Microwave-induced fluxon bunching in weakly coupled Josephson junctions

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We consider a pair of weakly coupled long Josephson junctions and investigate the conditions under which an external microwave field may phase-lock a single shuttling fluxon in each junction, and subsequently collapse the mutually repulsive fluxons into a phase-locked bunched state. Based on the coupled sine-Gordon model and the collective coordinate perturbation approach, we develop specific conditions for the physical parameters necessary to ensure the bunching of two fluxons in different junctions. The perturbation results are verified by direct numerical simulations.

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Coupled Josephson junctions have been extensively investigated both theoretically and experimentally in part because of the prospect of phase locking and mutual synchronization. One reason for the interest in phase locking is that an array of synchronized oscillators can dramatically enhance the emitted power and simultaneously retain a small linewidth of the resulting output signal.[1,2] One synchronization mechanism is a coupling through the boundaries such that phase locking may occur due to, e.g., reflecting fluxons.[1,3,4] Another is a spatially distributed coupling between neighboring long junctions[2,5–8] through which, e.g., fluxon dynamics is mutually coordinated between the junctions. The latter case may significantly change the basic properties of each junction as has been shown theoretically for short-range inductive[5–7] and long-range magnetic[8] coupling between long Josephson junctions. These studies have identified that a system of two coupled junctions will exhibit mode-dependent characteristic velocities in the dispersion relation, corresponding to characteristic voltages or frequencies in a physical system.[5,7] In a system of two coupled junctions, these modes, one faster and one slower than the usual characteristic velocity of a single junction, represent the asymptotic velocities for the superposition field and the difference field of the two junctions, respectively. Thus, synchronized modes of two coupled junctions have different properties according to whether they are in or out of phase. One particular example is when two coupled junctions each are operated in the single-fluxon mode. In this case, the in-phase mode is energetically unstable, while the out-of-phase fluxon mode is energetically stable. Nevertheless, the in-phase mode, once formed, has been shown to be stable because of the Cherenkov radiation arising from the incommensurate characteristic velocities of the coupled system.[9] Despite the existence and stability of the fast-moving bunched fluxon mode it is not clear how to experimentally obtain such a state, since fluxons of equal polarity in different junctions are mutually repulsive and experiments are usually initiated at low-fluxon velocity. In fact, while several other types of excitations have been shown to exhibit both branches of in-phase and out-of-phase modes,[10,11] the repulsive fluxon mode is not easily observed.

This paper investigates how the in-phase bunched fluxon mode, which has an energy formation barrier when the system is initiated in an energetically favorable state, can be obtained by experimentally controllable parameters. By applying a microwave magnetic field, which can phase lock a shuttling fluxon,[12,14] to two inductively and capacitively coupled junctions, we demonstrate how the system can go from the antiphase mode, with two mutually repulsive fluxons, to the in-phase mode, and we develop explicit expressions for experimental parameters such that this may happen in weakly coupled systems.

The normalized equations of motion for a model system of two coupled Josephson junctions is[6,7,15]

$$\phi_{xx} - \phi_{tt} - \sin\phi + \Delta_1 \psi_{xx} + \Delta_2 \psi_{tt} = \alpha \phi_t - \eta \tag{1}$$

$$\psi_{xx} - \psi_{tt} - \sin\psi + \Delta_1 \phi_{xx} + \Delta_2 \phi_{tt} = \alpha \psi_t - \eta, \tag{2}$$

where ϕ and ψ are the gauge-invariant phase differences between the quantum-mechanical wave functions of the two superconducters defining the two junctions, respectively. See Refs. 6 and 15 for details on parameters and normalization. The applied boundary conditions are

$$\phi_x(0) = \psi_x(0) = \phi_x(L) = \psi_x(L) = \Gamma \sin(\Omega t), \tag{3}$$

modeling a system with normalized length L embedded in a magnetic field with normalized amplitude Γ and frequency Ω.[13]

The corresponding normalized energy of the system is defined by[15] $H = H_\phi + H_\psi + H_I$ where

$$H_\phi = \int_0^L (\phi_t^2 + \phi_x^2 + 1 - \cos\phi)dx, \tag{4}$$

$$H_I = \int_0^L (\Delta_1 \phi_x \psi_x - \Delta_2 \phi_t \psi_t)dx, \tag{5}$$

and H_ψ is defined similar to H_ϕ.

