Quench dynamics of a superfluid Fermi gas

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With an eye toward the interpretation of so-called "cosmological" experiments performed on the low-temperature phases of ^{3}He, in which regions of the superfluid are destroyed by local heating with neutron radiation, we have studied the behavior of a Fermi gas subjected to uniform variations of an attractive BCS interaction parameter λ. In ^{3}He, the quenches induced by the rapid cooling of the "hot spots" back through the transition may lead to the formation of vortex loops via the Kibble-Zurek mechanism. A consideration of the free energy available in the quenched region for the production of such vortices reveals that the Kibble-Zurek scaling law gives at best a lower bound on the defect spacing. Further, for quenches that fall far outside the Ginzburg-Landau regime, the dynamics on the pair subspace, as initiated by quantum fluctuations, tends irreversibly to a self-driven steady state with a gap $\Delta_\infty = \epsilon_C (e^{2/N(0)\lambda} - 1)^{-1/2}$. In weak coupling, this is only half the BCS gap, the extra energy being taken up by the residual collective motion of the pairs.

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I. INTRODUCTION

It is widely believed that the vacuum immediately following the Big Bang proceeded from a state of high symmetry through a series of symmetry-breaking phase transitions during the subsequent expansion and cooling of the universe.[1] For sufficiently rapid expansion, the spatial extent of any order parameters emerging at such transitions would have been limited by the causal horizon. On this basis, it was proposed that the early universe spontaneously acquired a domain structure characterized by independently directed order parameters in each domain. The frustrated dynamics resulting from such a structure may have left behind measurable traces in the form of topological defects.

If we imagine, with Kibble,[2] the simplest case corresponding to the breaking of a global $U(1)$ gauge symmetry, it is then clear, provided we treat the fields classically, that these domains adopt uncorrelated $U(1)$ orientations. At junctions between three or more such domains, it will sometimes occur that the $U(1)$ phase of the order parameter winds by 2π about a filamentary region corresponding to the symmetry-unbroken state. In light of the topological constraint of quantized circulation in the new phase, this circumstance may be viewed as the "trapping" of a vortex core by the frustrated dynamics. According to this picture, the initial domain structure resolves itself very quickly into a tangle of vortex loops moving in the background of the new phase.

As suggested by Zurek,[3] one can test this idea by looking for topological defects following controlled quenches in condensed matter systems like ^{4}He, which exhibits exactly the kind of $U(1)$ symmetry breaking invoked by Kibble. Although subsequent tests[4] in this system failed to show any vortex formation associated with uniform pressure quenches through the lambda-line, the basic idea continues to motivate new experiments.[5–7]

So far, only one testable, quantitative output of the Kibble hypothesis has emerged, namely, the expected defect spacing d after the quench. Zurek estimates[3] that this should scale as $d \sim \xi_0(\tau_Q/\tau_0)^{1/4}$, where ξ_0 and τ_0 are the equilibrium correlation length and relaxation time, respectively, and τ_Q^{-1} is a constant quench rate. The argument rests on the generic phenomenon known as "critical slowing down" near second-order phase transitions, which for finite-time quenches will leave the system with a "frozen" value of the order parameter correlation length as it crosses the critical line.

In this connection, we are interested principally in the interpretation of a certain group of experiments[5,6] on low-temperature ^{3}He, in which samples of the superfluid A and B phases are bombarded with neutrons. These trigger the production of localized "hot spots" with effective temperatures $(10^2-10^3)T_c$. The detailed dynamical evolution of these hot spots is a matter of some debate;[8–11] however, it is generally agreed that they cool quite rapidly on the scale of the quasiparticle scattering rate. This leads to quench processes very like those envisioned by Kibble, and it has been argued[10] that the induced vorticity, unambiguously observed in Ref. 6, and inferred on calorimetric grounds in Ref. 5, is directly associated with the Kibble-Zurek mechanism.

There are, however, a number of fundamental difficulties with this interpretation. The validity of the time-dependent Ginzburg-Landau (TDGL) equation, upon which the Zurek estimates are based, requires that the quasiparticle inelastic scattering rate (greatly) exceed the gap frequency $\Delta/\hbar$, a condition which in ^{3}He holds only over a rather narrow strip of width $\sim 10^{-5} k_B T_c$ about the critical curve. As such, the dynamics in this region consists in the motion of an order parameter that is strongly overdamped by frequent quasiparticle collisions. The Ruutu experiments[6] are performed at temperatures much lower than T_c, and the corresponding quenches pass far outside this region; one may therefore expect the order parameter to obey a collisionless analog of these equations in which the long-wavelength components, damped out near T_c, make a substantial contribution. Furthermore, the TDGL equations explicitly fail to account for the conservation of energy, which is likely to impose rather strong constraints on the dynamics regardless of proximity to the transition.

In the following, we attempt to build a coherent physical picture of these quench phenomena by transplanting the relevant physics to the more tractable degenerate Fermi gas. In Sec. II we study the thermodynamics of a finite-time quench from the normal to the superfluid state of such a gas, with special attention given to the energy available for defect formation. We find that in order for the Kibble-Zurek scaling relation to hold, the conditions of the quench must be such that the interior of the quenched region remains thermally isolated from its environment during the entire process, and further that the system must cool all the way back to the ambient temperature. Both of these assumptions are questionable in the case of neutron-irradiated ^{3}He. Section III derives the zero-temperature dynamics that ensue from the sudden variation of the interaction parameter λ from zero to some finite (attractive) value. For a uniform quench, quantum fluctuations in the off-diagonal field avalanche into large, semiclassical oscillations of the order parameter, which tend at long times to a steady state. This state, while not the BCS ground state, is nevertheless characterized by a finite gap $\Delta_\infty = \epsilon_C (e^{2/N(0)\lambda} - 1)^{-1/2}$, with ϵ_C the usual BCS cutoffparameter; in weak coupling this is half the BCS gap. The gap is smaller than that of BCS because of residual collective motion of the pairs, which prevents full condensation.

II. THERMODYNAMICS OF A SHALLOW QUENCH

The two most prominent ^{3}He quench scenarios proposed thus far,[9,10] while differing in their approach to the question of energy transport away from a hot spot, have in common the notion that the region left behind must evolve in effective isolation from its immediate environment. Such a feature is a requirement of the condition that the choice of order parameter inside the cooling hot spot be made independently of that in the surrounding liquid. The spontaneous generation of vortices, if indeed it occurs, must therefore draw its energy from within the quenched region itself. In this section we identify the source of this energy and assess its consequences for the Kibble-Zurek scenario by a consideration of the relevant thermodynamic functions.

In the standard, and experimentally usually most relevant, analysis[12] of Fermi gas-superfluid transitions, thermal contact with a reservoir is tacitly assumed. Thus, if the temperature is made to drop very slowly from the normal state through the transition, the superfluid expels any condensation energy spontaneously. We will take a different route; first, by treating the gas as thermally isolated, and second, by tuning instead of temperature, the attractive interaction parameter itself.

