Cooperative effects in Josephson junctions in a cavity in the strong coupling regime

Marco Frasca*
Via Erasmo Gattamelata, 3, 00176 Roma, Italy

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We analyze the behavior of systems of two and three qubits made by Josephson junctions, treated in the two level approximation, driven by a radiation mode in a cavity. The regime we consider is a strong coupling one recently experimentally reached for a single junction. Rabi oscillations are obtained with the frequency proportional to integer order Bessel functions in the limit of a large photon number, similarly to the case of the single qubit. A selection rule is derived for the appearance of Rabi oscillations. A quantum amplifier built with a large number of Josephson junctions in a cavity in the strong coupling regime is also described.

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I. INTRODUCTION

Quantum computation [1–3] promises a large improvement on the ability to execute demanding algorithms due to the large parallelism involved. Presently, it is not yet clearly understood how the hardware for a quantum computer should be realized. Several proposal relying, e.g., on ion traps[4] and NMR[5] have been devised but solid state devices seem to be very promising for this aim. Josephson junctions have been largely used both experimentally and theoretically for this goal.[6,7]

In a couple of recent experiments Nakamura, Pashkin, and Tsai[8] and Chiorescu *et al.*[9] were able to display the behavior of a single Josephson junction strongly coupled to a radiation field and its suitability for quantum computation. The strong coupling regime has the advantage that can be theoretically described by a two-level model.[10,11]

Validity of the two-level model to describe the behavior of Josephson junctions has been proved by Stroud and Al-Saidi.[12,13] They considered the case of weak coupling where the so-called rotating wave approximation does apply.[14] They also proved that, in some cases, a correction term to the model is needed while collapse and revival of Rabi oscillations is observed as also happens to similar models in quantum optics.[15,16] Similarities between solid state devices and quantum optics systems are becoming increasingly meaningful.[17]

The model used by Stroud and Al-Saidi cannot be applied directly to the experiments of Nakamura's group. Rather, we need to use the Dicke model[18] without any other approximations than the numbers of radiation modes and two-level systems. A complete application of this kind has been given by Ref. 10 even if a study of this model in the strong coupling regime was started by Cohen-Tannoudji and his group[19] for a single radiation mode and a two-level atom. In this case all the terms in the interaction part of the Hamiltonian must be retained.

In our paper[10] we were able to obtain the equations for the probability amplitudes in the strong coupling regime given the proper set of states. Two bands of levels were obtained and both intraband and interband transitions were shown. In the limit of a large number of photons in the cavity, the Rabi frequency is proportional to the integer number Bessel functions. We will recover this result below. Rather interestingly, the square of the amplitudes of the levels involved in the Rabi oscillations is a Poisson distribution. Rabi oscillations arise from the crossing of the energy levels of the Dicke model and appear between macroscopic superposition of charge states (known in the current literature as Schrödinger cat states).

In the Dicke model, two-level systems are coupled by the radiation field. The same can happen with Josephson junctions.[20] So, our aim is to see how collective effects can emerge when more Josephson junctions are coupled by a cavity field. We will prove that Rabi oscillations also emerge in this case with the Rabi frequencies proportional to integer order Bessel functions. A selection rule arises for situations with more qubits as certain transitions are not allowed. In turn this means that the transition to some qubit configurations cannot be realized. Then, states involved in the Rabi oscillations are always macroscopic superpositions of entangled states between the radiation field, which can have a large number of photons, and the junctions.

Finally, we extend the analysis to the thermodynamic limit showing that, in this case, a large number of Josephson junctions coupled by a cavity field in the strong coupling regime can be used to amplify vacuum fluctuations of the field to a large macroscopic field, assuming all the junctions in their ground state. Decoherence effects may be prominent in this case, as we will discuss. This effect defines a quantum amplifier (QAMP) as proposed by us in the literature.[14,21,22]

The paper is structured as follows. In Sec. II we introduce the model describing a number of Josephson junctions interacting through a cavity field. In Sec. III we analyze explicitly the cases for one, two, and three coupled qubits obtaining the Rabi frequencies and proving, in the limit of a large number of photons, the proportionality with integer order Bessel functions. In Sec. IV we discuss the thermodynamic limit, that is, we see the physics of a large number of Josephson junctions in a cavity proving that an amplification of the vacuum fluctuations of the radiation field in the cavity is obtained. This device we term QAMP. Finally, in Sec. V conclusions are given.

