Fast partial decoherence of a superconducting flux qubit in a spin bath

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The superconducting flux qubit has two quantum states with opposite currents which generate different magnetic fields. Environment of nuclear spins can find out the magnetic field after a decoherence time τ_0 inversely proportional to the magnetic field and the square root of the number of spins. When the Hamiltonian of the qubit drives fast coherent Rabi oscillations between the states with opposite current, then the magnetic field is flipped at a constant rate ω and the decoherence time $\tau = \omega \tau_0^2$ is much longer than τ_0. However, on closer inspection decoherence actually takes place on two time scales. The long time τ is a time of full decoherence but a part of quantum coherence is lost already after the much shorter time π / ω. This fast partial decoherence biases coherent flux oscillations towards the initial flux direction and it can affect performance of the superconducting devices as qubits.

Quantum computers can perform certain tasks exponentially faster than their classical counterparts.[1] The basic unit of the quantum computer is a quantum bit or qubit. Several physical implementations of the qubit were proposed including ion traps,[2] nuclear magnetic resonance,[3] quantum dots,[4] single photon,[5] and two flavors of superconducting Josephson junction qubits.[6] The solid state implementations are in principle most easily scallable and most compatible with the existing classical electronics, but at the same time they are the most endangered by quantum decoherence. However, recent experiments[7,8] in both types of superconducting qubits show hundreds of coherent oscillations. This is still far from the minimal requirements of the fault tolerant quantum computation[9] but it makes superconducting qubits attractive candidates for scallable quantum technology. In order to push their performance even further it is essential to understand their quantum decoherence better.

I. SUPERCONDUCTING FLUX QUBIT

The superconducting flux qubit is a superconducting loop broken by one or more Josephson junctions.[6] The junctions are designed so that the lowest two quantum eigenstates are states with opposite currents and different magnetic fluxes. The flux difference can range from a tiny $10^{-3} \cdots 10^{-2}$ fraction of the flux quantum[10] to $1/2$ of the quantum.[11] The two states span a two-dimensional Hilbert space of the qubit. I assume convention that the state $|1\rangle$ ($|0\rangle$) with positive(negative) current is the $+1(-1)$ eigenstate of σ_z. Degeneracy of these two states is removed by coherent tunneling between the opposite current states driven by the qubit Hamiltonian:

$$H_Q = \frac{1}{2}\omega\sigma_x + \mu\sigma_z. \qquad (1)$$

For $\mu = 0$ the eigenstates of H_Q are the coherent superpositions $|\pm\rangle = (|1\rangle \pm |0\rangle)/\sqrt{2}$. These Schrödinger cats were observed in two independent experiments.[10,11] For $\mu = 0$ the qubit Hamiltonian (1) drives coherent oscillations between the states with opposite current. For example, the initial state $|1\rangle$

evolves into $|1\rangle\cos(\frac{1}{2}\omega t) - i|0\rangle\sin(\frac{1}{2}\omega t)$ with an expectation value of the flux operator

$$F(t) \equiv \langle\sigma_z\rangle = \cos(\omega t) \qquad (2)$$

oscillating with the Rabi frequency ω. Damped flux oscillations were observed in a number of experiments[7,8] in both types of the superconducting qubit.

The qubit Hamiltonian (1) is sufficient to perform any one-qubit operation. Two-qubit operations are possible thanks to inductive coupling between any two qubits.[6] Entangled eigenstates of two qubits were detected in Ref. 12 and coherent Rabi oscillations in a system of two inductively coupled qubits were observed in Ref. 8. It is possible to perform single-qubit NOT and two-qubit CNOT operations in an adiabatic way[13] that, in particular, precludes excitation beyond the truncated two-dimensional Hilbert space of the qubit. However, before it comes to quantum computation, quantum decoherence has to be overcome first.

