Universality of ac conduction in anisotropic disordered systems:
An effective-medium-approximation study

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Anisotropic disordered systems are studied in this work within the random barrier model. In such systems the transition probabilities in different directions have different probability density functions. The frequency-dependent conductivity at low temperatures is obtained using an effective medium approximation. It is shown that the isotropic universal ac-conduction law, $\tilde{u} = \tilde{\sigma} \ln \tilde{\sigma}$, is recovered if properly scaled conductivity ($\tilde{\sigma}$) and frequency ($\tilde{u}$) variables are used.

I. INTRODUCTION

In the past years, ac-conduction in isotropic disordered systems has been extensively studied.[1–11] Several experiments support the existence of a universal function governing the conductivity-frequency relation in a variety of materials, in either electronic[5–7] or ionic[5,6,8–11] systems. This universality is also supported by several theoretical studies.[6] One of the simplest models for studying conduction in disordered systems is the random barrier model, in which the energy barriers joining sites of a given network are selected at random from a given probability density function (PDF). For this model the universality of ac-conduction at low temperatures is well established. Dyre and co-workers studied the isotropic random barrier model within the effective medium approximation (EMA).[4–6,12] They found that a universal function is arrived at in a low-temperature-small-frequency expansion, which corresponds to the solution of the equation

$$\tilde{u} = \tilde{\sigma} \ln \tilde{\sigma}, \tag{1}$$

where $\tilde{\sigma} = \sigma(u)/\sigma(0)$ and $\tilde{u} = u \tilde{\beta} \ln \tilde{\beta} / \sigma(0)$ are scaled conductivity and frequency variables, respectively (with $\tilde{\beta} \propto \beta = 1/k_B T$, see below). This equation is universal in the sense that it does not depend on the characteristic disorder of the system. The same equation had been previously arrived at with other approximations, e.g., the macroscopic[1–3] and hopping models.[13] In addition, other approaches such as the percolation path approximation, the diffusion cluster approximation and the velocity autocorrelation method, predict universal functions which present a better data collapse.[5–7] However, EMA offers a simple systematic tool to study ac-conductivity of disordered systems analytically which, in addition, gives the expected universal behavior.

The conductivity properties of anisotropic disordered systems have also attracted attention in the last years.[14–24] Two relevant examples of anisotropic disordered systems are the superconductor cuprates and porous reservoir rocks. In the first case, conductivity properties are strongly anisotropic, with a remarkable difference between the conductivity in the *ab* plane and along the *c* axis. In the second example, a relation between permeability and electrical conductivity in isotropic fluid-saturated porous media is well estab-

lished,[25–27] and a universal behavior for the dynamical permeability, analogous to Eq. (1), was observed numerical and experimentally.[28,29] Since anisotropy is a key characteristic of porous media and fractured rock,[30] the behavior of anisotropic frequency-dependent conductivity in disordered media and its relation with the permeability tensor[27] is of key interest.

In view of these and others examples of anisotropic disordered media, is necessary to find a relation between anisotropic conductivity and frequency. The approach used here is to use the ideas and concepts used in isotropic problems and apply them to anisotropic systems. In this sense, the main purpose of this work is to extend and generalize the universal law, Eq. (1), to anisotropic systems. An anisotropic random barrier model in two dimensions is studied within an effective medium theory. The analytical procedure used by Dyre[4] is closely followed in order to obtain a consistent low-temperature-small-frequency expansion for the anisotropic ac-conductivity. Here, from the analysis of the low temperature limit, the isotropic universal function is recovered for properly scaled variables relating the conductivity in both directions. The paper is organized as follows: In Sec. II the anisotropic random barrier model is described and previous zero-frequency results are summarized. Section III briefly presents the main features of the anisotropic EMA and in Sec. IV the low-temperature-small-frequency expansion is performed. Finally, a summary is presented in Sec. V.

II. BACKGROUND

In the anisotropic random barrier model considered here, equal energy minima form a square lattice with its four nearest-neighbors separated by energy barriers, whose heights are randomly selected from anisotropic PDFs. Let 1 and 2 be the main directions of the square lattice, and the PDFs in each direction $\rho_1(E_1)$ and $\rho_2(E_2)$. Once the energy barrier $E_{\alpha\beta}$ between two nearest-neighbor sites α and β is selected from the corresponding PDF, the transition rate between sites α and β is given by $\omega_{\alpha\beta} = \omega_0 \exp(-\beta E_{\alpha\beta})$, where ω_0 is the constant jump rate and $\beta = 1/k_B T$ is the inverse temperature, with k_B being the Boltzmann constant. Since $E_{\alpha\beta} = E_{\beta\alpha}$, the forward ($\alpha \rightarrow \beta$) and backward ($\beta \rightarrow \alpha$) jumps

have the same transition rate. The energy PDFs, $\rho_1(E_1)$ and $\rho_2(E_2)$, are related to the transition rate PDFs, $\nu_1(\omega_1)$ and $\nu_2(\omega_2)$, through the corresponding transformation of random variables.