II. DESCRIPTION OF THE MODEL

The model we should consider for a single Josephson junction, treated in the two-level approximation, in a cavity field is (here and in the following $\hbar=1$)

$$H = \Delta S_z + \omega a^\dagger a + 2gS_x(a^\dagger + a), \quad (1)$$

Δ being the separation between the ground and the first excited state in a Cooper pair,[12,13] ω the frequency of the field in the cavity, g the coupling between the junction and the cavity, $a^\dagger$ and a creation and annihilation operators, and $S_x, S_z = \sigma_x/2, \sigma_z/2$ with σ_x, σ_z Pauli matrices.

Equation (1) is just the Dicke model that we have specialized to a single qubit with Pauli matrices. This generalizes immediately to any number of qubits putting

$$S_x = \frac{1}{2}\sum_{i=1}^{N} \sigma_{x,i},$$

$$S_z = \frac{1}{2}\sum_{i=1}^{N} \sigma_{z,i}, \quad (2)$$

N being the number of qubits.

One may think that Eq. (1) differs from the Hamiltonian currently used to describe Josephson junction qubits in experiments[8,9] but these Hamiltonians are equivalent as we can interchange σ_x and σ_z with the unitary transformation $e^{i(\pi/4)\sigma_y}$ and what one gets, in the analysis in the strong coupling regime, is a shift to the energy levels. Thus both Hamiltonians give rise to the same physics and we can safely work with the Dicke model as currently known.

In order to work out the analysis of the model in the strong coupling regime proper to the experiments carried out so far, we follow the same approach used in Ref. 10. We split the Hamiltonian (1) as

$$H_0 = \omega a^\dagger a + 2gS_x(a^\dagger + a),$$

$$H_1 = \Delta S_z, \quad (3)$$

H_0 being the unperturbed Hamiltonian and H_1 the perturbation. The unperturbed part H_0 can be immediately diagonalized by the eigenstates, as can be directly verified, treating S_x as a c number,

$$|[n];S,S_x\rangle = e^{(2g/\omega)S_x(a-a^\dagger)}|n\rangle|S,S_x\rangle \quad (4)$$

S_x being the component of the spin along the x axis chosen as the axis of quantization, $|S,S_x\rangle$ the corresponding eigenstates, and $a^\dagger a|n\rangle = n|n\rangle$ and $n=0,1,2,\ldots$ an integer. Energy eigenvalues are given by

$$E_{n,S_x} = n\omega - \frac{4g^2S_x^2}{\omega} \quad (5)$$

and we observe a degeneracy between positive and negative values of the spin components. Then we can introduce the unitary evolution operator

$$U_F(t) = \sum_{n,S_x} e^{-i[n\omega-(4g^2S_x^2/\omega)]t}|[n];S,S_x\rangle\langle[n];S,S_x|. \quad (6)$$

We can apply this unitary operator to the initial Hamiltonian (1) and the problem is reduced to the Hamiltonian

$$\tilde{H} = \sum_{m,S_x}\sum_{n,S'_x} e^{i[m\omega-(4g^2S_x^2/\omega)]t}e^{-i[n\omega-(4g^2S_x'^2/\omega)]t}\Delta|[m];S,S_x\rangle \times\langle[n];S,S'_x|\langle[m];S,S_x|S_z|[n];S,S'_x\rangle. \quad (7)$$

The matrix elements can be evaluated by noting that $S_z = (S_+ - S_-)/2i$ and one gets

$$\langle[m];S,S_x|S_z|[n];S,S'_x\rangle = \frac{1}{2i}\left(\delta_{S_x,S'_x+1}\sqrt{S(S+1)-S'_x(S'_x+1)}M^-_{mn} - \delta_{S_x,S'_x-1}\sqrt{S(S+1)-S'_x(S'_x-1)}M^+_{mn}\right) \quad (8)$$

where we have put[10]

$$M^\pm_{mn} = \langle m|e^{\pm(2g/\omega)(a-a^\dagger)}|n\rangle = \sqrt{\frac{n!}{m!}}e^{-2g^2/\omega^2}\left(\pm\frac{2g}{\omega}\right)^{m-n}L_n^{m-n}\left(\frac{4g^2}{\omega^2}\right) \quad (9)$$

L_n^{m-n} being Laguerre polynomials.

