Diffusion behavior of quasiparticles in two-dimensional disordered systems with *s*-wave pairing

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We investigate the diffusion behavior of quasiparticles in two-dimensional (2D) disordered systems with *s*-wave pairing by using the finite-size scaling analysis and transfer-matrix method. The disorder is introduced by random site energies, and the spatial fluctuations of the pairing potential due to this randomness are determined self-consistently. From the size dependence of the Lyapunov exponents, we show that the quasiparticle state in every channel is localized in such a 2D system. The calculated size dependence of the total transmittance of quasiparticles through all possible channels, however, shows a different scaling behavior that suggests the existence of a critical point. The associated critical behavior is studied and the relationship of the results to the Meissner effect and supercurrent is discussed.

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I. INTRODUCTION

Recently, the study of disorder in two-dimensional (2D) superconductors has remained an engaging topic. The effect of strong disorder on superconductivity has been of great interest for a long time,[1–3] but the nature of destruction of superconductivity by disorder and the localization of quasiparticles is not very clear. On the one hand, at zero temperature Fermi liquid suffers the Cooper instability even at an infinitesimal attractive interaction. Provided that the nonsuperconducting state remains a metal, the Anderson theorem guarantees that the nature of the transition from a Fermi liquid to an *s*-wave superconductor is unchanged by the presence of nonmagnetic impurities or disorder.[4] On the other hand, the scaling theory has predicted that an infinitesimal disorder can cause localization of all electron states in 2D, and in the thermodynamic limit there is no truly 2D metallic phase in the nonsuperconducting state.[5] In the coexistence of the superconducting pairing and disorder, a complicated situation may appear in 2D. It has been pointed out that the ground state of the system may be an insulator, a metal, or a superconductor, determined by an infinite-wavelength and zero-frequency current-current correlation function as a criterion discussed in Ref. 6. In the presence of strong disorder, the self-consistent calculations based on the Bogoliubov–de Gennes (BdG) framework show that the spectral gap persists, but the local pairing amplitude develops broad spatial fluctuations and off-diagonal correlations exhibit a substantial reduction.[7–9] From a finite-size scaling analysis, it is shown that a *d*-wave component of the pairing potential is necessary for the delocalization of quasiparticle states in a 2D disordered system.[10] The effects of the disorder on 2D or quasi-2D *s*-wave superconductors have also been investigated in experimental research.[11,12]

In the present paper, we consider a 2D disordered model with the *s*-wave pairing to investigate the combined effect of disorder and pairing potential on the diffusion properties of quasiparticles. The disorder is introduced with the randomness of site energies. In the BdG framework, we numerically determine the relationship between the spatial fluctuations of the pairing potential and the distribution of random site energies by using self-consistent calculations. With the transfer-matrix method, and taking into account the fluctuations of the pairing potential, we investigate the size dependence of the Lyapunov exponent of the most extended quasiparticle state and the total transmittance of quasiparticles through all possible channels. The results show that although the most extended quasiparticle is still localized in the thermodynamic limit, the calculated size dependence of the total transmittance of quasiparticles indicates the existence of the long-range transparent state. The relationship of the localization of quasiparticles in a single channel and the transparency of the total transmittance to the Meissner effect and the supercurrent are discussed.

The paper is organized as follows. In the next section, we describe the basic formalism in our calculations. In Sec. III, we present the main results and discuss their physical meanings. The final section is devoted to a brief summary of the conclusions.