At first sight this might seem a poor model for the quenching of a ^{3}He hot spot. However, upon closer inspection there is a close analogy between these two apparently distinct paths to the superfluid state. This consists of a kind of "duality" between the matrix elements that are tuned, and the phase space available for the scattering of Cooper pairs which causes the instability. Within a given hot spot, the distribution of excited quasiparticles acts to block this phase space, which opens up very rapidly upon cooling back to the ambient temperature. Thus, we surmise that the dynamical situation would be little changed if instead the matrix elements themselves were varied suddenly at a given temperature. Regardless, excepting the speed with which the quench is performed, one would not expect the physics to depend very much on the details of the approach to the critical line.

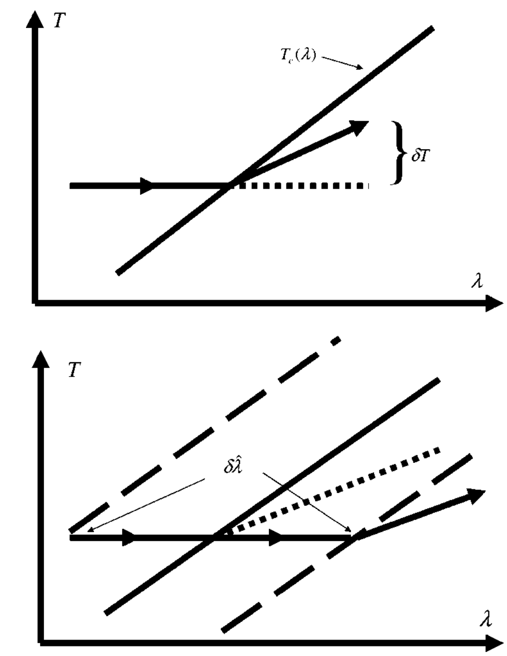

FIG. 1. Adiabatic vs finite-time quenches. (a) For adiabatic passage through the transition curve, the slope of the isentrope jumps discontinuously from zero at critical coupling. Thus, the superfluid emerges at a steadily increasing temperature. (b) For a finite-time passage, the system falls out of equilibrium upon crossing the first thick dashed line; this delays the slope discontinuity until the system intersects the dashed line on the other side.

We proceed by considering a degenerate Fermi gas subject to the BCS reduced Hamiltonian:

$$H = \sum_{k,\alpha} \left(\frac{\hbar^2 k^2}{2m} - \mu \right) \hat{c}^\dagger_{k,\alpha} \hat{c}_{k,\alpha} - \lambda \sum_{k,k'} \hat{c}^\dagger_{k\uparrow} \hat{c}^\dagger_{k\downarrow} \hat{c}_{k'\downarrow} \hat{c}_{k'\uparrow}. \quad (1)$$

We have kept only those terms associated with Cooper pair scattering, and neglect scattering away from or into the pair subspace. Thus, left to itself the system can never approach a true equilibrium; however, at ultralow temperatures far from the transition, the relaxation time scale is so long that this is a reasonable first approximation. As discussed above, we shall assume that the coupling λ can be tuned, as by a Feshbach resonance;[13] this opens up the very real possibility for experimental investigation of quench phenomena along the lines of the present discussion.

Let us first explore the situation for slow variation of the interaction parameter close to the transition. By taking the entropy $S = S(T,\lambda)$ and tracing the adiabat through the transition line [i.e., following the curve for which $\delta S = (\partial S/\partial T)|_\lambda \delta T + (\partial S/\partial \lambda)|_T \delta\lambda = 0$], we can determine δT as a function of $\delta\lambda$. For this purpose, consider a close-up on a portion of the phase diagram in the T-λ plane [Fig. 1(a)], where $T_c(\lambda) = .31 E_F e^{-2\pi^2\hbar^2/mk_F\lambda}$ (Ref. 14) separates the normal and superfluid phases. For the normal Fermi gas in equilibrium, the entropy $S \propto T$ for the low temperatures of interest; thus, for quasistatic variation of λ, the normal state remains at constant temperature. Upon crossing the critical line, however, the resulting superfluid must emerge with a slightly elevated temperature $T+\delta T$ to accommodate the condensation energy. The possibility arises that this energy will heat the nascent superfluid back into the normal state. However, this is not a problem as long as the temperature increase δT for a small variation $\delta\lambda$ is smaller than $(\partial T_c/\partial\lambda)\delta\lambda$, which indeed proves to be the case.

Beginning with the combinatorial expression for the entropy of a Fermi system

$$S=-2k_B\sum_{\mathbf{k}}[f_{\mathbf{k}}\ln f_{\mathbf{k}}+(1-f_{\mathbf{k}})\ln(1-f_{\mathbf{k}})]$$

and changing the sums to integrals, we obtain, after some algebraic manipulation and integration by parts, the following expression for S in the superconducting state:

$$S=2k_B^2N(0)T\int_{-\infty}^{\infty}d\epsilon\left[(\beta\Delta)^2\frac{f}{E}-2\beta\ln(1-f)\right], \tag{2}$$

where $N(0)$ is the Fermi surface density of states, Δ is the gap, and $f=1/(e^{\beta E}+1)$ with $E=\sqrt{\epsilon^2+\Delta^2}$.

Following the mathematical treatment of superconducting thermodynamics developed by Mühlschlegel,[15] we define the functions

$$a(x)=-\frac{2}{\pi}\int_{-\infty}^{\infty}du\,\ln(1+e^{-\pi\sqrt{u^2+x}})+x\left(\ln\gamma\sqrt{x}-\frac{1}{2}\right)-\frac{1}{3}, \tag{3}$$

$$a'(x)=\int_{-\infty}^{\infty}d\epsilon\frac{f}{E}+\ln\gamma\sqrt{x}, \tag{4}$$

where the dimensionless variables $u\equiv(\epsilon/\pi)\beta$ and $x\equiv(\Delta^2/\pi^2)\beta^2$, and $\gamma\approx.57\ldots$ is the Euler-Mascheroni constant. Thus defined, $a(x)$ and $a'(x)$ are regular functions from which the logarithmic singularities have been explicitly subtracted. A number of properties of these functions are tabulated in Ref. 15; for our purposes, we need only that $a(0)=a'(0)=0$ and $a''(0)=\frac{7}{8}\zeta(3)$.