Now the Hamiltonian (7) can be split into two parts to give a diagonal part

$$\tilde{H}_0 = \sum_{n,S_x}\frac{1}{2i}\Delta M_n\left(e^{-i[4g^2(2S_x-1)/\omega]t}\sqrt{S(S+1)-S_x(S_x-1)} \times|[n];S,S_x\rangle\langle[n];S,S_x-1| - e^{i[4g^2(2S_x+1)/\omega]t}\sqrt{S(S+1)-S_x(S_x+1)}|[n];S,S_x\rangle \times\langle[n];S,S_x+1|\right), \quad (10)$$

having set

$$M_n = M^\pm_{nn} = e^{-2g^2/\omega^2}L_n\left(\frac{4g^2}{\omega^2}\right), \quad (11)$$

and an off-diagonal part

$$\tilde{H}_1 = \sum_{\substack{m,n,S_x\\ m\neq n}}\frac{1}{2i}\Delta e^{-i(n-m)\omega t}\left(e^{-i[4g^2(2S_x-1)/\omega]t}M^-_{mn}\sqrt{S(S+1)-S_x(S_x-1)}|[m];S,S_x\rangle\langle[n];S,S_x - 1| - e^{i[4g^2(2S_x+1)/\omega]t}M^+_{mn}\sqrt{S(S+1)-S_x(S_x+1)}|[m];S,S_x\rangle\langle[n];S,S_x+1|\right). \quad (12)$$

In order to obtain the equations for the probability amplitudes, we diagonalize Hamiltonian (10) obtaining the energy eigenvalues, the eigenstates, and the geometrical part of the phase as we can have time-dependent eigenstates, in agreement to the general approach outlined in Refs. 10 and 23. Then, by interaction picture, the amplitude equations are obtained using Hamiltonian (12).

In the analysis we pursue, in the following we will see that Hamiltonian (10) has eigenvalues $s_x \Delta M_n$ having s_x the same values of the spin projection S_x and one can have transitions between the eigenstates only if the selection rule $\Delta s_x = 0, \pm 1$ holds. We recognize that each level s_x develops a band of levels numbered by integer n, so the selection rule provides intraband $(s_x = \tilde{s}_x)$ and interband $(s_x \neq \tilde{s}_x)$ permitted transitions at resonance, respectively.

In the limit of a large number of photons we can show, in all cases below, that the Rabi frequencies are proportional to integer order Bessel functions. This can be accomplished using the relation[10]

$$J_\alpha(2\sqrt{nx}) = e^{-x/2}\left(\frac{x}{n}\right)^{\alpha/2} L_n^\alpha(x) \tag{13}$$

that holds in the limit of n going to infinity.

III. ANALYSIS OF THE MODEL

We apply the procedure outlined in the previous section to one, two, and three qubits to see in detail the physics of the model. The one qubit case has been already discussed in literature[10] but we present it here again to show the way our method works to make this paper self-contained.

A. One qubit

In the one qubit case we have $S_x = \frac{1}{2}, -\frac{1}{2}$ being $S = \frac{1}{2}$. The diagonal part of the Hamiltonian will be

$$\tilde{H}_0 = \sum_n \frac{1}{2i}\Delta M_n \left(\left| [n]; \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle [n]; \frac{1}{2}, -\frac{1}{2} \right| - \left| [n]; \frac{1}{2}, -\frac{1}{2} \right\rangle \times \left\langle [n]; \frac{1}{2}, \frac{1}{2} \right| \right), \tag{14}$$

that is time-independent in this case. Similarly, we have the off-diagonal part

$$\tilde{H}_1 = \sum_{\substack{m,n \\ m \neq n}} \frac{1}{2i}\Delta e^{-i(n-m)\omega t}\left(M_{mn}^- \left| [m]; \frac{1}{2}, \frac{1}{2} \right\rangle \left\langle [n]; \frac{1}{2}, -\frac{1}{2} \right| - M_{mn}^+ \left| [m]; \frac{1}{2}, -\frac{1}{2} \right\rangle \left\langle [n]; \frac{1}{2}, \frac{1}{2} \right| \right). \tag{15}$$

This operator is Hermitian as can be verified using the fact that sum indexes are dummy.