II. MODEL AND BASIC FORMALISM

We consider a 2D square lattice with on-site disorder and on-site attractive interaction which is responsible for the BdG pairing potential,

$$H = \sum_{i,j;\sigma} \epsilon_{ij} c^{\dagger}_{ij,\sigma} c_{ij,\sigma} + \sum_{\langle i,j;i',j'\rangle,\sigma} (t_0 c^{\dagger}_{ij,\sigma} c_{i'j',\sigma} + \mathrm{H.c.}) - \sum_{i,j} V c^{\dagger}_{ij,\uparrow} c^{\dagger}_{ij,\downarrow} c_{ij,\downarrow} c_{ij,\uparrow}, \quad (1)$$

where $c_{ij,\sigma}$ is the annihilation operator for an electron on site (i,j) and of spin σ, with i and j labeling the site position in the x and y directions, respectively, t_0 is the nearest-neighbor hopping integral, ϵ_{ij}'s denote the random site energies uniformly distributed in $[\epsilon_0 - W/2, \epsilon_0 + W/2]$, with ϵ_0 being the average value of the site energies and W the strength of disorder, and V is the strength of the on-site attractive interaction. Here we set the Fermi level as the energy zero. The BdG *s*-wave pairing potential can be expressed as

$$\lambda_{ij} = -V\langle c^{\dagger}_{ij,\uparrow} c^{\dagger}_{ij,\downarrow}\rangle, \quad (2)$$

where $\langle\cdots\rangle$ denotes the statistical averaging. With such a mean-field treatment, one obtains the BdG Hamiltonian

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$$H = \sum_{i,j;\sigma} \epsilon_{ij} c^{\dagger}_{ij,\sigma} c_{ij,\sigma} + \sum_{\langle i,j;i',j' \rangle,\sigma} (t_0 c^{\dagger}_{ij,\sigma} c_{i'j',\sigma} + \mathrm{H.c.}) + \sum_{i,j} (\lambda_{ij} c_{ij,\downarrow} c_{ij,\uparrow} + \mathrm{H.c.}). \quad (3)$$

In order to investigate the diffusion behavior of quasiparticles, let us consider a strip with a finite width of M in the x direction and very long length L in the y direction. In the site representation, a wave function in such a system can be written as a linear superposition,

$$|\Psi\rangle = \sum_{i=1}^{M} \sum_{j=1}^{L} (a_{ij} c^{\dagger}_{ij,\sigma} + b_{ij} c_{ij,-\sigma}) |F\rangle, \quad (4)$$

where $|F\rangle$ denotes the Fermi sea in the normal state. The coefficients in the superposition satisfy the BdG equations

$$(\varepsilon - \epsilon_{ij}) a_{ij} = t_0 (a_{i+1,j} + a_{i-1,j} + a_{i,j+1} + a_{i,j-1}) + \lambda^*_{ij} b_{ij}, \quad (5)$$

$$(\varepsilon + \epsilon_{ij}) b_{ij} = -t_0 (b_{i+1,j} + b_{i-1,j} + b_{i,j+1} + b_{i,j-1}) + \lambda_{ij} a_{ij}, \quad (6)$$

where ε is the corresponding quasiparticle energy. For given values of $\{\lambda_{ij}\}$ and energy ε, one can rewrite Eqs. (5) and (6) in a transfer-matrix form,

$$\begin{pmatrix} \vec{a}_{j+1} \\ \vec{b}_{j+1} \\ \vec{a}_j \\ \vec{b}_j \end{pmatrix} = \hat{T}_j \begin{pmatrix} \vec{a}_j \\ \vec{b}_j \\ \vec{a}_{j-1} \\ \vec{b}_{j-1} \end{pmatrix}, \quad (7)$$

where vector $\vec{a}_j(\vec{b}_j)$ has M components $a_{1j}, a_{2j}, \ldots, a_{Mj}$ $(b_{1j}, b_{2j}, \ldots, b_{Mj})$, and $\hat{T}_j$ is a $4M \times 4M$ transfer matrix. From BdG equations, $\hat{T}_j$ can be written as

$$\hat{T}_j = \begin{pmatrix} \hat{u}_1 & \hat{v} & -\hat{1} & \hat{0} \\ \hat{v}^{\dagger} & \hat{u}_2 & \hat{0} & -\hat{1} \\ \hat{0} & \hat{0} & \hat{1} & \hat{0} \\ \hat{0} & \hat{0} & \hat{0} & \hat{1} \end{pmatrix}, \quad (8)$$