The entropy, as expressed in terms of these functions, becomes

$$S=2\pi^2k_B^2N(0)T\left\{1+3[xa'(x)-a(x)]-\frac{3}{2}x\right\}. \tag{5}$$

For a small increment $\delta\lambda$ beyond the critical curve, the proportion $\delta T/\delta T_C$ by which the superfluid must heat itself to compensate the condensation energy is simply

$$\chi=\lim_{\lambda\to\lambda_c^+}\left[\left(-\left.\frac{\partial S}{\partial T_c}\right|_T\right)\Big/\left(\left.\frac{\partial S}{\partial T}\right|_\lambda\right)\right].$$

The relevant derivatives are, from Eq. (5),

$$\frac{\partial S}{\partial T_c}=2\pi^2k_B^2N(0)T\left[xa''(x)-\frac{1}{2}\right]\frac{\partial x}{\partial T_c}$$

and

$$\frac{\partial S}{\partial T}=\frac{S}{T}+2\pi^2k_B^2N(0)T\left[xa''(x)-\frac{1}{2}\right]\frac{\partial x}{\partial T}.$$

In the $T\to T_c$ limit, $\Delta^2\approx[8\pi^2k_B^2/7\zeta(3)]T_c^2(1-T/T_c)$, so that

$$\frac{\partial x}{\partial T_c}\to\frac{8}{7\zeta(3)k_B^2T_c},$$

$$\frac{\partial x}{\partial T}\to-\frac{8}{7\zeta(3)k_B^2T_c},$$

and we obtain

$$\chi=\frac{1}{1+\frac{7\zeta(3)}{12}}\approx.588, \tag{6}$$

Thus, upon traversing the critical line, the superfluid must choose a "compromise temperature" $T^*=T+\chi(\partial T/\partial\lambda)\delta\lambda$, intermediate between the new value of T_c and the temperature T of the normal liquid from which it started.

Now let us imagine [Fig. 1(b)] a slight generalization of the preceding argument in which λ is made to vary at a constant finite rate τ_Q^{-1} through the transition, and define the associated "quench parameter" or "reduced coupling" $\epsilon=\delta\lambda/\lambda_c=t/\tau_Q$, where $\delta\lambda=\lambda_c-\lambda$ is the deviation from critical coupling. If, following Zurek, we allow that near T_c, the correlation length and velocity assume the scaling forms $\xi=\xi_0\epsilon^{-\nu}$ and $u=u_0\epsilon^{1-\nu}$, and thus that $\tau=\tau_0\epsilon^{-1}$, then the system "freezes out" at a time $\hat{t}=\tau(\hat{t})=\sqrt{\tau_0\tau_Q}$ before emerging finally at the same temperature a distance $\delta\lambda(\hat{t})$ from λ_c on the other side. This new state presumably contains a number of trapped vortex loops.

From this picture, it is clear that the energy required to create these loops must derive from the free energy difference between the superfluid states at T and T^*. This energy is evidently an overestimate predicated on the total freeze-out of thermodynamic variables at $-\hat{t}$. In reality, we might expect T to increase somewhat during this process; however, the condition of total freeze-out does allow us at least to put an upper bound on the defect density.

The relevant changes in the free energy density near T_c can be calculated from the Hellmann-Feynman theorem for the thermodynamic potential Ω

$$\left.\frac{\partial\Omega}{\partial\lambda}\right|_{T,V,\mu}=\left\langle\frac{\partial H}{\partial\lambda}\right\rangle=-\sum_{\mathbf{kk}'}u_{\mathbf{k}}v_{\mathbf{k}}u_{\mathbf{k}'}v_{\mathbf{k}'}(1-2f_{\mathbf{k}})(1-2f_{\mathbf{k}'}),$$

where $u_{\mathbf{k}},v_{\mathbf{k}}$ are the usual BCS coefficients arising from the average. Since the gap $\Delta=\lambda\Sigma_{\mathbf{k}}u_{\mathbf{k}}v_{\mathbf{k}}(1-2f_{\mathbf{k}})$, we have $\partial\Omega/\partial\lambda=-\Delta^2/\lambda^2$, or

$$\Omega_s-\Omega_n=-\int_0^{\lambda}d\lambda'\frac{\Delta^2}{\lambda'^2}, \tag{7}$$

which is valid for any thermodynamic potential.[12] For convenience we choose the Helmholtz free energy at constant temperature, again using the near-T_c expression for Δ:

$$\frac{\Delta F_{s-n}}{V} \cong \frac{2\pi^2}{7\zeta(3)} N(0) k_B^2 [T_c(\lambda_c + \delta\lambda) - T]^2. \tag{8}$$

If we start from T in the normal state, then clearly $T_c(\lambda_c + \delta\lambda) = T + (\partial T_c/\partial\lambda)\delta\lambda$.

The energy available to vortex production is thus $\Delta E = \Delta F_{s-n}(T^*) - \Delta F_{s-n}(T)$, or

$$\frac{\Delta E}{V} = \alpha N(0)(k_B T)^2 \left| \log \frac{k_B T}{.31 E_F} \right|^4 \hat{\epsilon}^2, \tag{9}$$

where λ_c has been expressed as a function of the starting temperature T, $\alpha = \chi(2-\chi)[2\pi^2/7\zeta(3)]$ is a numerical prefactor ~ 1.9, and $\hat{\epsilon}$ is the value of ϵ at freeze-out. An estimate of the initial defect density requires a comparison of Eq. (9) with the energy density of a particular vortex distribution, to which there are two principal contributions: that associated with the suppression of Δ in the core, and the kinetic energy of superflow about it. Only the latter can be expected to contribute substantially for distances somewhat larger than the core dimension. For a vortex loop of radius d, this is $\epsilon_L \sim 2\pi^2 \rho_s (\hbar^2/m^2) d \log(d/a)$, with a the dimension of the core. The energy density of a vortex line per volume d^3 is just ϵ_L/d^3, which gives a lower bound on the Zurek length scale d of the form

$$d \geq \frac{\hbar}{m} \left(\frac{2\pi^2 \rho_S(T)}{\alpha N(0)} \right)^{1/2} \frac{\sqrt{\log d/a}}{k_B T \left| \log \frac{k_B T}{.31 E_F} \right|^2} \left(\frac{\tau_Q}{\tau_0} \right)^{1/2}. \tag{10}$$

The superfluid density near the transition grows linearly as $\rho_S(T) = 2\rho[1 - T/T_c(\lambda_c + \delta\lambda)]$, or, expressed in terms of more appropriate variables,

$$\rho_S(\epsilon, T) = 2\rho \frac{\epsilon}{\epsilon + \left| \log\left(\frac{k_B T}{.31 E_F} \right) \right|^{-1}},$$

where $\epsilon \ll |\log(k_B T/.31 E_F)|^{-1}$. The combination $(\hbar/m) \times [\rho/N(0)]^{1/2} \sim \hbar v_F/k_B T$; hence, up to logarithmic factors and dimensionless constants of order 1, we have

$$d \geq \frac{\hbar v_F}{k_B T} \left(\frac{\tau_Q}{\tau_0} \right)^{1/4}. \tag{11}$$

For quenches in ^{3}He near the transition curve (i.e., for $T \sim T_c$), this is of course the coherence length ξ_0. This implies that for such quenches the Kibble-Zurek scaling law gives a lower bound to the defect spacing; it requires (1) that the quenched region does not expel any energy to its surroundings and (2) that the temperature of the liquid just after the quench drops all the way back to its original temperature (namely, the ambient temperature of the liquid). Both of these assumptions are dubious, and their violation tends strongly to decrease the free energy available to vortices. Further, for quenches induced in the superfluid at much lower temperatures, the prefactor will be somewhat larger than ξ_0, tending once again to suppress vortex production. We are led to conclude that the Kibble mechanism depends rather precariously on the availability of free energy in the quenched region, requiring perfect isolation and cooling, and reasonable proximity to the critical line.