The zero-frequency conductivity of the anisotropic random barrier model in two dimensions was recently studied by using an anisotropic generalization of the EMA.[24] The low temperature conductivity in each direction was shown to follow Arrhenius laws with the same activation energy E_c, which is determined by the anisotropic percolation properties of the lattice. For the square lattice studied, the bond percolation threshold is the critical surface $p_1+p_2-1=0$,[31] where p_i represents the probability of having a conducting link between two nearest-neighbor sites in the i direction. This implies an activation energy given by[24]

$$\int_0^{E_c} \rho_1(E_1)dE_1 + \int_0^{E_c} \rho_2(E_2)dE_2 = 1. \tag{2}$$

The zero-frequency conductivities, $\sigma_i(u=0)$, in each direction are thus given by

$$\sigma_1(0) = \gamma_{12}\omega_0 a^2 e^{-\beta E_c}, \quad \sigma_2(0) = \gamma_{21}\omega_0 a^2 e^{-\beta E_c}, \tag{3}$$

where a is the lattice constant and the prefactor

$$\gamma_{12} = \gamma_{21}^{-1} = \frac{\displaystyle\int_0^{E_c} \rho_1(E_1)dE_1}{\displaystyle\int_0^{E_c} \rho_2(E_2)dE_2}. \tag{4}$$

Note that, at low temperatures, the anisotropic character of the system reflects only in the prefactors γ_{12} and γ_{21} of the zero-frequency conductivity.

III. ANISOTROPIC EMA

The EMA consists in averaging the effects of disorder by defining an effective medium with effective transition rates, which depend on the Laplace variable u. These effective transition rates are self-consistently determined by the requirement that the difference between the propagator of the impurity and homogeneous problems should average to zero.[32–40] In anisotropic problems, two effective transition rates, one for each direction, are introduced. The effective frequency-dependent conductivities of the disordered system, $\sigma_1(u)$ and $\sigma_2(u)$, are proportional to the effective transition rates.[14,17] A *rationalized* unit system is used where all the prefactors are absorbed in the definition of effective conductivities, so that they are equivalent to effective transition rates.[4] These effective conductivities are then determined by two self-consistent conditions,[14,17]

$$\left\langle \frac{\sigma_1 - \omega_1}{1 + 2(\sigma_1 - \omega_1)[G^1(u) - G^0(u)]} \right\rangle_{\nu_1(\omega_1)} = 0,$$

$$\left\langle \frac{\sigma_2 - \omega_2}{1 + 2(\sigma_2 - \omega_2)[G^2(u) - G^0(u)]} \right\rangle_{\nu_2(\omega_2)} = 0. \tag{5}$$

Here, $G^{1(2)}$ and G^0 represent the nonperturbed anisotropic Greens functions related to the probabilities of moving from

the origin to one of its nearest neighbors in the 1(2) direction and the return probability, respectively. The impure bond connects two nearest neighbor sites of the lattice whose transition rates are equal to ω_1 if the impure bond lies in the 1 direction and ω_2 if the impure bond is in the other direction. The angular brackets denote averaging over the corresponding transition rate PDFs.

The real and imaginary parts of the frequency-dependent conductivity are obtained considering that the Laplace frequency u is actually an imaginary frequency related to the real frequency by $u=is$.[4,38] However, the Laplace frequency picture has proved to be a useful simplification for studying frequency-dependent conductivity.[4,6]

IV. UNIVERSAL FREQUENCY-DEPENDENT CONDUCTIVITY

In this section a low-temperature-small-frequency expansion[4] of the set of Eqs. (5) is performed. In the following, the subscripts $i,j=1,2$ are used to represent the two different directions of the lattice, noting that one of the self-consistent conditions in Eqs. (5) is obtained interchanging the subscripts i and j in the other equation. To the lowest order in u, the difference between the Greens functions in the i direction appearing in the denominator of Eqs. (5) may be written as[17]

$$(G^0 - G^i)_{u\to 0} = \frac{f_{ij}}{2\sigma_i} + \frac{ug_{ij}}{2\sigma_i}, \tag{6}$$

with

$$f_{ij} = \frac{2}{\pi} \arctan\sqrt{\frac{\sigma_i}{\sigma_j}}, \tag{7}$$

and

$$g_{ij} = \frac{1}{4\pi\sqrt{\sigma_i\sigma_j}} \ln\frac{64\sigma_i\sigma_j}{u(\sigma_i+\sigma_j)}. \tag{8}$$