Since we are interested in single fluxon modes and their interaction with an external magnetic field acting through the junction boundaries, we will consider the solutions to the semi-infinite $(L \to \infty)$ system[7] describing two single fluxons moving with asymptotic speed u and distance $r = u|\tau_1 - \tau_2|$, where τ_1 and τ_2 are the boundary collision times for the fluxons of ϕ and ψ, respectively:

$$\phi_\sigma = 4\sigma_\phi \tan^{-1}\left(\frac{c_\sigma}{u} \frac{\sinh\left(\frac{t-\tau_1}{\sqrt{1+\sigma\Delta_1}} u\gamma(u/c_\sigma)\right)}{\cosh\left(\frac{x}{\sqrt{1+\sigma\Delta_1}}\gamma(u/c_\sigma)\right)} \right) \quad (6)$$

with ψ_σ being defined similarly, $\sigma_\phi = \pm 1$, $\sigma_\psi = \pm 1$, and

$$c_\sigma = \sqrt{\frac{1+\sigma\Delta_1}{1-\sigma\Delta_2}}, \quad \gamma(u) = \frac{1}{\sqrt{1-u^2}}, \quad \sigma = \pm 1. \quad (7)$$

These functions are exact solutions to the left hand side of Eqs. (1) and (2) for $L \to \infty$, $\tau_1 = \tau_2$, and $\Gamma = 0$ when $\phi = \sigma\psi$ and $\sigma = \sigma_\phi \sigma_\psi$. Inserting the above ansatz into the expression for the total energy clearly reveals that the $\sigma = \sigma_\phi\sigma_\psi = 1$ solution is energetically unstable and that $\sigma = \sigma_\phi\sigma_\psi = -1$ provides a stable state for $\tau_1 = \tau_2$.

We will adopt the above ansatz for the perturbed problem such that the solutions for $\sigma = -1$ is used for all cases, where $\phi \neq \psi$; thus, we will generally assume $\sigma_\phi\sigma_\psi = 1$, regardless of σ. Since we investigate if phase locking to an external magnetic field can force $\tau_1 = \tau_2$, we will first explore the phase locking of the system by adapting the wave profiles [Eq. (6)] to the phase-locking analysis.[14]

To phase lock, a fluxon must travel a half period in the time it takes the external ac magnetic field to complete N quarter periods, leading to[14]

$$\frac{c_\sigma}{u} \sinh\left(\frac{\pi N u \gamma(u/c_\sigma)}{2\Omega\sqrt{1+\sigma\Delta_1}}\right) = \cosh\left(\frac{L\gamma(u/c_\sigma)}{2\sqrt{1+\sigma\Delta_1}}\right), \quad (8)$$

from which the asymptotic velocity u can be determined for a given set of parameters. In addition to this condition, the energy change averaged over one period of steady-state motion ΔH must be zero. Following the perturbation analysis for phase locking, a single junction to an external field[14] with the ansatz (6), we obtain

$$\Delta H = \eta I - \alpha \int \int_{t_0-(\pi N/2\Omega)}^{t_0+(\pi N/2\Omega)} (\phi_t^2 + \psi_t^2) dt dx - \Delta H_B, \quad (9)$$

where the spatial integration is understood to be over the system. The first term on the right hand side represents the energy absorption from the dc bias current, I being given by

$$\sinh\left(\frac{I\gamma(u/c_\sigma)}{8\pi\sqrt{1+\sigma\Delta_1}}\right) = \frac{c_\sigma}{u} \sinh\left(\frac{\pi N u\gamma(u/c_\sigma)}{2\Omega\sqrt{1+\sigma\Delta_1}}\right), \quad (10)$$

the second term is the energy loss, and the third term is the energy exchange with the external magnetic field,

$$\Delta H_B = 2\kappa\Gamma \sin(\Omega\tau_0)\cos\left(\Omega\frac{\delta\tau}{2}\right), \quad (11)$$

where $\tau_0 = \frac{1}{2}(\tau_1 + \tau_2)$, $\delta\tau = \tau_2 - \tau_1$, and

$$\kappa = 4\pi(1+\Delta_1)\frac{\cosh\left[\frac{\Omega\sqrt{1+\sigma\Delta_1}}{2u\gamma(u/c_\sigma)} \cos^{-1}\left(2\frac{u^2}{c_\sigma^2} - 1\right)\right]}{\cosh\left(\frac{\pi\Omega\sqrt{1+\sigma\Delta_1}}{2u\gamma(u/c_\sigma)}\right)}. \quad (12)$$