III. QUENCHING AND ORDER PARAMETER DYNAMICS

In this section, we develop a theory for the order parameter dynamics following a "deep," uniform quench through the superfluid transition in a Fermi gas. It is hoped that such considerations can shed light on quenching phenomena induced in ^{3}He in the context of so-called cosmological experiments. These involve the rapid decrease in effective temperature of a small region of the superfluid that has been heated back into the normal state by neutron irradiation and subsequently cools through the diffusion (anomalous or otherwise[10,11]) of quasiparticles. By 'deep,' we mean that the reduced temperature $\delta T/T_c$ calculated at freeze-out ($\pm\hat{t}$), which for the estimated quench times in ^{3}He yields values in the range from 1 to 10^{-2}, is much larger than that corresponding to the range over which the relaxation time scale is comparable to or greater the "gap frequency" $\Delta/\hbar$, a condition necessary to the validity of the TDGL description; as mentioned in Sec. I, this is given by $\delta T/T_c \sim 10^{-5}$.

From the above description, it is clear that the neutron-induced quenches in ^{3}He occur under highly inhomogeneous conditions; however, we are concerned here with the spontaneous emergence of a length scale which in this context has been predicted to be much smaller than the dimension of the quench itself.[10] Within the TDGL picture, upon which the prevailing theory of these quench phenomena is founded, collisions between thermal quasiparticles maintain local equilibrium and are therefore the dominant time scale. The motion of the order parameter is overdamped, killing contributions from the longest length scales; by contrast, in the deep quench scenario, $\hbar/\Delta$ is the largest time scale in the problem, and the long-wavelength fluctuations which freeze out in the Kibble-Zurek picture may here play a vital role. Therefore, in what follows we focus exclusively on these long-wavelength coherent dynamics. In particular, we show that the Fermi gas is unstable to even the smallest long-wavelength fluctuations, and that the ensuing dynamics leads to a steady state with a finite value of the gap.

A. Pseudospin representation, and initial conditions

Visualization of the order parameter dynamics following the sudden turn-on of a pairing potential is perhaps the chief difficulty of our problem. This can be mitigated to a large extent by making use of an analogy, noticed by Anderson,[16] between a BCS system on the pair subspace and a one-dimensional lattice of spins. That is, for each distinct $\mathbf{k}$ we can identify operators as $\sigma_z(k) \equiv 1 - n_k - n_{-k}$, $\sigma_+(k) \equiv c_k^\dagger c_{-k}^\dagger$, and $\sigma_-(k) \equiv c_{-k} c_k$, where $\sigma_\pm(k) = \frac{1}{2}[\sigma_x(k) \pm i\sigma_y(k)]$, and the σ_i are the 2×2 Pauli matrices. It is easily verified that these obey the $SU(2)$ commutator algebra $[\sigma_i(k), \sigma_j(k')] = i\delta_{k,k'} \epsilon_{ijl} \sigma_l(k)$. Expressed in terms of pseudospin degrees of freedom, the BCS reduced Hamiltonian (1) assumes the form

$$H = -\sum_k \epsilon_k \sigma_z(k) - \lambda \sum_{k \neq k'} \sigma_-(k) \sigma_+(k'). \tag{12}$$

The normal state $|N\rangle$ corresponds in this description to a domain wall centered at the Fermi energy, with occupied states below the Fermi surface pointing along the negative-z direction. This is clearly degenerate with respect to global rotations about the z axis; for $\lambda > 0$ the BCS ground state breaks this degeneracy and lowers the total energy by rotating the spins into the new effective field, and interpolating smoothly between the up and down orientations at $\pm\epsilon_C$.

We begin by writing the appropriate operator equations of motion, which are obtained from the usual Bloch relations $i\hbar\dot{\sigma}_i(k)=[\sigma_i(k),H]$. These yield

$$\dot{\sigma}_z(k) = \frac{-\lambda}{\hbar}\left\{\left[\sum_{k'\neq k}\sigma_x(k')\right]\sigma_y(k) + \sigma_y(k)\left[\sum_{k'\neq k}\sigma_x(k')\right] - \left[\sum_{k'\neq k}\sigma_y(k')\right]\sigma_x(k) - \sigma_x(k)\left[\sum_{k'\neq k}\sigma_y(k')\right]\right\},$$

$$\dot{\sigma}_x(k) = \frac{\epsilon_k}{\hbar}\sigma_y(k) - \frac{\lambda}{\hbar}\left\{\sigma_z(k)\left[\sum_{k'\neq k}\sigma_y(k')\right] + \left[\sum_{k'\neq k}\sigma_y(k')\right]\sigma_z(k)\right\}, \tag{13}$$

$$\dot{\sigma}_y(k) = \frac{-\epsilon_k}{\hbar}\sigma_x(k) + \frac{\lambda}{\hbar}\left\{\sigma_z(k)\left[\sum_{k'\neq k}\sigma_x(k')\right] + \left[\sum_{k'\neq k}\sigma_x(k')\right]\sigma_z(k)\right\}$$

It remains to consider the rather delicate question of initial conditions. A complete quantum-mechanical description requires taking the expectation value of the above operator equations with respect to $|\psi(t=0^+)\rangle$, the total wave function immediately following the variation of λ from zero to some finite value at $t=0$. In the sudden approximation this is just the normal state, implying that the off-diagonal effective field $\propto\langle\Sigma_k\sigma_x(k)\rangle$ vanishes, and thus that the pseudospins persist indefinitely in their original configuration. Since we know the normal state to be unstable for $\lambda > 0$, this cannot possibly be correct.

The difficulty stems from the global gauge symmetry of the normal state, and can be removed by taking as our initial wave function, instead of the normal state itself, its projection onto the Hilbert space appropriate to a particular direction of symmetry breaking. In spirit, this is not unlike the problem of a single particle in the ground state of an external potential that has been suddenly varied; the first step towards the solution of its subsequent dynamics is the projection of this initial state onto a basis appropriate to the new potential. In much the same way, we generate the dynamics for a "branch" of the many-body wave function along a given direction of symmetry-breaking ϕ. As we shall see, this makes sense when the number of pairs N participating in the wave function is large, so that each direction ϕ represents a quasidistinct Hilbert space, and the various branches therefore evolve independently. Note in particular that negative eigenvalues of the off-diagonal field operator along the ϕ direction belong properly to the $\phi+\pi$ sector, so that only positive expectation values contribute to ϕ.