where the symbols with a caret are $M \times M$ matrices and their elements are

$$\{\hat{u}_1\}_{ii'} = \delta_{i,i'} \frac{\varepsilon_l - \epsilon_{ij}}{t_0} - \delta_{i,i'-1} - \delta_{i,i'+1}, \quad (9)$$

$$\{\hat{u}_2\}_{ii'} = -\delta_{i,i'} \frac{\varepsilon_l + \epsilon_{ij}}{t_0} - \delta_{i,i'-1} - \delta_{i,i'+1}, \quad (10)$$

$$\{\hat{v}\}_{ii'} = -\delta_{i,i'} \frac{\lambda^*_{ij}}{t_0}. \quad (11)$$

For a strip with length L, the coefficients at one end are related to the coefficients at the other end with the transfer matrices

$$\begin{pmatrix} \vec{a}_L \\ \vec{b}_L \\ \vec{a}_{L-1} \\ \vec{b}_{L-1} \end{pmatrix} = \left(\prod_{j=1}^{L-1} \hat{T}_{L-j} \right) \begin{pmatrix} \vec{a}_1 \\ \vec{b}_1 \\ \vec{a}_0 \\ \vec{b}_0 \end{pmatrix}. \quad (12)$$

We can calculate the Lyapunov exponents of quasiparticle states by using the transfer-matrix method, in which the orthonormalization procedure is adopted.[13] The Lyapunov exponents are the natural logarithms of eigenvalues of the transfer matrix. In the present case, there are $4M$ eigenvalues for the transfer matrix, corresponding to M spatial transverse channels, two particle-hole channels, and two propagating directions (forward and backward). The natural logarithms of eigenvalues for the forward and backward waves have opposite signs, so we only keep the $2M$ positive ones, whose inverses, $\xi_l(\varepsilon, M)$ with $l = 1, 2, \ldots, 2M$, are the localization lengths of the corresponding channels. According to Ref. 14, the rescaled localization length is defined as

$$\Lambda_l(\varepsilon, M) = \xi_l(\varepsilon, M)/M. \quad (13)$$

The properties of the system, such as the superconductivity and the conductance, are closely related to the localization behavior of quasiparticles. Let us consider a square system of size $M \times M$. From Ref. 15, the off-diagonal long-range order (ODLRO) is defined as the probability of finding a Cooper pair at the right edge after it is injected from the left edge. If the Cooper pair is injected into the lth channel of the left edge, the probability of finding it at the right edge is related to the rescaled localization length as

$$P_l(\varepsilon, M) = \sum_{i=1}^{M} |a_{l;i0} b_{l;i0}|^2 \exp[-2/\Lambda_l(\varepsilon, M)], \quad (14)$$

where $a_{l;i0}$ and $b_{l;i0}$ are the components of the lth eigenvector of the transfer matrix. It can be seen that besides the prefactor $\Sigma_{i=1}^{M} |a_{l;i0} b_{l;i0}|^2$, which represents the strength of pairing in this channel, the ODLRO has the same scaling behavior as the rescaled localization length. Thus, from the M dependence of the largest $\Lambda_l(\varepsilon, M)$, we can determine whether the ODLRO in one channel vanishes in the thermodynamic limit. In some cases, the properties are determined by the total contribution from all the channels, so it is also interesting to investigate the M dependence of the following dimensionless quantity:

$$g(\varepsilon, M) = \sum_{l=1}^{M} \frac{1}{\pi} \left[\frac{\tau_l(\varepsilon, M)}{1 - \tau_l(\varepsilon, M)} \right], \quad (15)$$

where $\tau_l(\varepsilon, M)$ is the transmission coefficient of the lth channel,

$$\tau_l(\varepsilon, M) = \exp[-2/\Lambda_l(\varepsilon, M)].$$

$g(\varepsilon, M)$ can be regarded as the probability that a quasiparticle can travel through the system from the left to the right, no matter what channel is taken.