II. QUBIT IN A SPIN BATH

Various sources of decoherence for flux qubits have been discussed in Ref. 14. At relatively high temperatures the main source of decoherence is normal state quasiparticles. However, density of quasiparticles is exponentially suppressed at temperatures much less than the critical temperature. Other decoherence mechanisms including electromagnetic radiation from the qubit or ohmic dissipation in the environment are, respectively, negligible or tractable. Nuclear spins are argued to play a minor role. Due to relaxation the spins randomly flip their polarization. Random spins are a source of random magnetic field which couples to the magnetic moment of the qubit and randomizes its quantum state. This picture is further corroborated in Ref. 16 where the spins are assumed to be mutually noninteracting but each of them is coupled to a bosonic environment. The decoherence time is estimated in the range of milliseconds. The final Eq. (47) in Ref. 16 shows that in the limit of vanishing nuclear spin relaxation rate (γ_i in Ref. 16) the decoherence time tends to infinity. As expected, the external mag-

netic noise vanishes for vanishing nuclear spin relaxation rate.

However, apart from generating magnetic noise nuclear spins can be also silent witnesses of the quantum state of the qubit. In the limit of vanishing spin relaxation and for negligible spin-spin interaction each spin is simply precessing in the magnetic field of the qubit. The direction of the field and the direction of the precession depend on the state of the qubit. This way the spins can learn the quantum state of the qubit. Once they know the state any quantum coherence between the states with opposite current is lost. This elementary argument shows that decoherence does not vanish for vanishing spin relaxation.

In this paper I neglect nuclear spin relaxation. When the relaxation rate is longer than any other relevant time scale, like the frequency of the Rabi oscillations or frequency of the spin precession in the magnetic field of a qubit, then this is a very reasonable assumption. Spin lattice relaxation is also neglected in the paper of Prokofev and Stamp.[15] In contrast to the very general formalism employed in Ref. 15, in this paper I apply most elementary methods to a simple model Hamiltonian. This simple approach benefits with clear interpretation of the results and additional insights into dynamics of the decoherence process. In particular, the simple model makes it very clear that even in the absence of any spin lattice relaxation the spin environment decoheres the quantum state of a single qubit. The decoherence is not an artifact of ensemble average over different static spin configurations or an ensemble of qubits, as sometimes claimed in the literature, but a result of genuine entanglement between the qubit and the spins.

The qubit interacting with spins is described by a Hamiltonian

$$H = H_Q + V + H_S. \tag{3}$$

Here V is interaction between the qubit and N spins

$$V = \sigma_z \sum_{n=1}^{N} \vec{B}(\vec{r}_n) \vec{\sigma}^{(n)}. \tag{4}$$

$\vec{B}(\vec{r}_n)$ is a magnetic field of the qubit in the state $|1\rangle$ at the position $\vec{r}_n$ of the nth spin. Direction of the magnetic field is reversed in the state $|0\rangle$ as is accounted for by the operator σ_z. $\vec{\sigma}^{(n)}$ is a vector of Pauli matrices $(\sigma_x^{(n)}, \sigma_y^{(n)}, \sigma_z^{(n)})$ in the Hilbert space of the nth spin. A unitary transformation in the Hilbert space of each spin brings the interaction Hamiltonian to a more convenient form

$$V = \sigma_z \sum_{n=1}^{N} B_n \sigma_x^{(n)}. \tag{5}$$

Here $B_n = |\vec{B}(\vec{r}_n)| > 0$ is the strength of the qubit magnetic field at the location of the nth spin.

III. EXACTLY SOLVABLE MODEL

When the magnetic field from the qubit is stronger than magnetic fields from other spins, then spin-spin interaction can be neglected, $H_S = 0$ and

$$H = \frac{1}{2}\omega\sigma_x + \sigma_z\left(\mu + \sum_{n=1}^{N} B_n \sigma_x^{(n)}\right). \tag{6}$$

This exactly solvable spin-spin model was also considered in Ref. 17. The Hamiltonian (6) can be easily diagonalized. The spin part of any eigenstate is

$$|\vec{s}\rangle = |s_1\rangle \cdots |s_N\rangle. \tag{7}$$

Here $|s_n\rangle$ is an eigenstate of $\sigma_x^{(n)}$ with an eigenvalue $s_n \in \{+1, -1\}$. In the subspace of $|\vec{s}\rangle$ the Hamiltonian (6) reduces to an effective qubit Hamiltonian

$$H(\vec{s}) = \frac{1}{2}\omega\sigma_x + \sigma_z b(\vec{s}) \tag{8}$$

with an effective external "magnetic field" $b(\vec{s}) = \mu + \sum_{n=1}^{N} B_n s_n$. $H(\vec{s})$ has eigenvalues $\pm\Omega(\vec{s}) \equiv \pm\sqrt{\frac{1}{4}\omega^2 + b^2(\vec{s})}$ with corresponding eigenstates proportional to $b|+\rangle \pm [\Omega \mp \frac{1}{2}\omega]|-\rangle$.