To illustrate the general idea, let us consider the full manifold of states, which can be generated by operating on the normal state "domain wall" with appropriately defined spin-$1/2$ rotation operators. Let us fix the $\hat{x},\hat{y}$ axes and denote the various possible symmetry-breaking directions by ϕ, the deviation from $\hat{x}$. In terms of these conventions, our states become

$$|\Psi(\theta_k,\phi_k)\rangle = \prod_k \exp\left[-\frac{i}{2}\phi_k\sigma_z(k)\right]\exp\left[-\frac{i}{2}\theta_k\sigma_y(k)\right]|N\rangle. \tag{14}$$

Expanding the exponentials in σ_y, and then applying the z-rotation operators yields

$$=\prod_k\left\{\cos\frac{\theta_k}{2} - e^{i\phi_k}\sin\frac{\theta_k}{2}[i\sigma_y(k)]\right\}|N\rangle, \tag{15}$$

where we have used the fact that the wave function is defined only up to an overall phase. This coincides with the usual BCS wave function when $\phi_k=\phi$ for all k and $\theta_k=\tan^{-1}(\Delta/\epsilon_k)$, provided Δ is chosen self-consistently to satisfy $\Delta=\lambda\langle\Psi|[\Sigma_k\tilde{\sigma}_x(k)]|\Psi\rangle$, where $\tilde{\sigma}_x(k)$ is the x operator rotated by ϕ.

To a good approximation, we may restrict ourselves to states with particle-hole (p-h) symmetry, so that $\theta(\epsilon_k) = -\theta(-\epsilon_k)$ and $\phi(\epsilon_k)=-\phi(-\epsilon_k)$ for all k. The direction of symmetry breaking for a state of the form (14) obeying this symmetry is thus $\phi=(1/2N)\Sigma_k\phi_k$. The inner product between any such state $|\Psi(\phi)\rangle$ and a rotated version of itself $|\Psi(\phi')\rangle$, is easily calculated as $\langle\Psi(\phi)|\Psi(\phi')\rangle = \langle\Psi(\phi)|\exp[-(i/2)\delta\phi\Sigma_k\sigma_z(k)]|\Psi(\phi)\rangle=\cos^{2N}(\delta\phi/2)$, which for large N is practically zero. This is becausethe two states in question can be connected only by the simultaneous rotation of a macroscopic number of spins. The statement that states of different ϕ belong to distinct Hilbert spaces should therefore be interpreted in this sense.

For concreteness, let us consider the $\phi=0$ direction, and single out the state $|\Psi_x\rangle=|\Psi[\theta_k=\mathrm{sgn}(\epsilon_k)\pi/2,\phi_k=0]\rangle$. A complete basis on the pair subspace for $\phi=0$ can be constructed by acting on $|\Psi_x\rangle$ with operators of the form $\exp[(i/2)\pi\sigma_y(k')]$, as long as we make sure to rotate p-h symmetric partners together. Let us denote such a p-h symmetric rotation by $\hat{\pi}_k$; the projection of the normal state onto the $\phi=0$ Hilbert space may therefore be expanded as

$$\hat{P}_{\phi=0}|N\rangle = (\alpha_{2N} + \sum_{|k|<k_F}\alpha^k_{2N-2}\hat{\pi}_k + \cdots)|\Psi_x\rangle,$$

where, in each sum, we must be careful not to repeat indices. In this basis, all the coefficients equal $1/2^N$. The subscripts label the associated eigenvalues of the off-diagonal operator $\Sigma_k\sigma_x(k)$, which are $2m$ for $m=1,3,\ldots,N$ when N is odd, and $m=0,2,\ldots,N$ for N even. Thus, the probability for a nonzero, positive eigenvalue follows a distribution of the form

$$P(m) = \frac{1}{2^{N-1}} \frac{N!}{(N/2+m/2)\,!\,(N/2-m/2)!},$$

yielding a small but finite expectation value for the off-diagonal field $\sim\lambda N^{-1/2}$ along this direction.

States of the form (14) do not possess the full gauge symmetry of the Hamiltonian, and thus do not conserve particle number. As mentioned above, our starting state, and hence the many-body state that evolves from it, must preserve this symmetry. It can in fact be restored at any time by taking the superposition $(1/2\pi)\int_0^{2\pi} d\phi e^{-i/2\phi\Sigma_k\sigma_z(k)}|\psi_S(\theta_k,\phi_k)\rangle$, where $\phi=(1/2N)\Sigma_k\phi_k$, which is nothing but the usual prescription[16] for "projecting out" the N-particle subspace from a BCS wave function.

It is perhaps appropriate at this point to summarize the above discussion by appealing to a simpler, more intuitive picture of the quantum fluctuations. This picture gives some insight into the nature of these fluctuations without involving us in any mathematical complications. For a spin-1/2 system in an up or down eigenstate of σ_z, there is a component in the x-y plane associated with zero-point precession about the z axis. Thus, for a large group of such spins, the collective zero-point motion consists of a superposition of all possible configurations of these "extra" x-y components, the bulk of which largely cancel out. There will, however, be a small minority that make an enormous contribution, so that there is a nonzero expectation value along any given unit vector at the origin of the x-y plane sufficient to drive the system away from the normal state in that direction.

B. Semiclassical equations of motion

Having discussed the nature of quantum fluctuations in the immediate aftermath of the quench, let us now consider the evolution of each branch of the total wave function, arising from equations of motion for the associated expectation values of the pseudospin operators. Such equations constitute a semiclassical description of the pseudospins, expressing the self-consistent precession of each operator's "axis of quantization."

All of the wave functions of interest consist of products of factors, one for each k; this has the advantage that expectation values of operator products break into products of expectation values, provided the operators themselves are functions of distinct k. When the number of pairs N is large, we may identify terms as

$$\lim_{N\to\infty} \lambda \sum_{k\neq k'} \langle \sigma_x(k)\rangle = \lambda \sum_k \langle \sigma_x(k,t)\rangle = \Delta,$$

and $\Sigma_{k\neq k'}\langle\sigma_y(k)\rangle \to 0$, where again the direction of symmetry-breaking points by convention along the x axis. In all subsequent equations, we denote expectation values of the pseudospin operators by $\frac{1}{2}\langle\sigma_i\rangle \equiv s_i$. The semiclassical equations of motion are thus (dropping factors of $\hbar$),

$$\frac{d}{dt}s_x(k,t) = 2\epsilon_k s_y(k,t),$$

$$\frac{d}{dt}s_y(k,t) = 2\Delta s_z(k,t) - 2\epsilon_k s_x(k,t), \tag{16}$$

$$\frac{d}{dt}s_z(k,t) = -2\Delta s_y(k,t),$$

and the assumption of p-h symmetry takes the form

$$s_x(-\epsilon_k,t) = s_x(\epsilon_k,t),$$

$$s_y(-\epsilon_k,t) = -s_y(\epsilon_k,t),$$

$$s_z(-\epsilon_k,t) = -s_z(\epsilon_k,t).$$

As a consequence, the direction of symmetry-breaking ϕ is a constant of the motion, in harmony with our earlier quantum-mechanical reasoning. Hence, the dynamics may be interpreted as evolving along independent semiclassical trajectories.