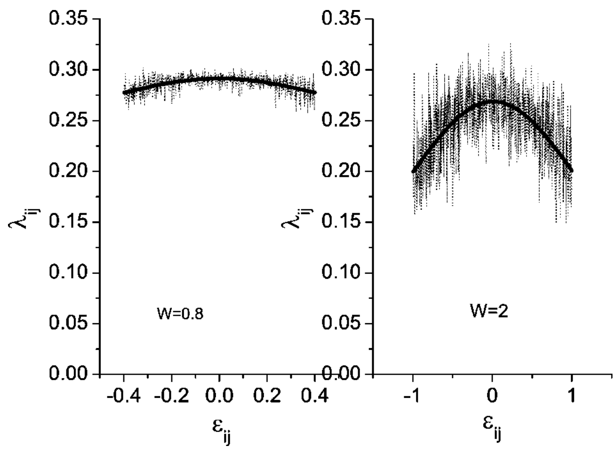

FIG. 1. Dependence of the self-consistent pairing potential λ_{ij} on the on-site energy ϵ_{ij} for different values of W. Dashed curves are the results from the solutions of the BdG equations in a 30×30 square lattice, and the solid curves are fitting ones. The attractive interaction $V=1.8t_0$. Energy units are set to be the hopping integral t_0.

III. SCALING BEHAVIOR OF QUASIPARTICLE DIFFUSION

The transfer-matrix calculations and the scaling analysis are performed on a very long strip of length $L=5\times 10^4$ in the x direction and with a varying width M in the y direction, for which the periodic boundary conditions are applied. For a given energy ε, the $4M\times 4M$ transfer matrix maps the amplitudes of a quasiparticle wave function at the left end of the strip to those at the right end. The propagation of quasiparticles along the strip is therefore determined by the Lyapunov exponents and the rescaled localization lengths of the transfer matrix obtained from the orthonormalization procedure. In this procedure, the self-averaging over the randomness is automatically achieved by the large length of the strip. Since the attractive interaction V in the Hamiltonian is a constant and the spatial fluctuations of pairing potential λ_{ij} are caused by the randomness of site energies ϵ_{ij}, we can first determine the dependence of $\{\lambda_{ij}\}$ on $\{\epsilon_{ij}\}$ with self-consistent calculations on an $M\times M$ square system. The obtained results show that this dependence has a local feature, i.e., the value of λ_{ij} is mainly determined by ϵ_{ij} on the same site, and only weakly dependent on $\epsilon_{i'j'}$'s with sites $(i'j')$ distant from it. The dependence of λ_{ij} on ϵ_{ij} is shown in Fig. 1. This dependence provides a way to determine the values of λ_{ij} in the transfer-matrix calculations of the long strip.

Figure 2 shows the largest rescaled localization length $\Lambda(\varepsilon,M)/M$ as a function of width M for given ε and various values of disorder strength W. It can be seen that, for all the investigated values of W, the largest rescaled localization length decreases almost monotonically with width M if the fluctuations are neglected. This means that all the quasiparticles are localized even in the case of very small W, consistent with the scaling theory for 2D systems.

The results in Fig. 2 reflect the localization behavior of quasiparticles in every channel. This corresponds to the properties that rely on the transport in one or a few channels.