IV. PURE INITIAL STATE OF SPINS

I open the discussion of decoherence with an example where the initial state is

$$|\psi(0)\rangle = (\alpha|+\rangle + \beta|-\rangle)|1_1, \ldots, 1_N\rangle. \tag{9}$$

Each spin is initially in the $+1$ eigenstate of its $\sigma_z^{(n)}$. This is a very special initial pure state of spins. However, in Sec. VIII I will show that this example is generic and that all the results derived in this special pure case literally pass without any modification to the case of the initial high temperature thermal state of spins, or to an ensemble average over all possible initial pure states of spins. In this section I choose this special pure state to make it more transparent that decoherence is due to genuine entanglement between the qubit and the spins, and that it is not an artifact of an ensemble average over different qubits or over different static spin configurations.

The Hamiltonian (6) evolves the initial state into

$$|\psi(t)\rangle = \frac{1}{2^{N/2}}\sum_{\vec{s}} [\mathcal{A}(t,\vec{s})|+\rangle + \mathcal{B}(t,\vec{s})|-\rangle]|\vec{s}\rangle,$$

$$\mathcal{A}(t,\vec{s}) = e^{-i\Omega t} b \frac{\alpha b - \beta\left(\frac{1}{2}\omega - \Omega\right)}{b^2 + \left(\frac{1}{2}\omega - \Omega\right)^2}$$

$$+ e^{+i\Omega t} b \frac{\alpha b - \beta\left(\frac{1}{2}\omega + \Omega\right)}{b^2 + \left(\frac{1}{2}\omega + \Omega\right)^2}, \tag{10}$$

$$\mathcal{B}(t,\vec{s}) = -e^{-i\Omega t}\left(\frac{1}{2}\omega - \Omega\right)\frac{\alpha b - \beta\left(\frac{1}{2}\omega - \Omega\right)}{b^2 + \left(\frac{1}{2}\omega - \Omega\right)^2}$$

$$+ -e^{+i\Omega t}\left(\Omega + \frac{1}{2}\omega\right)\frac{\alpha b - \beta\left(\frac{1}{2}\omega + \Omega\right)}{b^2 + \left(\frac{1}{2}\omega + \Omega\right)^2}. \tag{11}$$

This state is an entangled state of the qubit and the spins: the state of the qubit $\mathcal{A}(t,\vec{s})|+\rangle + \mathcal{B}(t,\vec{s})|-\rangle$ depends on the state

of spins $|\vec{s}\rangle$. The state of the qubit can be described by reduced density matrix $\rho(t)$ obtained by taking a partial trace over the spins:

$$\rho(t) \equiv \mathrm{Tr}_S |\psi(t)\rangle\langle\psi(t)| = \frac{1}{2^N}\sum_{\vec{s}}\begin{pmatrix} \mathcal{A}^*\mathcal{A} & \mathcal{A}\mathcal{B}^* \\ \mathcal{A}^*\mathcal{B} & \mathcal{B}\mathcal{B}^* \end{pmatrix}. \quad (12)$$

This matrix form is given in the basis of $|\pm\rangle$ states of the qubit. $\rho(t)$ is in general a mixed state as can be most conveniently measured by its "quadratic entropy" $S(t)=1-\mathrm{Tr}\,\rho^2(t)\in[0,\frac{1}{2}]$ introduced in Ref. 18. The initial state is pure, $S(0)=0$, but in general $S(t)$ grows as the interacting qubit and spins get entangled and the state of the qubit is loosing its initial coherence.

