

Rigorous Born approximation and beyond for the spin-boson model

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Within the lowest-order Born approximation, we present an exact calculation of the time dynamics of the spin-boson model in the Ohmic regime. We observe non-Markovian effects at zero temperature that scale with the system-bath coupling strength and cause qualitative changes in the evolution of coherence at intermediate times of order of the oscillation period. These changes could significantly affect the performance of these systems as qubits. In the biased case, we find a prompt loss of coherence at these intermediate times, whose decay rate is set by $\sqrt{\alpha}$, where α is the coupling strength to the environment. We also explore the calculation of the next-order Born approximation: we show that, at the expense of very large computational complexity, interesting physical quantities can be rigorously computed at fourth order using computer algebra, presented completely in an accompanying MATHEMATICA file. We compute the $O(\alpha)$ corrections to the long time behavior of the system density matrix; the result is identical to the reduced density matrix of the equilibrium state to the same order in α . All these calculations indicate precision experimental tests that could confirm or refute the validity of the spin-boson model in a variety of systems.

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I. INTRODUCTION

Novel solid-state devices that can control spin degrees of freedom of individual electrons,^{1,2} or discrete quantum states in superconducting circuits,³⁻⁶ show promise in realizing the ideal of the completely controllable two-state quantum system, weakly coupled to its environment, that is the essential starting point for qubit operation in quantum computation. From a fundamental point of view, these experimental successes also bring us close to embodying the ideal test of quantum coherence as envisioned by Leggett and co-workers many years ago,^{7,8} in which a simple quantum system is placed in a known initial state, is allowed to evolve for a definite time t under the action of its own Hamiltonian and under the influence of decoherence from the environment, and is then measured.

Recent experiments, starting with Ref. 4, show that this ideal test can be implemented in practice. The decay of quantum oscillations due to environmental decoherence is now³⁻⁶ sufficiently weak that some tens of coherent oscillations can be observed. If quantum computation is to become a reality, it is believed⁹ that these systems will eventually need to achieve even lower levels of decoherence, such that thousands or tens of thousands of coherent oscillations could be observed. This prospect of producing experiments with ultralong coherence times in quantum two-state systems offers a new challenge for theoretical modeling of decoherence. Despite the many years of work^{10,11} following Leggett's initial proposals, there has never been a full, systematic analysis of the most popular description of these systems, the spin-boson model, in the limit of very weak coupling to the environment.

In this paper, we provide an exact analysis of the weak-coupling limit of the spin-boson model for the Ohmic heat bath, and in the low-temperature limit. In this limit, the Born approximation (to the self-energy) should become essentially exact, and we make no other approximations in our

solutions—in particular, no Markov approximation is made. As other workers have recently emphasized,^{12,13} understanding the details of the short-time dynamics of this model is especially crucial for the operation of these systems as qubits. It is clear that the spin-boson model will be an important tool in understanding imperfections in quantum gate operation.^{9,14}

We find important non-Markovian effects in this regime. At lowest order in the Born expansion of the self-energy superoperator, the time dynamics of the model rigorously separates into a sum of strictly exponential pieces (the usual “ T_1 ” and “ T_2 ” decays of the Bloch-Redfield model) plus two distinct nonexponential pieces that arise, technically speaking, from two different kinds of branch cuts in the Laplace transform of the solution of the generalized master equation that we obtain.

These two contributions both have power-law forms at long times, $t > T_1, T_2$, and thus formally dominate the exponentially decaying parts. But more interesting is that they both give new structure to the time evolution at intermediate times t , $1/\omega_c < t < T_1, T_2$; this structure typically occurs for t on the order of the oscillation period. (Here, ω_c is a high-frequency cutoff of the bath modes, defining the very short time regime, $t < 1/\omega_c$, which is of no interest here.) We can explain our results in the language of the double-well potential, where the two quantum states are “left” and “right” (L/R), the $t=0$ state is pure L , and the system oscillates in time via tunneling from L to R . The first branch-cut contribution is most important in the unbiased case (L and R energies degenerate) and it causes the system, starting immediately in the first quantum oscillation, to spend more time in the R well, that is, the *opposite* well from the one the system is in initially. The second branch-cut contribution, present when the system is biased, adds to the amplitude of the coherent oscillation, but dies out after an intermediate time which scales like the inverse square root of the interaction

strength α with the bath. This *prompt loss of coherence*, whose amplitude is proportional to α , changes qualitatively the picture of the initial decay of coherence that is so important for discussions of fault-tolerant quantum computation.¹⁵

Finally, we set up the next-order Born approximation and do some initial calculations with it. This involves computing the self-energy of the master equation to fourth order in the system-bath coupling. At this order, the self-energy is a sum of thousands of separate terms; but we find that it is feasible to compute various quantities of physical interest with the aid of MATHEMATICA. As an illustration, we provide a full calculation of the steady-state system density matrix to order α in the limit of low temperature, which requires the fourth-order self-energy. Given the enormous complexity of the calculation, we find a very simple result for the corrections to steady state; they turn out to be identical to those for the thermodynamic equilibrium state calculated to the same order in α . Thus, we are able to establish rigorously a very strong form of ergodicity for the spin-boson model at this order.

