again at work here [notice however that, given that the integrand is here a polynom of degree 3, the terms like (B7) are not present here]. This yields

$$
\mathfrak{I} = C^{\mathrm{st}} + \mathcal{O}(T^4) \tag{B17}
$$

as given in the text (36).

2. Handling the peculiar term $\propto (1 - \delta_{n,0})$

We generalize these properties to the case where the solution of the variational equations is of the form shown in Eq. (46). We use here the same strategy as presented in Appendix A. We isolate the mode $i\nu_n = 0$ as in (A23) and (A24),

and given that $(\widetilde{\mathcal{G}}(\tau) - g)$ or $(\widetilde{\mathcal{G}}(-\tau) - g)$ do not depend on this $i\nu_n = 0$ mode, one obtains straightforwardly,

$$\mathcal{I} = \int_0^{\beta\hbar} d\tau [(\widetilde{\mathcal{G}}(\tau) - g)^2 (\widetilde{\mathcal{G}}(-\tau) - g)^2 - g^4]$$

$$= C^{\text{st}} - 4\frac{\Theta}{J} g^{(0)} (T^2 g^{(2)} + g^{(3)} T^3) + \mathcal{O}(T^4), \qquad \text{(B18)}$$

$$\mathfrak{I} = \int_0^{\beta} d\tau e^{-i\nu_n \tau} [(\widetilde{\mathcal{G}}(\tau) - g)^2 (\widetilde{\mathcal{G}}(-\tau) - g) + g^3]$$

$$= C^{\text{st}} + \delta_{n,0} \frac{3\Theta}{J} (T^2 g^{(2)} + g^{(3)} T^3) + \mathcal{O}(T^4), \qquad \text{(B19)}$$

as announced in the text (47) and (61).

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