Hamiltonian (14) can be immediately diagonalized with eigenvalues $E_{n,s_x} = s_x \Delta M_n$, $s_x = \frac{1}{2}, -\frac{1}{2}$ corresponding to the eigenstates

$$|n; s_x\rangle = \frac{1}{\sqrt{2}}\left(\left| [n]; \frac{1}{2}, -\frac{1}{2} \right\rangle - \frac{i}{2s_x} \left| [n]; \frac{1}{2}, \frac{1}{2} \right\rangle \right) \tag{16}$$

that are not eigenstates of the spin projection S_x. Then, we can write the equation for the probability amplitudes by looking for a time-dependent solution in the form

$$|\psi(t)\rangle = \sum_{k,s_x} e^{-iE_{k,s_x}t} c_{k,s_x}(t) |k; s_x\rangle \tag{17}$$

where no contribution enters due to geometrical phases as Hamiltonian (14) is time-independent. The Schrödinger equation takes the form

$$i\dot{c}_{m,\tilde{s}_x}(t) = \frac{\Delta}{2}\sum_{n,s_x} e^{-i[E_{n,s_x} - E_{m,\tilde{s}_x} + (n-m)\omega]t} \times (\tilde{s}_x M_{mn}^- + s_x M_{mn}^+) c_{n,s_x}(t). \tag{18}$$

We recognize the resonance conditions

$$E_{n,s_x} - E_{m,\tilde{s}_x} + (n-m)\omega = 0 \tag{19}$$

with $\Delta s_x = 0$ for intraband transitions and $\Delta s_x = \pm 1$ for interband transitions as said above. These resonance conditions correspond to crossing of the energy levels of the Dicke model. Energy levels are given by[24]

$$E_{n,s_x} = n\omega - \frac{4s_x^2 g^2}{\omega} + s_x \Delta M_n \tag{20}$$

where we can recognize the band structure and the degeneracy for $s_x = \pm\frac{1}{2}$ that is removed by the last term. Then resonance conditions can be straightforwardly interpreted as crossings of energy levels. Such crossings produce Rabi oscillations as experimentally observed.

Finally, let us check the Rabi frequencies for the single qubit case. It is not difficult to realize that we have for intraband transitions $\Delta s_x = 0$ from Eq. (18), assuming the rotating wave approximation due to the resonant terms,

$$\mathcal{R}_1 = \frac{1}{2}\Delta(M_{mn}^+ + M_{mn}^-) = \Delta \left\langle m \left| \cosh\left[\frac{2g}{\omega}(a - a^\dagger) \right] \right| n \right\rangle \tag{21}$$

and for interband transitions $\Delta s_x = \pm 1$

$$\mathcal{R}_1' = \frac{1}{2}\Delta(M_{mn}^+ - M_{mn}^-) = \Delta \left\langle m \left| \sinh\left[\frac{2g}{\omega}(a - a^\dagger) \right] \right| n \right\rangle. \tag{22}$$

Now we can use Eq. (9) to show that transitions with states differing by an odd number of photons, $m = n + 2N + 1$, are permitted for intraband transitions and an even number of photons are involved, $m = n + 2N$, in interband transitions.[10] Using Sterling formula and Eq. (13) one gets, in the limit of Fock states with a large number of photons,

$$\mathcal{R}_1 \approx \Delta J_{2N+1}\left(\frac{4\sqrt{n}g}{\omega} \right) \tag{23}$$

and

$$\mathcal{R}_1' \approx \Delta J_{2N}\left(\frac{4\sqrt{n}g}{\omega}\right), \tag{24}$$

completely in agreement with experimental results by Nakamura, Pashkin, and Tsai.[8] As also experimentally observed, crossing at $m=n$ can happen, belonging to interband transitions with $N=0$. Further details for a single qubit case are given in Ref. 10.