For a given asymptotic distance, $r = u\delta\tau$, between the fluxons, they can be phase locked if the bias η satisfies

$$|\eta - \eta_0| \leq 2\Gamma\frac{\kappa}{I}\cos\left(\Omega\frac{\delta\tau}{2}\right) \equiv \frac{1}{2}\Delta\eta, \quad (13)$$

$$\eta_0 \equiv \frac{\alpha}{I} \int \int_0^{\pi N/\Omega} (\phi_t^2 + \psi_t^2) dt dx. \quad (14)$$

To investigate if the phase-locked state will collapse into a bunched, $r = \delta\tau = 0$, state, we must evaluate

$$F = \frac{-\Omega}{N\pi}\left(\Omega\kappa\Gamma \sin(\Omega\tau_0)\sin\left(\frac{\Omega r}{2u}\right) + \int_0^{N\pi/\Omega} \frac{\partial H_I}{\partial r} dt\right) \quad (15)$$

for $\sigma = -1$. Since F is an expression of the effective force between the two fluxons, steady-state dynamics must imply that $F = 0$ for a given steady-state distance r. A simple traveling wave solution, which is valid when the fluxons are far from the boundaries, yields the expression[16]

$$H_I^\infty = \frac{8\sigma_\phi\sigma_\psi\gamma^2(u/c_\sigma)r}{\sinh[\gamma(u/c_\sigma)r/\sqrt{1+\sigma\Delta_1}]}\frac{\Delta_1 - \Delta_2 u^2}{1+\sigma\Delta_1}. \quad (16)$$

Note that we are using this expression as a repulsive interaction ($\sigma_\phi\sigma_\psi = 1$) regardless of the value of σ. Assuming that the time the fluxons interact with the boundary is small compared to a half period of motion, we will proceed using $\int H_I dt \approx \int H_I^\infty dt$. Then, $F = 0$ implies

$$\kappa\Gamma\cos(\Omega\tau_0)\sin\left(\Omega\frac{r}{2u}\right) = \frac{-8\sigma_\phi\sigma_\psi\pi u\gamma^2(u/c_\sigma)}{\Omega\sinh\left(\frac{\gamma(u/c_\sigma)r}{\sqrt{1+\sigma\Delta_1}}\right)} \times \left(1 - \frac{\gamma(u/c_\sigma)r\cosh\left(\frac{\gamma(u/c_\sigma)r}{\sqrt{1+\sigma\Delta_1}}\right)}{\sqrt{1+\sigma\Delta_1}\sinh\left(\frac{\gamma(u/c_\sigma)r}{\sqrt{1+\sigma\Delta_1}}\right)}\right)\frac{\Delta_1 - \Delta_2 u^2}{\sqrt{1+\sigma\Delta_1}}. \quad (17)$$

The maximum value of $\Gamma\cos(\tau_0\Omega)$, necessary to ensure $r \to 0$, is found for $r = 0$ (for $\sigma = -1$).

Expanding Eq. (17) for small r, we obtain for $r = 0$ the critical value Γ_c of the ac-amplitude of the ac magnetic field beyond which two phase-locked, mutually repulsive fluxons will collapse into a bunched state

$$\Gamma_c = \frac{16\sigma_\phi\sigma_\psi\pi u^2\gamma^3(u/c_\sigma)}{3\kappa\Omega^2\cos(\Omega\tau_0)}\frac{\Delta_1 - \Delta_2 u^2}{1+\sigma\Delta_1}. \quad (18)$$

This result shows that the ac-magnetic-field-induced bunching of fluxons is most effective at the center of a phase-

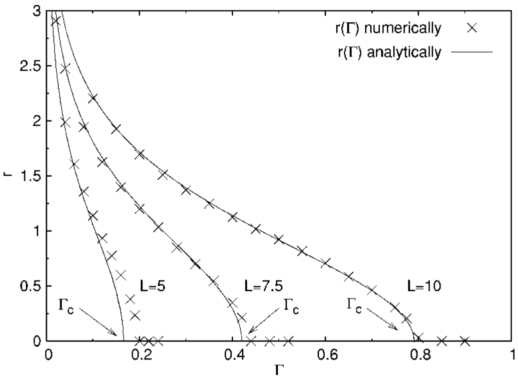

FIG. 1. Distance, r, between two phase-locked fluxons as a function of magnetic-field amplitude Γ. Continuous curves are obtained from Eq. (17) and markers represent the corresponding results of numerical simulations. Parameters are $\alpha=0.1$, $\eta\approx\eta_0^{\sigma=-1}$, $\Omega=2.5/L$, $\Delta_1=0.015$, and $\Delta_2=0$.

locked step in the current-voltage characteristics ($\eta\approx\eta_0$) since $\tau_0=0$.

