Equilibrium entanglement vanishes at finite temperature

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We show that the equilibrium entanglement of a bipartite system having a finite number of quantum states vanishes at finite temperature, for arbitrary interactions between its constituents and with the environment.

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The interplay between the properties of condensed matter systems and the notion of entanglement has recently come into the focus of active research.[1–12] In the present paper, we exploit the general topology of state space and, thereby, formulate a rigorous statement on the disappearance of entanglement in equilibrium condensed matter systems as a function of temperature. Below, in order to be specific, we first consider the simplest case of two quantum two-level systems (TLS's), and then generalize the result to the case of bipartite systems with arbitrary (finite) dimension of the Hilbert space.

Consider two TLS's interacting with each other and with some environment. The coupling to the environment may be strong or weak. Typical examples are two coupled spins $1/2$ belonging to a larger spin lattice, or two two-level defects in an environment of phonons. The two TLS's may become entangled with each other either as a result of direct interaction, or via interaction with their common environment. We focus on the family of the reduced equilibrium density matrices of these two TLS's obtained by tracing out the environmental degrees of freedom. The entanglement present in the equilibrium density matrix of the two TLS's will be called "equilibrium entanglement."

In principle, there exist theoretical settings in which equilibrium entanglement is absent at all temperatures. However, in another very common situation, when two interacting TLS's are entangled at sufficiently low temperatures, the following statement holds:

Equilibrium entanglement between two TLS's always vanishes at a finite temperature.

The constructive meaning of this observation is that the disappearance of entanglement never exhibits the character of continuous crossover extending to the infinite temperature. This is consistent with all results so far reported on specific physical systems that admit an analytical or numerical computation of some entanglement measure.[1,2,6,7,13,14] In general, however, the accurate calculation of the reduced density matrix for two TLS's, which strongly interact with the environment, is either very difficult or impossible. Notwithstanding, the above statement remains valid, no matter how complex the TLS's-plus-environment Hamiltonian may be. Therefore, it is applicable to all kinds of *experimentally relevant* TLS's.

It is also worth mentioning that the rule formulated above does not constitute a piece of common knowledge in the field of condensed matter physics. Unlike the case of pure states, the notion of entanglement for mixed state density matrices is not intuitive, as it discriminates correlations that can be described classically from those that cannot. *A priori*, one may have a conflicting intuition about the onset of entanglement, since, on the one hand, it is associated with some kind of quantum coherence, which may require a phase transition. On the other hand, if we focus on the case of two two-level defects surrounded by a phonon bath, the finiteness of the problem excludes the notion of a phase transition. In turn, this suggests that the onset of entanglement may have a a crossoverlike character, extending to the infinite temperature.

Our statement is based on the following argument:

Let us consider a family of 4×4 equilibrium density matrices ϱ defined in the basis $\{|\uparrow,\uparrow\rangle, |\uparrow,\downarrow\rangle, |\downarrow,\uparrow\rangle, |\downarrow,\downarrow\rangle\}$, where arrows $\uparrow$ and $\downarrow$ encode the two states of each TLS. As a measure of entanglement, we use concurrence, which can be calculated as[15]

$$c(\varrho) = \max[\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0], \tag{1}$$

where λ_i are the square roots of the eigenvalues of

$$\varrho\sigma_{1y}\sigma_{2y}\varrho^*\sigma_{1y}\sigma_{2y}, \tag{2}$$

and λ_1 is the largest among them. Here ϱ^* is the complex conjugate of ϱ, and σ_{1y} and σ_{2y} are the second Pauli matrices

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

acting on the first and the second TLS, respectively. The two TLS's are entangled only when $c>0$, which, in turn, is only possible when $\lambda_1-\lambda_2-\lambda_3-\lambda_4>0$.

At infinite temperature, the equilibrium density matrix has a particularly simple form

$$\varrho_\infty = \mathbb{1}/4, \tag{3}$$

which does not depend on the form of the TLS's-plus-environment Hamiltonian. In practice, the infinite temperature limit is reached when the temperature is much higher than the typical energy per one of the TLS's. (This energy is the sum of the interaction energies with the other TLS and with the environment.)

Upon substitution of ϱ_∞ given by Eq. (3) into Eq. (2), we obtain a matrix of the form $\mathbb{1}/16$, such that $\lambda_i=1/4$ ($i=1,\ldots,4$), and $\lambda_1-\lambda_2-\lambda_3-\lambda_4=-1/2$. Since the values of λ_i depend continuously on the matrix elements of ϱ, which, in turn, have a continuous dependence on the inverse temperature $1/T$, the value of $\lambda_1-\lambda_2-\lambda_3-\lambda_4$ can only turn positive after increasing the inverse temperature from zero to a finite value. Given Eq. (1), this is the precise equivalent of what was stated initially.

The generalization to equilibrium entanglement of bipartite quantum systems of arbitrary finite dimension follows from the topological observation[16] stating that, in the space of all possible $N \times N$ density matrices, the (infinite temperature) density matrix $\mathbb{1}/N$ is surrounded by a region of separable quantum states, for any bipartite decomposition of the N-level system. (See also Ref. 17.) Therefore, an infinite temperature density matrix cannot be transformed into an entangled density matrix by an infinitesimal change of its matrix elements.

A natural question arising in the above context is: Does the entanglement transition imply any observable effect? Promising evidence for one such an effect has recently emerged from the experimental and theoretical study[8] of insulating magnetic salt $LiHo_xY_{1-x}F_4$, which, for our present theoretical purposes, can be described as a set of spins $1/2$ randomly placed on a crystal lattice and coupled to each other by magnetic dipolar interaction. This study has shown that the onset of entanglement may explain the qualitative difference between the structured behavior of specific heat and the featureless behavior of magnetic susceptibility. It was also shown that the onset of entanglement is accompanied by rapidly growing deviations between predictions from classical and quantum theory.

Another interesting possibility is that the entanglement transition may signify a limit beyond which various high-temperature expansion techniques become unreliable. Since any high-temperature expansion starts from the infinite temperature, and since there is no entanglement in a finite range around infinite temperature, it remains an open question whether such techniques can possibly generate an entangled density matrix at finite temperature.

In conclusion, we hope that the fact that, in a large class of condensed matter systems, the entanglement transition occurs at finite temperature will stimulate further experimental and theoretical efforts to characterize this transition.

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