B. Two qubits

For the case of two qubits one has

$$\begin{aligned}\widetilde{H}_0 = \sum_n \frac{1}{2i}\sqrt{2}\Delta M_n(&e^{-i(4g^2/\omega)t}|[n];1,1\rangle\langle[n];1,0| \\ &+ e^{i(4g^2/\omega)t}|[n];1,0\rangle\langle[n];1,-1| - e^{i(4g^2/\omega)t}|[n];1,0\rangle \\ &\times\langle[n];1,1| - e^{-i(4g^2/\omega)t}|[n];1,-1\rangle\langle[n];1,0|).\end{aligned} \tag{25}$$

Now the Hamiltonian shows time-dependence and it is not straightforward to interpret the resonance conditions as energy level crossings. On the same ground one has

$$\begin{aligned}\widetilde{H}_1 = \sum_{\substack{m,n \\ m\neq n}} e^{-i(n-m)\omega t}\frac{1}{2i}\sqrt{2}\Delta(&e^{-i(4g^2/\omega)t}M^-_{mn}|[m];1,1\rangle\langle[n];1,0| \\ &+ e^{i(4g^2/\omega)t}M^-_{mn}|[m];1,0\rangle\langle[n];1,-1| \\ &- e^{i(4g^2/\omega)t}M^+_{mn}|[m];1,0\rangle\langle[n];1,1| \\ &- e^{-i(4g^2/\omega)t}M^+_{mn}|[m];1,-1\rangle\langle[n];1,0|).\end{aligned} \tag{26}$$

One can immediately diagonalize Hamiltonian (25) obtaining eigenvalues $E_{n,s_x}=s_x\Delta M_n$ with $s_x=1,0,-1$. The corresponding eigenvectors are

$$\begin{aligned}|n;1,t\rangle &= \frac{1}{2i}e^{-i(4g^2/\omega)t}|[n];1,1\rangle + \frac{1}{\sqrt{2}}|[n];1,0\rangle \\ &\quad - \frac{1}{2i}e^{-i(4g^2/\omega)t}|[n];1,-1\rangle,\end{aligned}$$

$$|n;0,t\rangle = \frac{1}{\sqrt{2}}|[n];1,1\rangle + \frac{1}{\sqrt{2}}|[n];1,-1\rangle,$$

$$\begin{aligned}|n;-1,t\rangle &= -\frac{1}{2i}e^{-i(4g^2/\omega)t}|[n];1,1\rangle + \frac{1}{\sqrt{2}}|[n];1,0\rangle \\ &\quad + \frac{1}{2i}e^{-i(4g^2/\omega)t}|[n];1,-1\rangle\end{aligned} \tag{27}$$

that do not give any geometrical phase as $\langle n;1,t|i\partial_t|n;1,t\rangle =\langle n;0,t|i\partial_t|n;0,t\rangle=\langle n;-1,t|i\partial_t|n;-1,t\rangle=0$. Thus we can put again

$$|\psi(t)\rangle = \sum_{k,s_x} e^{-iE_{k,s_x}t}c_{k,s_x}(t)|k;s_x,t\rangle \tag{28}$$

and obtain the equations for the amplitudes

$$\begin{aligned}i\dot{c}_{k,1}(t) &= \sum_n \frac{\Delta}{4}(M^-_{kn}+M^+_{kn})e^{-i[E_{n,1}-E_{k,1}+(n-k)\omega]t}c_{n,1}(t) \\ &\quad + \sqrt{2}\frac{\Delta}{4i}(M^-_{kn}-M^+_{kn})e^{i[E_{k,1}+4g^2/\omega-(n-k)\omega]t}c_{n,0}(t), \\ i\dot{c}_{k,0}(t) &= \sum_n \sqrt{2}\frac{\Delta}{4i}(M^-_{kn}-M^+_{kn})e^{-i[E_{k,1}+4g^2/\omega+(n-k)\omega]t}c_{n,1}(t) \\ &\quad + e^{-i[E_{k,-1}+4g^2/\omega+(n-k)\omega]t}c_{n,-1}(t), \\ i\dot{c}_{k,-1}(t) &= \sum_n -\frac{\Delta}{4}(M^-_{kn}+M^+_{kn})e^{-i[E_{n,-1}-E_{k,-1}+(n-k)\omega]t}c_{n,-1}(t) \\ &\quad + \sqrt{2}\frac{\Delta}{4i}(M^-_{kn}-M^+_{kn})e^{i[E_{k,-1}+4g^2/\omega-(n-k)\omega]t}c_{n,0}(t)\end{aligned} \tag{29}$$

where we can recognize again the effect of the selection rules $\Delta s_x=0,\pm1$ on the permitted transitions, originating from the resonance conditions

$$E_{n,\pm1} - E_{k,\pm1} + (n-k)\omega = 0,$$
$$E_{k,\pm1} + \frac{4g^2}{\omega} - (n-k)\omega = 0 \tag{30}$$