To induce bunching through phase locking, the resulting bunched state must fulfill the phase-locking conditions for $\sigma=1$ with the parameters used to phase lock and collapse the $\sigma=-1$ fluxons. This can be investigated directly by inserting $\sigma=1$ and $\delta\tau=0$ into Eqs. (13) and (14) and determining the threshold value Γ_b of the magnetic field. Thus, magnetic-field-induced bunching requires that $\Gamma\geqslant\Gamma_c$ *and* $\Gamma\geqslant\Gamma_b$. We have found that for most relevant cases $\Gamma\geqslant\Gamma_c$ results in $\Gamma\geqslant\Gamma_b$; only for relatively high values of the damping parameter have we observed this not always to be the case.

Direct numerical simulations of Eqs. (1) and (2) have been conducted, using a second-order central-difference approximation to the spatial derivatives with spatial discretization $dx=0.025$ and temporal discretization $dt\lesssim dx$. The initial condition has in all cases been an antiphase configuration of the two fluxons. Transient evolution of >100 periods of motion is conducted before the relative position between the fluxons is determined. Because of the physical relevance of the inductive coupling Δ_1 for Josephson junctions, all simulations are conducted for $\Delta_2=0$, and we will limit the study further to $\eta\approx\eta_0^{\sigma=-1}$, since the anticipated effect is most profound at the center of the phase-locked step.

Figure 1 shows the steady-state distance between two fluxons of different junctions, both phase locked to a magnetic field with frequency $\Omega=2.5/L$, as a function of the magnetic-field amplitude Γ. Solid curves represent the perturbation result [Eq. (18) ($\sigma=-1$)], where $\tau_0\approx 0$ and u is given by Eq. (8). Markers are the results of numerical simulations, where $r=u\delta\tau$ is determined from numerically measuring $\delta\tau$ and evaluating the asymptotic velocity from Eq. (8). The figure clearly shows how the steady-state distance between the fluxons decreases monotonically with increasing magnetic-field amplitude until a bunched state ($r=0$) is obtained at $\Gamma=\Gamma_c$, whereafter the state remains bunched. It is clear from the figure that the agreement between the simple perturbation treatment and the simulations results is very good for all three system lengths. It is also noticeable that the agreement is best for the longer junction. This is in agreement with the assumptions made in the analytical treatment. One assumption is that the ansatz [Eq. (6)] represent a half period of motion during a reflection at a boundary. Another is that the mutual repulsion between the fluxons due to the inductive coupling can be well described without the boundary effects. Both these assumptions are poor for a very short system. Nevertheless, the figure shows very convincing agreement between analytical and numerical results.

The value of magnetic-field amplitude Γ_c, for which the mutual fluxon distance r becomes zero, was studied in detail and the results shown in Fig. 2. Here, the two upper continuous curves show the critical-field amplitude Γ_c as given by Eq. (18), for two values of applied frequency, as a function of the inductive-coupling constant Δ_1. The two lower continuous curves represent the critical value Γ_b necessary for sustaining phase locking of the bunched state ($\sigma=1$). All results are obtained for $\alpha=0.1$, and the three plots represent three different system lengths, $L=2.5,5,10$. Markers indicate the results of numerical simulations. As in Fig. 1, we find very consistent and good agreement between the analytical results of the perturbation method and the direct numerical simulations over a wide range of parameters. And again, we find that the shorter systems are generally showing less good agreement than the longer systems. These consistent discrepancies observed in Figs. 2, $L=2.5$ and $L=5$ (slow modes), are most likely due to a poor determination of $\eta_0^{\sigma=-1}$, resulting in a factor of $(\cos\Omega\tau_0)^{-1}$ in Γ_c, which is consistent with the observed discrepancies in Figs. 2(a) and 2(b). We note that in spite of these discrepancies, the agreement is consistently good and the perturbation results have predictive capabilities.

We have determined a method for manipulating a system of coupled-overlap and open-ended Josephson junctions, each with a single shuttling fluxon such that an initial state of repulsive, out-of-phase fluxons will collapse into the energetically unfavorable bunched state. The results show that applying an external ac field, to which the fluxon motions may phase lock, acts as an effective restoring force between the fluxons, thereby countering the mutual repulsion from the coupling. The method has been explained analytically through a standard perturbation method, and the results verified through direct numerical simulations. We point out that although the results have demonstrated a method to bunch fluxons in inductively and capacitively coupled systems, they also show that this method is relevant only to weakly coupled systems, since the necessary magnetic-field amplitude Γ_c for bunching grows super-linearly with the coupling constant. It is clear from the results that shorter systems will be more easily manipulated by this method than longer systems because the relative contact time between the fluxons and the microwave field increases with decreasing L.