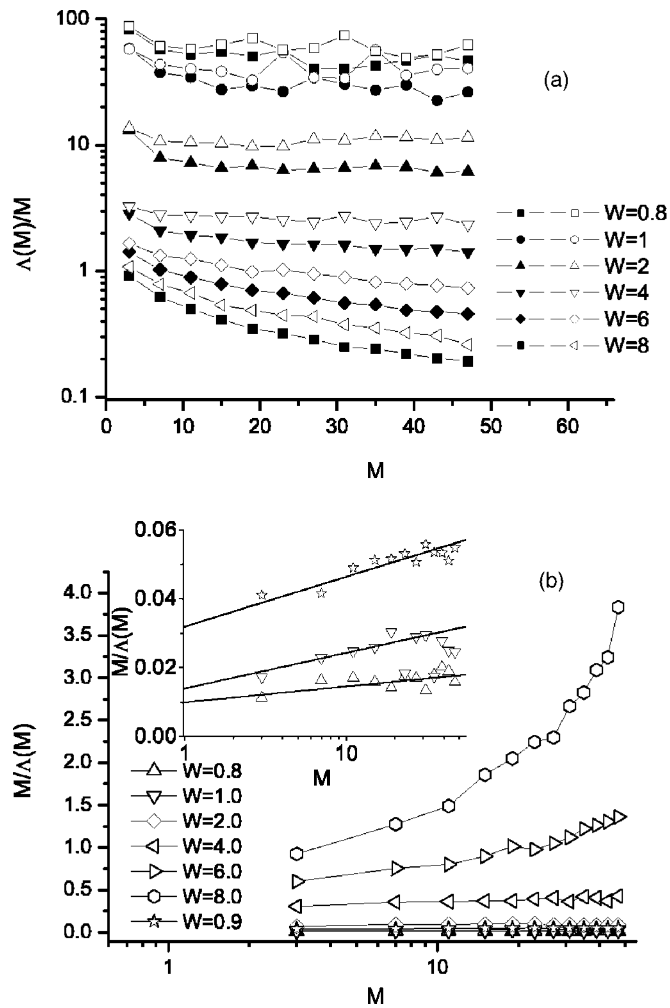

FIG. 2. Largest rescaled localization length as a function of width M. The energy $\varepsilon=-0.5$ and the average pairing potential $\bar{\lambda}=0.2$. The curves with filled symbols are obtained by using the average pairing potential for all sites (non-self-consistent calculations), and the curves with empty symbols are for the system with spatially fluctuated pairing potentials obtained from the self-consistent calculations.

For other properties, the contribution from all possible channels, which can be reflected with the dimensionless quantity g in Eq. (15), is more relevant. Figure 3 shows the M dependence of g for different values of disorder strength W. It can be seen that in spite of the statistical fluctuations, for $W>2$, g is decreased with increasing M, while for $W<2$, it is increased with M. For an $M\times M$ square system in the limit $M\to\infty$, there exists a region $(W<2)$ in which the transmission probability of quasiparticles from one edge to the other via all possible channels is nonzero. The size independence of g about $W=2$ corresponds to a critical point of a continuous phase transition.[16]

The dimensionless quantity g for a finite $M\times M$ square system near the critical point can be expressed with a universal function $g(M/\zeta)$ by using the finite-size scaling ansatz.[13,14] We make a further assumption that this scaling parameter ζ varies as $\zeta\propto|W-W_c|^{-\nu}$ in the vicinity of the critical point, obeying the scaling law in a continuous phase

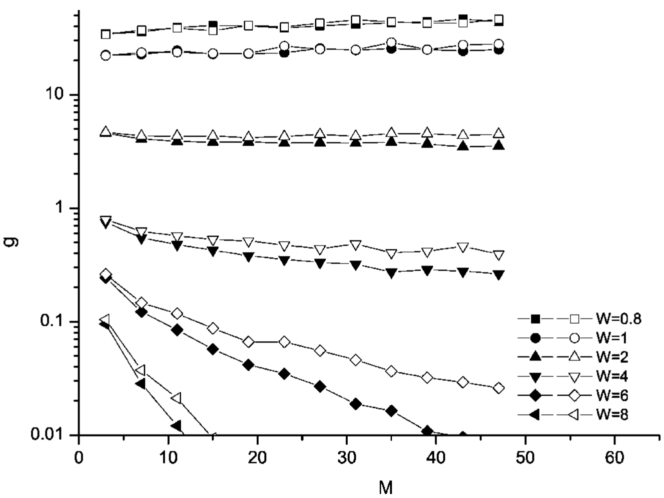

FIG. 3. Dimensionless quantity g as a function of width M for different values of W. Other parameters are the same as those in Fig. 2.

transition. In Fig. 4, we display this scaling function for different values of W. We also plot the scaling parameter ζ as a function of $|W-W_c|$ in the inset. From the data, the values of the exponent ν and the critical point W_c are fitted as $\nu=1.80$ and $W_c=1.8t_0$.