Using this expansion for the Greens functions near $u=0$, the frequency-dependent self-consistent condition Eqs. (5) read

$$\left\langle \frac{\omega_i - \sigma_i}{\omega_i + [(f_{ij}-ug_{ij})^{-1} - 1]\sigma_i} \right\rangle_{\nu_i(\omega_i)} = 0. \tag{9}$$

In order to emphasize the energy dependence of the transition rates, one may average over the energy distributions and write, rearranging terms in the previous equation,

$$\left\langle \frac{1}{\omega_i + [(f_{ij}-ug_{ij})^{-1} - 1]\sigma_i} \right\rangle_{\rho_i(E_i)} = \frac{f_{ij} - ug_{ij}}{\sigma_i}. \tag{10}$$

Two regimes may be identified for each direction: $\omega_i(E_i) \ll [(f_{ij}-ug_{ij})^{-1}-1]\sigma_i$, and $\omega_i(E_i) \gg [(f_{ij}-ug_{ij})^{-1}-1]\sigma_i$.[24] The energy separating these two cases, $E_i^g(u)$, is defined by

$$\omega_i[E_i^g(u)] = [(f_{ij}-ug_{ij})^{-1} - 1]\sigma_i, \tag{11}$$

which yields

$$E_i^g(u) = -\frac{1}{\beta} \ln\left\{ [(f_{ij} - ug_{ij})^{-1} - 1]\frac{\sigma_i}{\omega_0} \right\}. \qquad (12)$$

Using these different ranges of energy, separated by E_i^g, and the fact that transition rates and energies are related through an Arrhenius law, the integral of the energy average in Eq. (10) may be approximated, and the self-consistent condition, Eq. (10), then reads

$$\int_{E_i^g(u)}^{\infty} \frac{\rho_i(E_i)}{[(f_{ij} - ug_{ij})^{-1} - 1]\sigma_i} dE_i = \frac{f_{ij} - ug_{ij}}{\sigma_i}, \qquad (13)$$

or, equivalently,

$$\int_{E_i^g(u)}^{\infty} \rho_i(E_i) dE_i = 1 - f_{ij} + ug_{ij}. \qquad (14)$$

In the zero-frequency case, E_i^g becomes the same for the two directions,[24] and is given by Eq. (2) with $E_i^g(u=0) \equiv E_c$. Noting that for $u=0$, f_{ij} is given by

$$f_{ij} = \int_0^{E_c} \rho_i(E_i) dE_i \qquad (15)$$

and inserting Eq. (15) into Eq. (14) one obtains

$$\int_{E_i^g(u)}^{E_c} \rho_i(E_i) dE_i = ug_{ij}, \qquad (16)$$

which may be approximated by

$$\int_{E_i^g(u)}^{E_c} \rho_i(E_i) dE_i \simeq q_i[E_i^g(0) - E_i^g(u)], \qquad (17)$$

with $q_i = \rho_i[E_i^g(0)] = \rho_i[E_c]$. Evaluating $E_i^g(u)$ from Eq. (12) and combining Eq. (16) with Eq. (17) one obtains, to the lowest order in u,

$$ug_{ij} = \frac{q_i}{\beta} \ln \frac{[(f_{ij} - ug_{ij})^{-1} - 1]\sigma_i}{(f_{ij}^{-1} - 1)\sigma_i(0)} \simeq \frac{q_i}{\beta} \ln \frac{\sigma_i}{\sigma_i(0)}, \qquad (18)$$

where $\sigma_i(0)$ is the zero-frequency conductivity in the i direction, and is given by Eq. (3). Then, using the definition of g_{ij}, Eq. (8), the previous equation my be written as

$$\ln \frac{\sigma_i}{\sigma_i(0)} = \widetilde{\beta_i} \frac{u}{\sqrt{\sigma_i \sigma_j}} \ln \frac{64\sigma_i\sigma_j}{u(\sigma_i + \sigma_j)}, \qquad (19)$$

with $\widetilde{\beta_i} = \beta/(4\pi q_i)$. This last equation gives the general frequency-dependent conductivity for all temperatures and in the small frequency limit.