We will finally comment on the experimental task of obtaining a bunched state, after the microwave signal has been turned off. Since the repulsive force is increasing with u, the fluxon velocity must be kept relatively small to also keep Γ small. Unfortunately, without microwaves, the bunched state is only stable above c_{-1},[9] so all the numerical solutions shown may decay to the nonbunched state when the microwaves are turned off. Thus, to ensure a stable bunched state

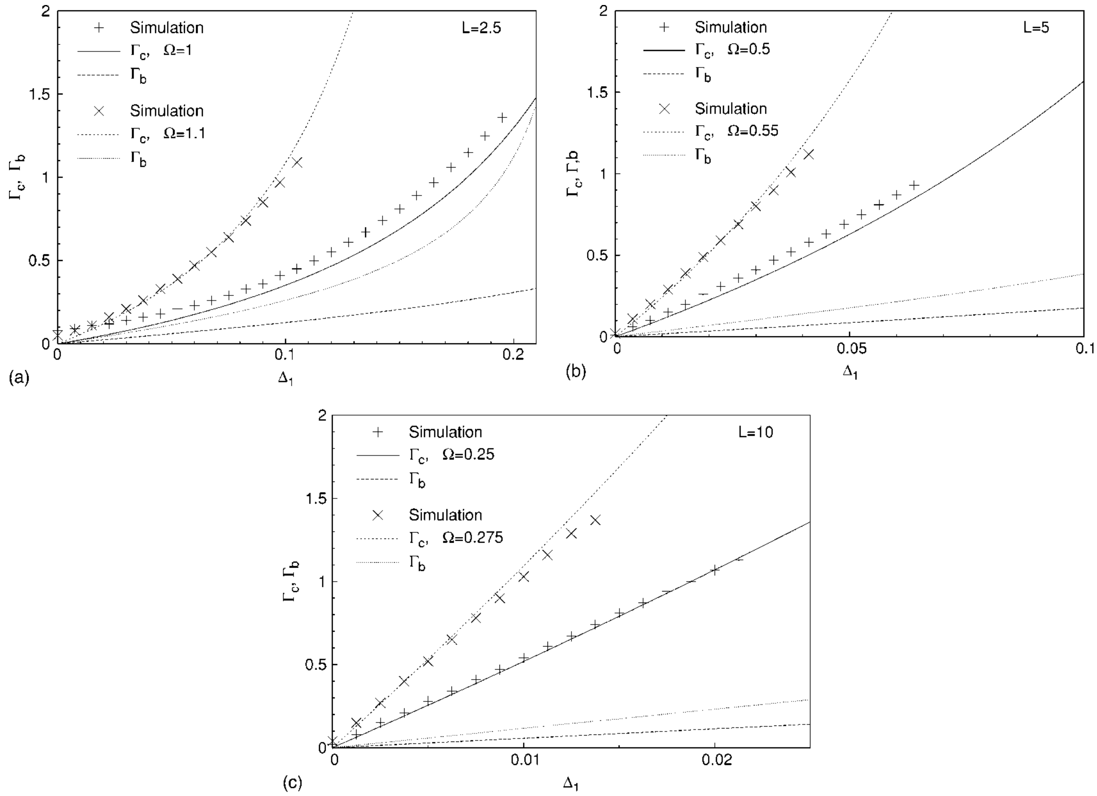

FIG. 2. Critical magnetic-field amplitude Γ_c for which phase-locked, mutually repulsive fluxons are forced to bunch. Continuous curves represent the perturbation result [Eq. (18)] for two different applied microwave frequencies (upper curves), and two lower curves represent the magnetic-field amplitude Γ_b, necessary for phase locking the resulting bunched $\sigma=1$ state. Markers represent the results of numerical simulations. Parameters are $\Delta_2=0$, $\alpha=0.1$, and $\eta\approx\eta_0^{\sigma=-1}$.

after microwaves are turned off one should increase the frequency of the microwaves, while keeping the system in the locking range given by Eqs. (13) and (14) with $\sigma=1$ until u is above c_{-1}. Then, the bunched mode will be stable without microwaves.

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