Thus, from the scaling behavior of the largest localization length and the total transmittance probability g, we are led to different conclusions on the existence of the critical point. To explore the origin of this difference, in Fig. 5 we display the distribution of Lyapunov exponents for all channels. It is interesting to note that, differently from the usual log-normal distribution in disordered system without superconducting pairing, there are two plateaus in the distribution corresponding to different regions of Lyapunov exponents. Especially, for Lyapunov exponents approaching the smallest value, corresponding to the limit of largest localization length, the distribution density is saturated at a finite value. This means that there exists a finite fraction of channels that have localization

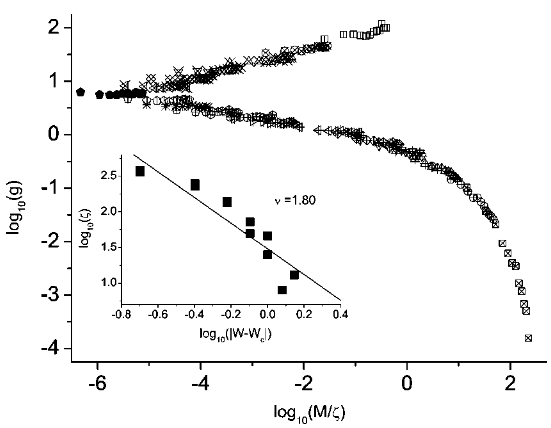

FIG. 4. Scaling function of g for a 2D system. Inset: $\log_{10}\zeta$ as a function of $\log_{10}(|W-W_c|)$, where the square symbols represent values from the data and the straight line is the fitting function $\zeta \propto |W-W_c|^{-\nu}$ with the shown value of ν. Other parameters are the same as those in Fig. 2.

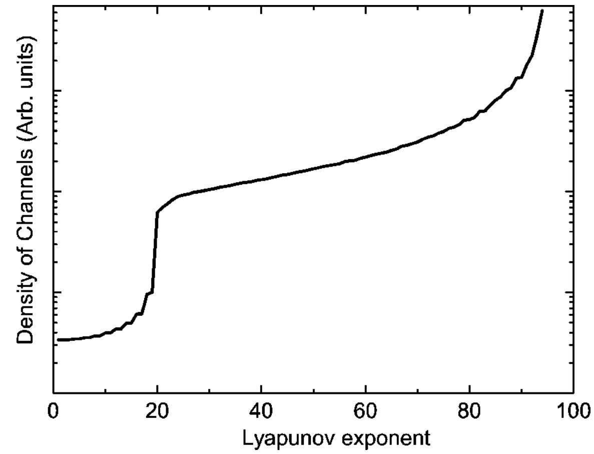

FIG. 5. Distribution density of channels as a function of Lyapunov exponent. Parameters are $E=-0.5t_0$, $W=0.8t_0$, $M=47$, and the pairing potential is $0.2t_0$.

lengths near the largest one. As a result, the total transmittance probability via all channels g may be increased with increasing M, although the largest localization length is decreased, because by increasing M the number of available channels is also increased. The physics here is that, although all quasiparticles are localized due to the random reflections, a finite supercurrent can still be supported by the multiple Andreev reflections between the local "islands." Such a theory has already been proposed by Ma and Lee and Kotliar and Kapitulnik.[17,18] The results presented here provide a numerical version of the theory.