The most right-hand side of Eq. (12) can be formally interpreted as an average over all possible states of spins, or over all possible sets $\vec{s}$ of N independent random variables s_n. In the limit of large N the effective magnetic field $b(\vec{s})=\mu+\Sigma_{n=1}^N B_n s_n$ becomes a Gaussian variable with a mean of μ and a variance of $NB^2 \equiv \Sigma_{n=1}^N B_n^2$. In this limit the sum in Eq. (12) can be approximated by an integral,

$$\rho(t) = \int_{-\infty}^{+\infty} db\, \frac{e^{-(b-\mu)^2/2NB^2}}{\sqrt{2\pi NB^2}}\begin{pmatrix} \mathcal{A}^*\mathcal{A} & \mathcal{A}\mathcal{B}^* \\ \mathcal{A}^*\mathcal{B} & \mathcal{B}\mathcal{B}^* \end{pmatrix}. \quad (13)$$

This density matrix is formally an average over different values of the static magnetic noise b generated by different static spin configurations. The mixed state of the qubit results from an average over different pure states of the qubit, each of them evolved from the same initial state but with a different value of b in the Hamiltonian (8). However, as the derivation of Eq. (13) demonstrates, this is not an average over different realizations of the experiment with different static configurations of nuclear spins, but it is an average over a superposition of different states of spins present in a single realization of the experiment. This is genuine decoherence due to entanglement with the spin bath.

In the following sections special cases of this exact solution are worked out in more detail. I begin with the simplest case when the frequency of Rabi oscillations $\omega=0$.

V. NO RABI OSCILLATIONS

The density matrix (13) becomes particularly simple in the absence of flux oscillations when we can set $\Omega(\vec{s})=b(\vec{s})$ and

$$\mathcal{A} = \alpha\cos bt - i\beta\sin bt,$$

$$\mathcal{B} = \beta\cos bt - i\alpha\cos bt.$$

The products $\mathcal{A}^*\mathcal{A}$ or $\mathcal{A}^*\mathcal{B}$ in the density matrix (13) contain oscillatory terms proportional to $e^{\pm 2ibt}$. The average over b in Eq. (13) dephases these oscillations to zero,

$$\int_{-\infty}^{+\infty} db\, \frac{e^{-(b-\mu)^2/2NB^2}}{\sqrt{2\pi NB^2}} e^{\pm 2ibt} = e^{\pm 2i\mu t}e^{-2NB^2 t^2}, \quad (14)$$

after decoherence time

$$\tau_0 = \frac{1}{2\sqrt{NB^2}}. \quad (15)$$

It is instructive to write the density matrix (13) in the basis of σ_z eigenstates

$$\rho(t) = \frac{1}{2}$$
$$\times \begin{pmatrix} |\alpha+\beta|^2 & (\alpha+\beta)(\alpha^*-\beta^*)e^{-t^2/2\tau_0^2+2i\mu t} \\ (\alpha^*+\beta^*)(\alpha-\beta)e^{-t^2/2\tau_0^2-2i\mu t} & |\alpha-\beta|^2 \end{pmatrix}. \quad (16)$$

The off-diagonal coherences between the σ_z eigenstates are destroyed after the decoherence time of τ_0 when the density matrix becomes diagonal in this basis. This is not quite surprising because the σ_z eigenstates are eigenstates of both the interaction Hamiltonian V and the qubit Hamiltonian ($\omega=0$)—they are the ideal pointer states of Ref. 19. Quadratic entropy of the state (16) grows like

$$S(t) = \frac{1}{2}|\alpha+\beta|^2|\alpha-\beta|^2\left(1-e^{-t^2/\tau_0^2}\right). \quad (17)$$

The only case when the state of the qubit remains pure, or $S(t)=0$, is when the initial state is one of the σ_z eigenstates

($\alpha=\pm\beta$). This is another fundamental property of the pointer states.

VI. FAST RABI OSCILLATIONS

When Rabi oscillations are much faster than the interaction with the spin bath, then we have a small parameter

$$\epsilon^2 = \frac{4NB^2}{\omega^2} \ll 1. \quad (18)$$

The matrix elements in the density matrix (13) contain oscillatory terms which can be approximated as

$$e^{\pm 2i\Omega t} = e^{\pm 2it\sqrt{1/4\omega^2 + b^2}} \approx e^{\pm i\omega t[1 + 4\mu/\omega_\mu(b-\mu)/\omega_\mu + 2\omega^2/\omega_\mu^2(b-\mu)^2/\omega_\mu^2]}$$

(19)

with $\omega_\mu^2 = \omega^2 + 4\mu^2$. In this expansion I use the assumption (18) that $(b-\mu)/\omega_\mu \simeq \epsilon \ll 1$.