We note in passing that the equations of motion (16) may also be derived in terms of the interaction of each pseudospin with a local effective field, written $(d/dt)\mathbf{s}(k,t)=\mathbf{s}(k,t)\times\mathbf{H}_k(t)$, where the effective field takes the form

$$\mathbf{H}_k(t) = 2\epsilon_k\hat{z} + 2\Delta\hat{x}.$$

Since we are interested here only in the gap dynamics, it proves convenient to collapse the three equations of motion into one for s_x alone. Taking an extra time derivative of the x component, and substituting into this the equation for the y component, gives

$$\frac{d^2}{dt^2}s_x(k,t) + 4\epsilon_k^2 s_x(k,t) = 4\epsilon_k\Delta s_z(k,t). \tag{17}$$

To eliminate s_z from this equation, we must combine the x and z components of Eq. (16) and find a way to express s_z in terms of s_x. Combining, we have

$$\frac{d}{dt}\epsilon_k s_z(k,t) + \Delta\frac{d}{dt}s_x(k,t) = 0. \tag{18}$$

Integrating with respect to time (and doing the x part by parts) we obtain

$$s_z(k,t) = s_z(k,0) + \frac{1}{\epsilon_k}\int^t dt' s_x(k,t')\dot{\Delta} - \frac{\Delta}{\epsilon_k}s_x(k,t). \tag{19}$$

Noting that $s_z(k,0)=\frac{1}{2}\mathrm{sgn}(\epsilon_k)$, we obtain the desired equation upon substitution into Eq. (19):

$$\left[\frac{d^2}{dt^2} + 4(\epsilon_k^2+\Delta^2)\right]s_x(k,t) = 4\Delta\left[\frac{1}{2}|\epsilon_k| + \int^t dt' s_x(k,t')\dot{\Delta}\right]. \tag{20}$$

The left-hand side is of course the harmonic oscillator equation, but with a time-dependent frequency. The right-hand side consists of a driving term proportional to Δ representing feedback from the aggregate of pseudospins in the mean-field. Equation (20), together with the self-consistency relation

$$\Delta = \lambda \sum_k s_x(k,t), \tag{21}$$

gives us the complete order parameter dynamics at the semiclassical level.

There are two conserved quantities of note; in terms of the effective field, the total energy on the pair subspace becomes

$$E = -2\sum_k \left[\epsilon_k s_z(k,t) + \lambda \sum_{k' \neq k} s_x(k,t) s_x(k',t) \right]$$
$$= -2\sum_k \left[\epsilon_k s_z(k,t) + \frac{1}{2}\Delta s_x(k,t) \right]. \tag{22}$$

This means in particular that

$$\frac{d}{dt}E = -2\sum_k \left[\epsilon_k \frac{d}{dt} s_z + \frac{1}{2}\Delta \frac{d}{dt} s_x + \frac{1}{2} s_x \frac{d}{dt}\Delta \right].$$

Applying the equations of motion we find

$$\frac{d}{dt}E = -\sum_k \left[s_x(k,t)\frac{d}{dt}\Delta - \Delta \frac{d}{dt} s_x(k,t) \right] = 0.$$

Hence, the semiclassical trajectories following from Eqs. (20) and (21) are energy conserving. The equations of motion also conserve mean particle number, which may be seen by summing over k in the last equation of Eqs. (16).

C. Early- and long-time behavior of solutions

Consider again a particular branch of the Fermi gas wave function. The normal ground state corresponds, on the semiclassical level, to a domain wall with all spins pointing along their local effective fields. The system would persist in this state were it not for the presence of the small quantum fluctuation at $t=0$, which drives the precession of each pseudospin at its natural frequency ϵ_k about a now slightly perturbed effective field. It is not difficult to see, on essentially geometric grounds, that such precession will tend to reinforce the initial fluctuation, leading to the growth of the off-diagonal field.

This can be demonstrated explicitly by solving the linearized version of Eq. (20), which is valid at very early times:

$$\left(\frac{d^2}{dt^2} + 4\epsilon_k^2 \right) s_x(k,t) = 2|\epsilon_k| [\Delta_{QF} + \Delta(t)].$$

The key point here is that Δ_{QF}, the initial quantum fluctuation in the gap, is determined at a level independent of the semiclassical description; as such, it is not subject to the self-consistency condition and therefore constitutes an "external" input to the semiclassical equations. $\Delta(t)$ is the self-consistent contribution defined by $\Delta(t)=\lambda\Sigma_k s_x(k,t)$, and thus must satisfy the initial conditions $\Delta(0)=0$, $\dot{\Delta}(0)=0$. Absent any singular terms in these linearized equations, their solutions are regular functions of time and may be expanded about $t=0$ as $s_x(k,t)=\Sigma_{n=0}^{\infty}\gamma_n(k)t^n$; hence, $\Delta(t)=\lambda\Sigma_{n=0}[\Sigma_k \gamma_n(k)]t^n$. The initial conditions imply $\gamma_0(k), \gamma_1(k)=0$. Substitution into the linearized equations reveals a hierarchy of relations for $n>2$ of the form

$$\gamma_n(k) = \frac{2|\epsilon_k|}{n(n-1)} \sum_{k'} \gamma_{n-2}(k') - \frac{4\epsilon_k^2}{n(n-1)} \gamma_{n-2}(k) \tag{23}$$

and

$$\gamma_2(k) = |\epsilon_k| \Delta_{QF}, \tag{24}$$

obtained by matching powers of t. These determine all $\gamma_n(k)$ uniquely, and automatically satisfy the self-consistency constraint. The first thing to note about these relations is that all odd n coefficients vanish since $\gamma_1=0$, and second, that the resulting gap varies as $\Delta(t)=\lambda\Delta_{QF}N(0)\epsilon_C^2 t^2+\mathcal{O}(t^4)$. This clearly demonstrates the instability of the pseudospin system to even the smallest (in this case quantum) fluctuations.

We can expect this growth to carry the system very quickly into the semiclassical regime, where $\Delta_{QF}/\Delta \ll 1$. However, we must bear in mind that within this level of description, Δ is built up from a large number of individual pseudospin oscillations spanning a quasicontinuous range of frequencies. This suggests that at long times, even in the absence of dissipation, interference between these various contributions will lead to a nonequilibrium steady state characterized by a static gap Δ_∞.