for intraband and interband transitions, respectively, giving rise to Rabi oscillations in the two-qubit system. Again, we can interpret these resonance conditions as originating from the crossings of the energy levels for the Dicke model

$$E_{n,s_x} = n\omega - \frac{4s_x^2g^2}{\omega} + s_x\Delta M_n \tag{31}$$

that is degenerate with respect to $s_x=\pm1$ but the degeneracy is removed by the last term. The Rabi frequencies can be computed from Eqs. (29) with the rotating wave approximation imposing the resonance conditions (30) and are given by, for intraband transitions,

$$\mathcal{R}_2 = \mathcal{R}_1 \tag{32}$$

and for interband transitions

$$\mathcal{R}_2' = \sqrt{2}\mathcal{R}_1'. \tag{33}$$

These frequencies, in the limit of a large number of photons, display dependence on the integer order Bessel functions, being proportional to $\mathcal{R}_1$ and $\mathcal{R}_1'$ as in the single qubit case. A cooperative effect arises from the coherent behavior of both the junctions entering entangled states and producing Rabi oscillations.

Thus the main conclusion of this section is that the two qubits act collectively producing Rabi oscillations, exactly as in the case of a single qubit. Such oscillations appear between entangled states to be considered macroscopic.

C. Three qubits

In order to gain further insight into the physics of Josephson junctions in this case, we analyze a system with three qubits. This case is rather different from the preceding ones, as H_0 depends on time and we have to apply a theorem for strong coupling that imposes formally the adiabatic theorem at the leading order.[23] So there is a geometrical contribution to the phases. This does not make it straightforward to interpret the resonance conditions as energy level crossings. We have

$$\begin{aligned}\tilde{H}_0 = \sum_n \frac{1}{2i}\Delta M_n\bigg(& e^{-i(8g^2/\omega)t}\sqrt{3}\left|[n];\frac{3}{2},\frac{3}{2}\right\rangle\left\langle[n];\frac{3}{2},\frac{1}{2}\right| \\ & + 2\left|[n];\frac{3}{2},\frac{1}{2}\right\rangle\left\langle[n];\frac{3}{2},-\frac{1}{2}\right| - e^{i(8g^2/\omega)t}\sqrt{3}\left|[n];\frac{3}{2},\frac{1}{2}\right\rangle \\ & \times\left\langle[n];\frac{3}{2},\frac{3}{2}\right| + e^{i(8g^2/\omega)t}\sqrt{3}\left|[n];\frac{3}{2},-\frac{1}{2}\right\rangle\left\langle[n];\frac{3}{2},-\frac{3}{2}\right| \\ & - 2\left|[n];\frac{3}{2},-\frac{1}{2}\right\rangle\left\langle[n];\frac{3}{2},\frac{1}{2}\right| - e^{-i(8g^2/\omega)t}\sqrt{3}\left|[n];\frac{3}{2},-\frac{3}{2}\right\rangle \\ & \times\left\langle[n];\frac{3}{2},-\frac{1}{2}\right|\bigg)\end{aligned} \tag{34}$$

and

$$\begin{aligned}\tilde{H}_1 = \sum_{\substack{m,n\\ m\neq n}} e^{-i(n-m)\omega t}\frac{1}{2i}\Delta\bigg(& e^{-i(8g^2/\omega)t}\sqrt{3}M_{mn}^-\left|[m];\frac{3}{2},\frac{3}{2}\right\rangle \\ & \times\left\langle[n];\frac{3}{2},\frac{1}{2}\right| + 2M_{mn}^-\left|[m];\frac{3}{2},\frac{1}{2}\right\rangle\left\langle[n];\frac{3}{2},-\frac{1}{2}\right| \\ & - e^{i(8g^2/\omega)t}\sqrt{3}M_{mn}^+\left|[m];\frac{3}{2},\frac{1}{2}\right\rangle\left\langle[n];\frac{3}{2},\frac{3}{2}\right| \\ & + e^{i(8g^2/\omega)t}\sqrt{3}M_{mn}^-\left|[m];\frac{3}{2},-\frac{1}{2}\right\rangle\left\langle[n];\frac{3}{2},-\frac{3}{2}\right| \\ & - 2M_{mn}^+\left|[m];\frac{3}{2},-\frac{1}{2}\right\rangle\left\langle[n];\frac{3}{2},\frac{1}{2}\right| \\ & - e^{-i(8g^2/\omega)t}\sqrt{3}M_{mn}^+\left|[m];\frac{3}{2},-\frac{3}{2}\right\rangle\left\langle[n];\frac{3}{2},-\frac{1}{2}\right|\bigg).\end{aligned} \tag{35}$$