Assuming further that the bias μ is weak as compared to ω,

$$\mu^2 \ll \omega^2,$$

(20)

the density matrix can be approximated by its leading order term in both μ and $(b-\mu)$:

$$\rho(t) = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* z^*(t) e^{-i\omega t} \\ \alpha^* \beta z(t) e^{i\omega t} & |\beta|^2 \end{pmatrix}$$

(21)

with the coherence function

$$z(t) = \frac{\exp\left(-\dfrac{t^2}{2\tau_\mu^2\left(1 - i\dfrac{t}{\tau}\right)}\right)}{\left(1 - i\dfrac{t}{\tau}\right)^{1/2}}.$$

(22)

This function comes from an average of the approximate (19) over the Gaussian variable b. $z(t)$ has two time scales

$$\tau = \frac{\tau_0}{\epsilon} \ll \tau_0,$$

(23)

$$\tau_\mu = \frac{\tau_0}{\left(\dfrac{2\mu}{\omega}\right)} \ll \tau_0,$$

(24)

both of them are much longer than the decoherence time τ_0 in the absence of Rabi oscillations. The coherence function $z(t)$ determines both the entropy growth

$$S(t) = 2|\alpha|^2|\beta|^2\left(1 - |z(t)|^2\right)$$

(25)

and coherent flux oscillations

$$F(t) = \operatorname{Tr}\rho(t)\sigma_z = \frac{1}{2}\left[z(t)e^{i\omega t} + \text{c.c.}\right]$$

(26)

starting from the initial $+1$ eigenstate σ_z. The exact solution (21) can be best understood in limiting cases.

For times $t \gg \tau$ the coherence function can be approximated by $z(t \gg \tau) = e^{-\tau^2/2\tau_\mu^2}\sqrt{\tau/t}\exp i(\pi/4 - \tau/2\tau_\mu^2 t)$ and the coherence decay is characterized by the power laws:

$$S(t \gg \tau) = 2|\alpha|^2|\beta|^2\left(1 - e^{-\tau^2/\tau_\mu^2}\frac{\tau}{t}\right),$$

(27)

$$F(t \gg \tau) = e^{-\tau^2/2\tau_\mu^2}\sqrt{\frac{\tau}{t}}\cos\left[\frac{\pi}{4} + \left(\omega - \frac{\tau}{2\tau_\mu^2}\right)t\right].$$

(28)

Apart from the shift in the frequency of oscillations, and the prefactor $e^{-\tau^2/2\tau_\mu^2}$ which is close to 1 when $\mu \ll NB^2$, in the regime of $t \gg \tau$ the coherence decay does not depend on τ_μ. This power law decay of flux oscillations was also derived in Ref. 17. Both the entropy growth and the decay of flux oscillations are due to the decay of the same coherence function $z(t)$ in the density matrix (21). Thus, contrary to Ref. 17, it is not possible to see oscillations after decay of quantum coherence: flux oscillations are coherent flux oscillations.

When $t \ll \tau$ the coherence function is $z(t \ll \tau) = \exp\times\left(-t^2/2\tau_\mu^2\right)$ and the decay of coherence is Gaussian:

$$S(t \ll \tau) = 2|\alpha|^2|\beta|^2\left(1 - e^{-t^2/\tau_\mu^2}\right),$$

(29)

$$F(t \ll \tau) = e^{-t^2/2\tau_\mu^2}\cos\omega t.$$

(30)

When the bias μ is stronger than the influence of spins, $NB^2 \ll \mu^2 \ll \omega^2$, then $\tau_\mu \ll \tau$ and the Gaussian decay of coherence is completed before crossover to the power law decay after τ. For a weak bias, $\mu^2 \ll NB^2 \ll \omega^2$, the coherence decay is a power law. The coherence decays most slowly when the bias $\mu = 0$. The power law (28) and Gaussian (30) decays are consistent with the decays derived in Ref. 20 from a model with a non-Markovian external noise, compare also to Ref. 22. Here, these results follow from a microscopic description of quantum decoherence in a bath of noninteracting spins.