From the different scaling behaviors of the largest localization length and the quantity g, one may expect some basic features of superconducting and transport properties of such 2D systems in the thermodynamic limit. (i) For the Meissner effect, the induced diamagnetic supercurrent is within the penetration depth. If the penetration depth is microscopically small, this supercurrent is only within a few channels. According to the localization behavior of quasiparticles in a single channel, in this case the Meissner effect could disappear due to the disorder. In this sense, the system is not a true superconductor even though the local pairing potential is nonzero. However, if the penetration depth is in a macroscopic range, the diamagnetic supercurrent is determined by the scaling behavior of g. In this case the Meissner effect cannot be suppressed by the disorder. (ii) Since the current is related to the contribution of all channels in the case of applying a dc bias, one may expect the existence of a supercurrent (or a normal current) under the dc bias. In this sense, the system may be regarded as a metal and a superconductor. This situation is certainly different from that of the usual 2D disordered systems.

IV. CONCLUSION AND DISCUSSION

We have investigated the diffusion behavior of quasiparticles in 2D systems with on-site energy disorder and s-wave pairing. By applying the transfer-matrix method and the finite size scaling analysis, it is found that although every quasiparticle channel is localized in the thermodynamic limit, the total transmittance probability via all available channels g can still be finite in the case of relatively weak disorder. Numerically, this is due to specific distribution of Lyapunov exponents in the channels in the presence of pairing. Physically, the origin is the multiple Andreev reflections that connect the localized regions with each other. The finite scaling ansatz on g gives the corresponding critical point of the disorder strength and the critical exponent. The results suggest that the Meissner effect may be absent for microscopically small penetration depth, but may exist in the case of macroscopic penetration depth. With applying the dc bias, the supercurrent exists.

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[1] D. Belitz and T. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).
[2] M. Sadovskii, Phys. Rep. **282**, 225 (1997).
[3] V. C. Hui and C. J. Lambert, J. Phys.: Condens. Matter **5**, 697 (1993).
[4] P. W. Anderson, J. Phys. Chem. Solids **11**, 26 (1959).
[5] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
[6] D. J. Scalapino, S. R. White, and Shoucheng Zhang, Phys. Rev. B **47**, 7995 (1993).
[7] N. Trivedi, R. T. Scalettar, and M. Randeria, Phys. Rev. B **54**, R3756 (1996).
[8] A. Ghosal, M. Randeria, and N. Trivedi, Phys. Rev. Lett. **81**, 3940 (1998).
[9] A. Ghosal, M. Randeria, and N. Trivedi, Phys. Rev. B **65**, 014501 (2001).
[10] Shi-Jie Xiong and Ye Xiong, Physica C **333**, 86 (2000).
[11] C.-T. Chen, P. Seneor, N.-C. Yeh, R. P. Vasquez, L. D. Bell, C. U. Jung, J. Y. Kim, Min-Seok Park, Heon-Jung Kim, and Sung-Ik Lee, Phys. Rev. Lett. **88**, 227002 (2002).
[12] E. Boaknin, M. A. Tanatar, J. Paglione, D. Hawthorn, F. Ronning, R. W. Hill, M. Sutherland, L. Taillefer, J. Sonier, S. M. Hayden, and J. W. Brill, Phys. Rev. Lett. **90**, 117003 (2003).
[13] A. MacKinnon and B. Kramer, Phys. Rev. Lett. **47**, 1546 (1981).
[14] A. MacKinnon and B. Kramer, Z. Phys. B: Condens. Matter **53**, 1 (1983).
[15] C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).
[16] M. N. Barber, in *Phase Transition and Critical Phenomena*, Vol. 8, edited by C. Domb and M. S. Green (Academic Press, London, 1983), p. 145.
[17] M. Ma and P. A. Lee, Phys. Rev. B **32**, 5658 (1985).
[18] G. Kotliar and A. Kapitulnik, Phys. Rev. B **33**, 3146 (1986).