To establish the existence of such steady-state solutions, we must go back to the basic equations of motion (16), and ask ourselves whether, by replacing Δ everywhere in these equations with a static value $\Delta_\infty > 0$, it is possible to find self-consistent solutions in the $t\to\infty$ limit. Making this substitution, and carrying through the steps (17)–(19) as before, we arrive at an equation of motion for $s_x(k,t)$ of the form

$$\left[\frac{d^2}{dt^2} + 4(\epsilon_k^2 + \Delta_\infty^2) \right] s_x(k,t) = 2\Delta_\infty |\epsilon_k|, \tag{25}$$

with general solutions

$$s_x(k,t) = \frac{\frac{1}{2}|\epsilon_k|\Delta_\infty}{\epsilon_k^2 + \Delta_\infty^2} [1 - \eta_k \sin(2\sqrt{\epsilon_k^2 + \Delta_\infty^2}\, t + \delta_k)]. \tag{26}$$

The δ_k are arbitrary phase factors; since we are interested only in times $t \gg \Delta_\infty^{-1}, \epsilon_C^{-1}$, they are irrelevant to the actual value of the gap, and are henceforth dropped. The dimensionless parameters η depend on the precise details of the pseusospin trajectory over the entire course of its dynamical history. Despite our ignorance of these parameters, physically there is little reason to believe that the trajectories of two nearly degenerate pseudospins will differ appreciably if the system approaches a steady state. Hence we shall assume that the parameters η_k form a continuous function of ϵ_k.

Substitution of these solutions into the self-consistency equation (21) leads, when expressed in terms of appropriate energy integrals, to the gap equation

$$\frac{1}{N(0)\lambda} = \frac{1}{2}\ln[1 + (\epsilon_C/\Delta_\infty)^2] + \int_{\Delta_\infty}^{\sqrt{\epsilon_C^2+\Delta_\infty^2}} \frac{dE}{E}\eta(E)\sin(2Et). \tag{27}$$

As shown in the Appendix, for $\eta(E)$ continuous the integral term toends to zero as $t\to\infty$; thus, a self-consistent solution emerges precisely in the long time limit of interest for a value of the static gap

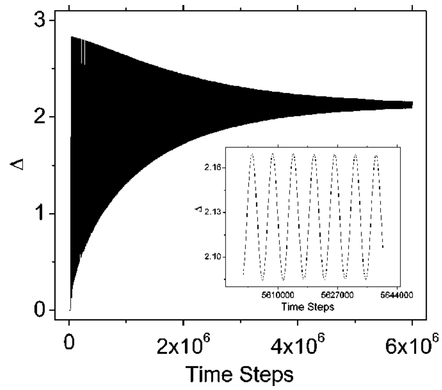

FIG. 2. Gap dynamics for 1000 pairs following a weak fluctuation ($\sim 10^{-7}$) for a coupling parameter $N(0)\lambda = 5$. The curve appears black because of the high frequency of the oscillations on this scale. However, the envelope is clearly discernible and shows that the amplitude of the oscillations decreases monotonically during the approach to the asymptotic steady state. The inset depicts a close-up of the curve, making the gap oscillations visible.

$$\Delta_\infty = \epsilon_C (e^{\frac{2}{N(0)\lambda}} - 1)^{-1/2}. \tag{28}$$

In weak coupling this is $\epsilon_C e^{-1/[N(0)\lambda]}$, or half the equilibrium result. Physically, this solution represents a collective state of motion, in which the pseudospins precess individually on fixed cones about static local fields. The energy locked up in this motion cannot be expelled in the absence of collisions, and the steady-state gap is thus somewhat less than the value it would have in equilibrium under otherwise identical conditions. It should be noted that although this state is only partially condensed and exhibits phase fluctuations, it contains no vortices.

The structure of the gap equation (27) suggests certain trends in the dynamical behavior, which it may be useful to enumerate. First, for weak coupling the gap oscillations are dominated by the cutoff ϵ_C, and for strong coupling by the gap parameter itself; second, the approach to the steady state is faster for deeper (i.e., larger λ) quenches; and third, the ultimate steady state to which the system tends does not depend on the size of the initial fluctuation. All of these trends, in addition to the actual values of the steady-state gap, have been checked by numerical integration, as described in the following section.

D. Numerical integration of the semiclassical equations

The preceding sections address both initial conditions and the physical state at asymptotically long times, but not the intermediate behavior. In the absence of exact analytical expressions, we must resort to numerical integration of the semiclassical equations of motion (20) and (21). These have been performed over a wide range of interaction parameters and initial fluctuation sizes. A typical output is shown in Fig. 2, corresponding to a value $N(0)\lambda = 5$. To start things off, we introduce a tiny off-diagonal field and evolve away from the normal state for a single time step; all subsequent evolution occurs self-consistently using the off-diagonal field produced by the pseudospins themselves in the previous time step.

The basic shape of the curve shown in in Fig. 2 appears to be generic; there are a large number of high-frequency oscillations and a slow decay to a constant value. All of the trends noted in the previous section have been observed; particularly interesting is the fact that the long-time value of the gap corresponds rather well with Eq. (28) despite variation in the initial fluctuation size over seven orders of magnitude. The early t^2 behavior can also be seen, which shows that the early- and long-time behaviors derived above are indeed connected by exact solutions of Eq. (20).

E. Related work

In the course of this work, the authors became aware of the related unpublished calculations of Shumenko[17] and more recently of Barankov, Levitov, and Spivak (SBLS)[18] regarding quench-induced dynamics in a BCS system. There are substantial qualitative differences between our own findings and those of SBLS, despite the fact that both proceed from identical semiclassical equations of motion (16). To clarify the situation, we offer here an interpretation of their results and an analysis of the discrepancies.

Looking again at Eqs. (20) and (21), we notice that they describe what is essentially a system of independent oscillators subject to a universal driving force, which force just happens to be determined self-consistently by the oscillators themselves. For a linear, damped oscillator driven harmonically and off resonance, the long-time solutions consist of pure oscillations at the frequency of the driving force. This suggests by analogy that the semiclassical equations of motion can exhibit solutions in which each pseudospin follows the motion of the gap, subject of course to the constraint of overall self-consistency. Such solutions would be separable in energy and time:

$$s_x(k,t) = A_k \Delta(t), \tag{29}$$

which is precisely the *ansatz* proposed by SBLS.