After decoherence the density matrix (21) becomes diagonal in the basis of σ_x eigenstates. What is more the entropy remains zero, $S(t) = 0$, only when $\alpha = 0$ or $\beta = 0$, i.e., when the initial state of the qubit is an eigenstate of σ_x. This is not quite surprising because when H_Q dominates over V and H_S, then the eigenstates of H_Q are expected to be the pointer states.[21]

VII. EARLY TIME PARTIAL DECOHERENCE

The density matrix (21) is the leading order term in the expansion of the exact density matrix (13) in powers of $b/\omega \simeq \epsilon$. In this section I include higher order terms. For the sake of simplicity I consider only the optimal case of zero bias when $\mu = 0$.

The products $\mathcal{A}^*\mathcal{B}$, etc. in the exact density matrix (13) are expanded up to second order in powers of b/ω. The oscillatory factors $e^{\pm i\Omega t}$ are approximated as in Eq. (19). After these two approximations the Gaussian integral over b in Eq. (13) gives the approximate density matrix accurate up to second order in ϵ

$$\rho(t) = \begin{pmatrix} P & C^* \\ C & 1 - P \end{pmatrix}$$

(31)

with matrix elements

$$P(t) = |\alpha|^2 - \frac{1}{2}\epsilon^2\left(|\alpha|^2 - |\beta|^2\right)\left[1 - \frac{e^{i\omega t}z^3(t) + \text{c.c.}}{2}\right],$$

(32)

$$C(t) = \alpha^*\beta z(t)e^{i\omega t} + \frac{1}{4}\epsilon^2(\alpha^*\beta + \text{c.c.})\left[2 - e^{i\omega t}z^3(t)\right]$$

$$- \frac{1}{4}\epsilon^2\left[\alpha^*\beta e^{i\omega t}z^3(t) + \text{c.c.}\right].$$

(33)

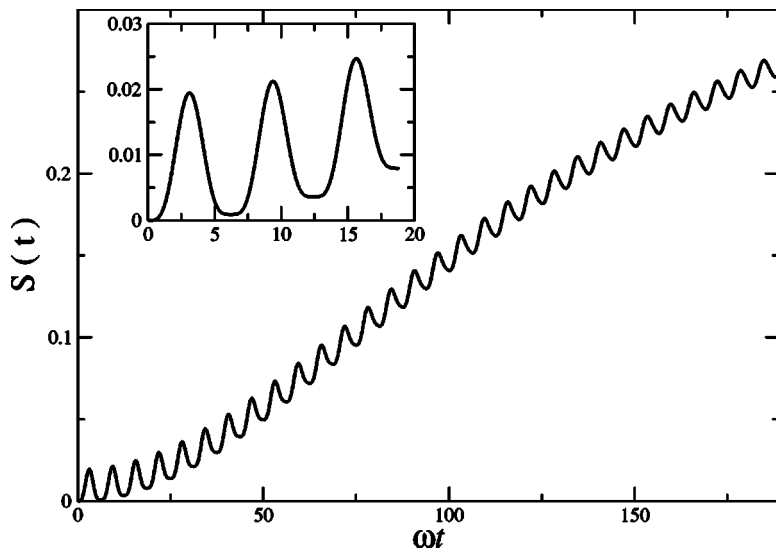Here P (or $1 - P$) is a probability to find the qubit in the state

FIG. 1. Entropy $S(t)$ according to the exact Eq. (13) for $\omega=1$, $\mu=0$, $\epsilon=0.1$, and $\alpha=\beta = 1/\sqrt{2}$. The time scale for full decoherence is $\tau =400$ and the early time partial decoherence sets in on the time scale $\tau_0=10$. The inset shows the entropy at early times. The nonvanishing early time entropy means partial decoherence.

$|+\rangle$ (or $|-\rangle$) and C is quantum coherence between these states. The $z(t)$ is the function (22) with zero bias $\mu=0$ or $\tau_\mu=\infty$.

For $\mu=0$ the function $z(t)$ has only one time scale τ. The regime $t \gg \tau$ has been considered in the preceding section. In the opposite regime of early time we can approximate $z(t \ll \tau)=1$ in Eqs. (32) and (33) and get the entropy

$$S(t \ll \tau), \tag{34}$$

$$\epsilon^2(1 - \cos \omega t)\{|\alpha|^4 + |\beta|^4 - [(\alpha^*\beta)^2 e^{i\omega t} + \text{c.c.}]\}. \tag{35}$$

No matter what is the initial state, defined by α and β, this entropy does not remain zero but oscillates with an amplitude proportional to ϵ^2, see Fig. 1. This fluctuating entropy means partial loss of coherence.