It is straightforward to diagonalize Hamiltonian (34) with the eigenstates

$$\begin{aligned}|n;s_x,t\rangle = \beta(s_x)\bigg[& -ie^{-i(8g^2/\omega)t}\frac{\sqrt{3}}{2s_x}\left|[n];\frac{3}{2},\frac{3}{2}\right\rangle \\ & + \left|[n];\frac{3}{2},\frac{1}{2}\right\rangle + i\left(s_x-\frac{3}{4s_x}\right)\left|[n];\frac{3}{2},-\frac{1}{2}\right\rangle \\ & - e^{-i(8g^2\omega)t}\frac{\sqrt{3}}{2}\left(1-\frac{3}{4s_x^2}\right)\left|[n];\frac{3}{2},\frac{3}{2}\right\rangle\bigg]\end{aligned} \tag{36}$$

being

$$\beta(s_x) = \frac{1}{\sqrt{s_x^2+\dfrac{3}{16s_x^2}+\dfrac{27}{64s_x^4}+\dfrac{1}{4}}} \tag{37}$$

and $s_x=\frac{3}{2},\frac{1}{2},-\frac{1}{2},-\frac{3}{2}$. As in the other cases, we get entangled macroscopic states. In this case we have geometrical phases as given by

$$\dot{\gamma}(s_x) = \frac{2g^2}{\omega}\frac{3-\dfrac{3}{2s_x^2}+\dfrac{27}{16s_x^4}}{s_x^2+\dfrac{3}{16s_x^2}+\dfrac{27}{64s_x^4}+\dfrac{1}{4}} \tag{38}$$

that reduces to $2g^2/\omega$ for $s_x=\frac{3}{2}$, $-\frac{3}{2}$ and $6g^2/\omega$ for $s_x=\frac{1}{2},-\frac{1}{2}$.

For the given $\tilde{H}_0$ and $\tilde{H}_1$ we seek a solution like

$$|\psi(t)\rangle = \sum_{k,s_x} e^{-iE_{k,s_x}t}e^{-i\dot{\gamma}_{s_x}t}c_{k,s_x}(t)|k;s_x,t\rangle \tag{39}$$

yielding the equations for the amplitudes

$$\begin{aligned}i\dot{c}_{m,\tilde{s}_x}(t) = \sum_{n,s_x} & e^{-i[E_{n,s_x}-E_{m,\tilde{s}_x}+(n-m)\omega+\dot{\gamma}_{\tilde{s}_x}-\dot{\gamma}_{s_x}]t}\Delta\beta(s_x)\beta(\tilde{s}_x) \\ & \times[\alpha(s_x,\tilde{s}_x)M_{mn}^+ + \alpha(\tilde{s}_x,s_x)M_{mn}^-]c_{n,s_x}(t)\end{aligned} \tag{40}$$

having set

$$\alpha(s_x,\tilde{s}_x) = \frac{3}{4s_x}+\tilde{s}_x-\frac{3}{4\tilde{s}_x}+\frac{3}{4}\left(1-\frac{3}{4\tilde{s}_x^2}\right)\left(s_x-\frac{3}{4s_x}\right). \tag{41}$$

Also in this case, it is not difficult to verify that the selection rules $\Delta s_x=0,\pm 1$ hold corresponding to intraband ($s_x=\tilde{s}_x$) and interband ($s_x\neq\tilde{s}_x$) transitions, respectively, with the resonance conditions

$$E_{n,s_x}-E_{m,\tilde{s}_x}+(n-m)\omega+\dot{\gamma}_{\tilde{s}_x}-\dot{\gamma}_{s_x}=0. \tag{42}$$