Substitution into Eqs. (20) and (21) yields immediately

$$\frac{d^2}{dt^2}\Delta + \left(4\epsilon_k^2 - \frac{2|\epsilon_k|}{A_k}\right)\Delta + 2\Delta^3 = 0 \tag{30}$$

and

$$1 - \lambda \sum_k A_k = 0. \tag{31}$$

If we choose A_k such that $4\epsilon_k^2 - 2|\epsilon_k|/A_k = C$, a constant independent of k, then each pseudospin obeys the same equation of motion. Thus, the *ansatz* can indeed be made self-consistent. Multiplying the first of these equations by $\dot{\Delta}$ and integrating over t, we obtain

$$\dot{\Delta}^2 + (\Delta^2 - \Delta_0^2)^2 = \Gamma^2,$$

where $\Gamma \equiv \dot{\Delta}(0)$, $\Delta_0 \equiv \Delta(0)$, and we have chosen $C = -2\Delta_0^2$, which yields for the self-consistency equation

$$1 - \lambda \sum_k \frac{|\epsilon_k|}{2\epsilon^2 + \Delta_0^2} = 0.$$

As discussed by SBLS, these equations clearly demonstrate the existence of a class of periodic solutions $\Delta(t)$. While consistent mathematically, physically they leave out an equally important class of solutions characterized by the response of individual pseudospins at their natural frequencies, and the concomitant dephasing of the pseudospin orientations with time. The nonlinearity of the semiclassical equations of motion, and the absence of substantial damping in the zero-temperature limit, both indicate that these solutions cannot be safely ignored. It is in fact extremely plausible on physical grounds, and clear from the analytical and numerical results in previous sections, that dephasing will affect rather strongly both the frequency of the gap oscillations and the nature of their long-time behavior, except perhaps for special choices of initial conditions. This consideration acquires even greater urgency in the limit of weak coupling $\Delta/\epsilon_C \ll 1$, in which case the majority of pseudospins precess very far from resonance. In such a circumstance it is very difficult to see how the full pseudospin system, when subjected to a sudden perturbation of the type described in Sec. III A, could respond with such a high degree of synchronicity. Indeed, the early-time solutions in Sec. III indicate otherwise, and appear to be inconsistent with those of SBLS.

Thus, it is reasonable to locate the essential differences between our own formulation of the quenching problem, and that of SBLS, in their respective approaches to the question of initial conditions. The former theory invokes latent quantum fluctuations of the normal state which, in the wake of the quench, provide a "kick" within each symmetry-broken Hilbert space. The sudden appearance of a finite, if minute, off-diagonal field will of necessity introduce some small degree of dephasing, leading inevitably to a nonequilibrium steady state as described above. SBLS appear to assume the complete absence of dephasing from the start. In the full space of possible initial conditions, this represents a rather special class, which leads us to conclude that the associated periodic behavior of the gap, while intriguing, is unlikely to be observed. Finally, there is no discussion by SBLS of the apparent breaking of gauge symmetry within their theory immediately after the quench. The quench by itself cannot break this symmetry; instead, the system evolves simultaneously along each direction of symmetry breaking in such a manner that the overall gauge symmetry remains unbroken.

IV. CONCLUSIONS

We have studied the behavior of a Fermi gas following the sudden turn-on of an attractive BCS interaction parameter λ, as a model for analogous processes that are thought to occur in the low-temperature superfluid phases of ^{3}He upon exposure to neutron radiation. For values of the parameters appropriate to ^{3}He near the transition line, a study of the free energy available for vortex generation in the wake of a quench reveals that the Kibble-Zurek scaling law gives, at best, a lower bound on the defect spacing. That is, the Kibble-Zurek law assumes that a maximum quantity of free energy is available for vortices, a condition requiring that the quenched region cools all the way back to its original temperature and does not expel any condensation energy to its surroundings. Both of these requirements are questionable.

Further, the reduced temperature at freeze-out far exceeds that at which the quasiparticle inelastic scattering rate ${\tau_{qp}}^{-1}$ becomes comparable to the gap frequency $\Delta/\hbar$, a condition which should be well satisfied in the Ginzburg-Landau regime on which the Kibble-Zurek scenario is based. This suggests that the dynamics following such a quench must be treated, to a first approximation, in the absence of collisions. Under these conditions, for a sudden turn-on of the interaction parameter, we demonstrate an absolute instability to uniform quantum fluctuations already present in the normal state. These subsequently amplify into large, semiclassical oscillations along each direction of symmetry breaking contained in the full wave function; even in the absence of collisions, these oscillations eventually settle to a self-driven steady state with gap $\Delta_\infty = \epsilon_C (e^{2/N(0)\lambda} - 1)^{-1/2}$. In weak coupling, this is half the BCS result, due to the persistence of incoherent collective motion of the pairs.

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APPENDIX

In this Appendix we consider the asymptotic behavior of the integral

$$I(t) = \int_A^B dE \frac{\eta(E)}{E} \sin 2Et,$$

with $A = \Delta_\infty$, $B = \int \sqrt{\epsilon_C^2 + \Delta_\infty^2}$, appearing in the gap equation (27). η is a dimensionless function of E; without loss of generality we may take it to vary with a dimensionless parameter $x = \frac{E}{E_C}$ where E_C is an arbitrary constant with the dimensions of energy. Since η is by assumption a continuous function of x on the interval $\left[\frac{A}{E_C}, \frac{A}{E_C}\right]$, it can, according to the Weierstrass theorem, be approximated to any desired degree of accuracy by a polynomial of finite degree. That is, for agiven error allowance $\epsilon > 0$, there exist polynomials $P_M(x) = \Sigma_{n=0}^M a_n x^n$, such that

$$|P_M(x) - \eta(x)| < \epsilon.$$

Substituting such a polynomial for η, the integral in question can be approximated by a weighted sum of integrals I_n of the form

$$I_n(t) = \frac{1}{E_C^n} \int_A^B dE E^{n-1} \sin 2Et \qquad \text{(A1)}$$

with n integer. Making the variable substitution $z=2Et$, we find $I_n(t)=\frac{1}{(2E_C t)^n}\int_{2At}^{2Bt} dz z^{n-1}\sin z$. There are two relevant cases: $n>0$, and $n=0$. In the first case, integration by parts yields

$$I_n(t) = \frac{1}{2E_C^n t}[B^{n-1}\cos 2Bt - A^{n-1}\cos 2At] + \mathcal{O}(t^{-2}).$$

For $n=0$, we have $I_0(t)=\mathrm{Si}(2Bt)-\mathrm{Si}(2At)$, where $\mathrm{Si}(x)$ is the sine integral. These are known to have the asymptotic property that $\mathrm{Si}(x)\to\frac{\pi}{2}$ uniformly as $x\to\infty$; hence we find that $I_n(t)\to 0$ as $t\to\infty$ for all $n\geqslant 0$. Since the polynomials used to approximate $\eta(E/E_C)$ were of finite order, the resulting approximate expression for $I(t)$ will vanish in the long time limit. Further, since this result holds for any choice of ϵ, we conclude that $I(t)$ itself must vanish in the limit.

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