This partial early time decoherence also shows in flux oscillations. The initial state $\alpha=\beta=1/\sqrt{2}$ leads to flux oscillations

$$F(t) \equiv \text{Tr} \, \sigma_z \rho(t) = \epsilon^2 + (1 - \epsilon^2)\cos \omega t. \tag{36}$$

Unlike the coherent flux oscillation between $+1$ and -1 in Eq. (2) this flux oscillates between $+1$ and $-1+2\epsilon^2$, see Fig. 2. The amplitude of the coherent oscillation is reduced due to the partial decoherence. The oscillation is also biased towards the initial positive value of F. Similar effect was also predicted recently in the spin-boson model.[23] Here it follows from the simple and explicitly solvable spin-spin model. The effect seems to be a generic feature of any non-Markovian environment.

Fast Rabi oscillations make the time of full decoherence τ much longer than the decoherence time τ_0 without Rabi oscillations. When the flux oscillates many times before the spins can learn their orientation, then it takes the spins a much longer time to learn the state of the qubit than in the static case. On a long time scale covering many flux oscillations the oscillating magnetic field measured by the spins effectively averages out to zero. This is the key idea of the quantum bang-bang control introduced in Ref. 24. However,
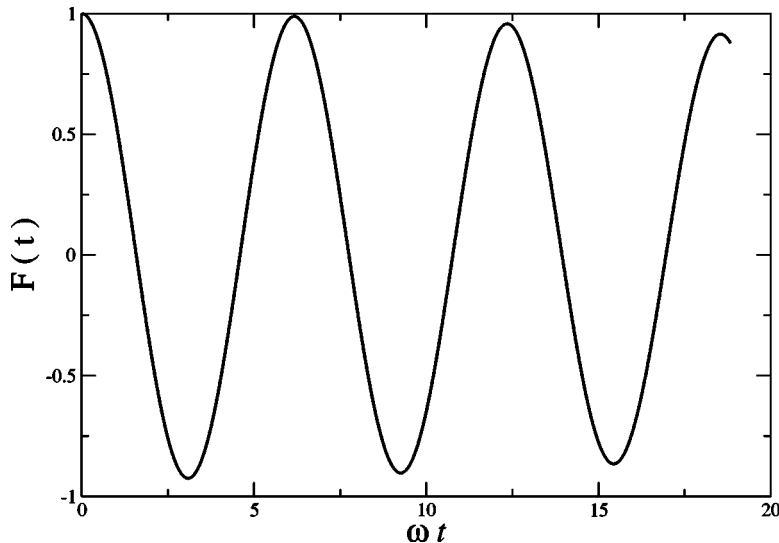

FIG. 2. Flux $F(t)$ according to the exact Eq. (13) for $\omega=1$, $\mu=0$, $\epsilon=0.2$, and $\alpha=\beta=1/\sqrt{2}$. As a result of the fast partial decoherence the flux oscillation is biased towards the initial $F(0)=+1$.

this argument does not apply before the first period of oscillation is completed. On this short time scale the magnetic field of the qubit is effectively static and during this short time the spins have a chance to get a rough idea about its orientation. The spins more or less realize that the initial flux in Eq. (36) is $+1$ and this fuzzy knowledge biases the following oscillations in $F(t)$ towards this positive value. The state of the qubit is partially "collapsed" towards the state with positive flux. This qualitative argument becomes more substantial when we take as an initial state the equal superposition of states with opposite current $|+\rangle$ (or $\alpha=1, \beta=0$) and compare the two expressions for entropy, one valid for fast Rabi oscillations (35) at an early time long before the first period of oscillations is completed, i.e., for $\omega t \ll 1$, and the other in the absence of any oscillations (17). For such early times both formulas are the same,

$$S(\omega t \ll 1) = \frac{t^2}{2\tau_0^2}, \qquad (37)$$

demonstrating that the early time decoherence takes place on the time scale τ_0 as if there were no Rabi oscillations at all. The partial decoherence reaches its first maximum at the time

$$\tau_{\max} = \frac{\pi}{\omega}, \qquad (38)$$

after which we observe periodic coherence revivals, see the inset in Fig. 1. The time of maximal partial decoherence $\tau_{\max}$ is much shorter that the time of full decoherence τ and even the time τ_0.