The Rabi frequencies can be computed from Eqs. (40) with the rotating wave approximation imposing the resonance conditions (42) and can be given explicitly as follows

$$\mathcal{R}_{3/2,3/2}=\mathcal{R}_{-3/2,-3/2}=\frac{3}{2}\mathcal{R}_1,$$

$$\mathcal{R}_{1/2,1/2}=\mathcal{R}_{-1/2,-1/2}=\frac{1}{2}\mathcal{R}_1 \tag{43}$$

for intraband transitions and

$$\mathcal{R}_{3/2,1/2}=\mathcal{R}_{1/2,3/2}=\frac{\sqrt{3}}{2}\mathcal{R}_1',$$

$$\mathcal{R}_{-3/2,-1/2}=\mathcal{R}_{-1/2,-3/2}=\frac{\sqrt{3}}{2}\mathcal{R}_1',$$

$$\mathcal{R}_{1/2,-1/2}=\mathcal{R}_{-1/2,1/2}=\mathcal{R}_1' \tag{44}$$

for interband transitions. Also, for three qubit the Rabi frequencies are proportional to $\mathcal{R}_1$ and $\mathcal{R}_1'$, so in the limit of a large number of photons one has that these frequencies are given by integer order Bessel functions. This rule applies to all cases.

As also seen for one and two qubits we can see again a collective effect. The behavior of the three qubits is coherent, producing entangled states with Rabi oscillations. Apart from numerical factors, the Rabi frequencies are the same as in the single qubit case. It would be interesting to extend the analysis to higher spins to see if this is a rule.

IV. THERMODYNAMIC LIMIT: QUANTUM AMPLIFIER

To analyze this case, we limit the study to the leading order as higher orders become increasingly less important as the number of Josephson junctions increases.[22] This permits us to write down immediately the unitary evolution being given by H_0 in Eq. (6). It is important to emphasize that, in order to unveil this effect, we should avoid random phases putting all the junctions in the same state. This makes the case somewhat different from above but shows again a collective effect possible to be observed.

Now let us assume that the field in the cavity is in the ground state and *all the Josephson junctions are in their lower state* externally imposed. Unitary evolution gives

$$|\phi(t)\rangle = \sum_n e^{-i(n\omega - g^2N^2/\omega)t}\left|[n];\frac{N}{2},-\frac{N}{2}\right\rangle \times e^{-N^2g^2/\omega^2}\left(\frac{Ng}{\omega}\right)^n\frac{1}{\sqrt{n!}}. \tag{45}$$

So, leaving aside the contributions of the Josephson junctions, one is left with

$$|N\chi(t)\rangle = e^{-(Ng/\omega)(a-a^\dagger)}\sum_n e^{-i(n\omega - g^2N^2/\omega)t} \times e^{-N^2g^2/\omega^2}\left(\frac{Ng}{\omega}\right)^n\frac{1}{\sqrt{n!}}|n\rangle, \tag{46}$$

that is, a coherent state with a parameter proportional to N. In the large N limit this represents a classical state of the radiation field[25] and we see that we have amplified quantum fluctuations of the ground state of the radiation field to a classical level. The interesting aspect of this already known result is that it can be practically realized through Josephson junctions that lend themselves to realize this kind of classical state out of a fully quantum initial state. One realizes a QAMP (quantum amplifier).[14,22] The device can be realized without too much care about taking just the limit $N\to\infty$ or $N\to\infty$, $V\to\infty$, V being the volume of the cavity, and N/V = const as we are in the strong coupling limit.

This represents a very peculiar collective effect of a large number of Josephson junctions in a cavity in the strong coupling regime, the same regime devised in the experiments of Nakamura's group[6,7] for a single qubit.

V. DISCUSSION AND CONCLUSIONS

The above discussed collective effects can have their limit in the appearance of decoherence. The experiments of Nakamura's group display the decay of Rabi oscillation but the decay time can be long enough to permit quantum computation. Neither the nature of this decoherence effect nor the way it scales with the number of Josephson junctions are known as far as we know. It should be expected that the same should happen for the QAMP producing a classical radiation field. The origin of decoherence on Josephson junctions is to be understood but it could not be excluded that revival and collapse effects of Rabi oscillations may be at work as happens in the weak coupling regime.[12,13]

We have shown how, in the strong coupling limit as devised by Nakamura's group, several collective effects such as Rabi oscillations and quantum amplification could appear in Josephson junctions coupled by a radiation field. Rabi oscillations, in the limit of a large number of photons, have frequencies proportional to integer order Bessel functions and a selection rule applies limiting transitions. These effects could be useful for applications in quantum computation or to use a large number of Josephson junctions to produce a laser field.

*Email address: marcofrasca@mclink.it

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