In order to obtain the low temperature limit for the frequency-dependent conductivity, scaled conductivity and frequency variables for each direction are introduced, namely,

$$\widetilde{\sigma_i} = \frac{\sigma_i}{\sqrt{\sigma_i(0)\sigma_j(0)}}, \qquad (20)$$

and

$$\widetilde{u_i} = \frac{\widetilde{\beta_i} \ln \widetilde{\beta_i}}{\sqrt{\sigma_i(0)\sigma_j(0)}} u. \qquad (21)$$

With this scaled variables Eq. (19) may be written as

$$\ln \widetilde{\sigma_i} + \frac{1}{2} \ln \frac{\sigma_j(0)}{\sigma_i(0)}$$
$$= \frac{\widetilde{u_i}}{\sqrt{\widetilde{\sigma_i}\widetilde{\sigma_j}} \ln \widetilde{\beta_i}} \left[\ln \frac{64\widetilde{\sigma_i}\widetilde{\sigma_j}}{\widetilde{u_i}(\widetilde{\sigma_i} + \widetilde{\sigma_j})} + \ln \widetilde{\beta_i} + \ln(\ln \widetilde{\beta_i}) \right]. \qquad (22)$$

By taking the low temperature limit $\beta \to \infty$ for fixed $\widetilde{u_i}$ and $\widetilde{\sigma_i}$, the following set of coupled equations is arrived at for the scaled conductivities as functions of the scaled frequencies

$$\widetilde{u_1} = \sqrt{\widetilde{\sigma_1}\widetilde{\sigma_2}}(\ln \widetilde{\sigma_1} + \ln \gamma_{21}),$$
$$\widetilde{u_2} = \sqrt{\widetilde{\sigma_2}\widetilde{\sigma_1}}(\ln \widetilde{\sigma_2} + \ln \gamma_{12}), \qquad (23)$$

where Eqs. (3) and (4) were used.

This set of equations represent the complex relation between the conductivity in each direction. Although they may be regarded as universal equations for the scaled conductivities, the use of two different frequencies, one for each direction, is not suitable for a frequency-dependent description. In addition, the terms containing γ_{12} still depend on the specific PDFs used. Still, Eqs. (23) are useful to obtain the frequency-dependent conductivities: given the PDFs for each direction, E_c and γ_{12} are calculated through Eqs. (2) and (4), then through the definitions of $\widetilde{u_i}$ and $\widetilde{\beta_i}$ the scaled conductivities in each direction may be calculated numerically by solving the set of coupled equations (23).

As the analytical derivation of Eqs. (23) closely follows the previous derivation of Eq. (1) from EMA,[4] the isotropic result is obviously recovered by setting the same PDF, $\rho(E)$, for the two directions, which implies $\widetilde{\sigma_1} = \widetilde{\sigma_2} = \widetilde{\sigma} = \sigma(u)/\sigma(0)$ and $\widetilde{u_1} = \widetilde{u_2} = \widetilde{u} = u\widetilde{\beta} \ln \widetilde{\beta}/\sigma(0)$, with $\widetilde{\beta} = \beta/[4\pi\rho(E_c)]$. However, by adding the two Eqs. (23) and setting $\widetilde{u} = (\widetilde{u_1} + \widetilde{u_2})/2$ and $\widetilde{\sigma} = \sqrt{\widetilde{\sigma_1}\widetilde{\sigma_2}}$, expression (1) is obtained again. This is a nontrivial result and establish that although the anisotropic conductivities are coupled through the complex relation given by Eqs. (23), the frequency dependence of the geometric mean conductivity, $\widetilde{\sigma}$, in the anisotropic problem is simple given by Eq. (1). This result given by the EMA should be tested by more complex theories.

V. SUMMARY

In summary, by means of the frequency-dependent EMA, conductivity in an anisotropic random barrier model has been studied. It was shown that in a small-frequency expansion, the low temperature limit is characterized by an universal law, i.e., independent of the anisotropic PDFs, relating scaled conductivity and frequency variables. This scaled quantities are obtained by a proper combination of the con-

ductivities and energy properties in each direction. Although the universal law is obtained for a two-dimensional system, it is expected to hold in three dimensions for appropriately scaled variables. Direct comparison with experiments on the conductivity of superconductor cuprates is not possible at present because the available data corresponds, to the best of the author's knowledge, to conductivity measurements in only one of the anisotropic directions of the system. Unfortunately, for a comparison with the scaling function, the conductivity in both directions should be available. The results presented can be of relevance for a complete theory of the anisotropic permeability of porous reservoir rocks. Given the relation between the anisotropic conductivity and permeability tensors,[27] the dynamical permeability in each relevant direction can be obtained through Eqs. (23). Finally, is worth noting that a perfect agreement between the EMA universal law and experimental or simulation data should not be expected, as this is the case even for isotropic problems.[5,6] Other theoretical and simulation methods were shown to predict a better universal law, which collapses experimental data from various disordered systems.[5,6] However, EMA still provides a simple analytical tool for a first exploration of the properties of ac-conduction. In the present work, a first insight on the emergence of a universal law for anisotropic disorder systems has been presented.

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