Fast Rabi oscillations postpone full decoherence after τ but they do not prevent partial decoherence to grow on the time scale of τ_0. This effect may affect scalability of the superconducting flux qubit technology. The partial loss of coherence $\simeq \epsilon^2$ is small in a single qubit, but it scales together with the number of qubits N_{qubits} as $N_{\text{qubits}}\epsilon^2$ and limits the number of qubits to

$$N_{\text{qubits}} \ll \frac{1}{\epsilon^2}. \qquad (39)$$

Obviously, this upper bound does not apply when quantum error correction is performed much faster than the time scale τ_0 of the initial entropy growth.

VIII. ENSEMBLE OF SPINS STATES

The initial state of spins considered so far was a very special choice of a pure state. This choice was made to emphasize that the decoherence results from genuine entanglement between the qubit and the spins and that it is not an artifact of any ensemble average over qubits or spin configurations.

In this section I assume that the initial state of spins is a mixed state described by the density matrix $1/2^N 1_S$ proportional to the unit matrix 1_S. This state can have at least two-fold interpretation as either

(1) the realistic high temperature thermal state of spins,

(2) or an ensemble average over all possible pure states of spins.

The initial state of the qubit and the spins is

$$\rho_{QS}(0) = (\alpha|+\rangle + \beta|-\rangle)(\alpha^*\langle+| + \beta^*\langle-|) \otimes \frac{1}{2^N}1_S. \quad (40)$$

The unit matrix 1_S can be expressed in any basis of spins' states but the representation $1_s = \Sigma_{\vec{s}}|\vec{s}\rangle\langle\vec{s}|$ most directly leads to the reduced density matrix of the qubit

$$\rho(t) \equiv \text{Tr}_S\, \rho_{QS}(t) = \frac{1}{2^N}\sum_{\vec{s}} \begin{pmatrix} \mathcal{A}^*\mathcal{A} & \mathcal{A}\mathcal{B}^* \\ \mathcal{A}^*\mathcal{B} & \mathcal{B}\mathcal{B}^* \end{pmatrix}. \quad (41)$$

This density matrix is identical with the density matrix (12) obtained from the special initial pure state in Eq. (9) and also all the results derived for the pure state, including the fast partial decoherence, literally pass without any modification to the mixed initial state. This means that the decoherence in the special pure state is exactly the same as in

(1) the realistic high temperature thermal state,

(2) or the ensemble average over pure states.

From the point of view of decoherence the special pure state is a generic example of spins' state.

IX. ESTIMATES

In this section I attempt to make some numerical estimates. For example, in the qubit design considered in Ref. 16 the parameters are estimated as $N=10^8$, $B=10^4$ to 10^5 s^{-1} and $\omega=10^9$ s^{-1}, compare Ref. 16. These numbers lead to $\tau_0 = 10^{-8}$ to 10^{-9} s, $\epsilon^2=10^{-2}$ to 10, and $\tau=10^{-7}$ to 10^{-9} s. In the best case of $\epsilon^2=10^{-2}$ one can see $\omega\tau=\epsilon^{-2}=10^2$ of coherent flux oscillations before the decoherence time τ. Thus it is possible to see hundreds of oscillations as in the experiments.[7] For the same parameters the fast partial decoherence limits the number of qubits to $N_{\text{qubits}} \ll 10^2$. These numbers leave, however, encouraging room for current experiments.

X. CONCLUSION

The environment of spins is a witness to the quantum state of the superconducting flux qubit. The qubit gets entangled with spins after a decoherence time τ_0. Coherent flux oscillations driven by the qubit Hamiltonian postpone full decoherence after a much longer time τ, but they do not prevent the fast partial decoherence from reaching its first maximum already after the time $\tau_{\max}= \pi/\omega$ which is even shorter than τ_0. This partial decoherence biases the flux oscillations towards the initial orientation of the flux and may seriously limit scalability of the superconducting flux qubit technology.

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