II. GENERALIZED MASTER EQUATION

We are interested in studying the time dependence of the system density matrix $\rho_S(t) = \text{Tr}_B \rho(t)$ with a time-independent system Hamiltonian, and in the presence of a fixed coupling to an environment. An exact equation for ρ_S —the generalized master equation (GME)—is¹⁶

$$\dot{\rho}_S(t) = -iL_S \rho_S(t) - i \int_0^t dt' \Sigma_S(t-t') \rho_S(t'), \quad (1)$$

$$\Sigma_S(t) = -i \text{Tr}_B L_{SB} e^{-iQ_L t} L_{SB} \rho_B. \quad (2)$$

Here the kernel $\Sigma_S(t)$ is the self-energy superoperator, the system-bath Hamiltonian is written

$$H = H_S + H_{SB} + H_B \quad (3)$$

(S =system, B =bath), the Liouvillian superoperator is defined by $L_x \rho = [H_x, \rho]$, $\rho_B = e^{-\beta H_B} / Z$, $\beta = 1/k_B T$, T is the temperature, and Q is the projection superoperator $Q = 1 - \rho_B \text{Tr}_B$. Equation (1) is written for the case $\text{Tr}_B H_{SB} \rho_B = 0$, and the total initial state is taken to be of the form $\rho(0) = \rho_S(0) \otimes \rho_B$, for an arbitrary $\rho_S(0)$. Since we are interested in the case of weak coupling to the bath, we will consider a systematic expansion in powers of this coupling L_{SB} in the self-energy operator $\Sigma_S(t)$.

Retention of only the lowest-order term in this expansion, giving the first Born approximation, is obtained¹⁷ by the replacement $e^{-iQ_L t} \rightarrow e^{-i\hat{Q}(L_S + L_B)t}$ in Eq. (2). Thus, in the lowest Born approximation, the self-energy becomes

$$\Sigma_S^{(2)}(t) = -i \text{Tr}_B L_{SB} e^{-i(L_S + L_B)t} L_{SB} \rho_B. \quad (4)$$

We have used the fact here that the expression is unaffected if the Q superoperator is dropped in the exponential.

We now proceed to solve the GME *with no further approximations*.¹⁸ This distinguishes our work from previous efforts, in which various other approximations (secular, ro-

tating wave, Markov, “noninteracting blips,” short time) are made (see, e.g., Refs. 7 and 10–13). We will find that, in particular, avoidance of the Markov approximation endows the solution with qualitatively new features.

We will work out all our results for the Ohmic spin-boson model, for which the Hamiltonian is

$$H_S = \frac{\Delta}{2} \sigma_x + \frac{\epsilon}{2} \sigma_z, \quad (5)$$

$$H_{SB} = \sigma_z \otimes \left[\sum_n c_n (b_n^\dagger + b_n) \right], \quad (6)$$

$$H_B = \sum_n \omega_n b_n^\dagger b_n. \quad (7)$$

Here $\sigma_{x,y,z}$ are the Pauli operators; we will use $\sigma_0 = I$ (identity operator). Also, b_n^\dagger and b_n are the creation and annihilation operators of harmonic oscillator n of the bath. With the spectral density defined as

$$J(\omega) \equiv \sum_n c_n^2 \delta(\omega - \omega_n), \quad (8)$$

the “Ohmic” case is defined by choosing the coefficients c_n and the oscillator frequencies ω_n such that, in the limit of a continuous spectrum,

$$J(\omega) = \frac{\alpha}{2} \omega e^{-\omega/\omega_c}. \quad (9)$$

Here ω_c is an ultraviolet cutoff frequency.

The first few steps of the solution of the GME do not depend on the details of this model; we need only assume that the system Hilbert space is two-dimensional, and the system-bath coupling has the bilinear form, $H_{SB} = S \otimes X [S(X)$ is an operator in the system (bath) space]. Under these general circumstances, the GME (1) in the Born approximation can be rewritten in an ordinary operator form,

$$\langle \dot{\sigma}_\mu(t) \rangle = -i \text{Tr}_S \sigma_\mu [H_S, \rho_S(t)] - \int_0^t dt' I_\mu(t, t'), \quad (10)$$

$$I_\mu(t, t') = I_{\mu 0}(t') + \sum_{\nu=1}^3 I_{\mu\nu}(t') \langle \sigma_\nu(t-t') \rangle, \quad (11)$$

$$I_{\mu\nu}(t') = \text{Re}\{C(-t') \text{Tr}_S \sigma_\nu(-t') [\sigma_\mu, S] S(-t')\}. \quad (12)$$

Here $\langle x \rangle \equiv \text{Tr}_S x \rho_S$, and the bath correlation function is

$$C(t) \equiv \text{Tr}_B [X X(t) \rho_B] = C'(t) + iC''(t). \quad (13)$$

C' and C'' denote the real and imaginary parts of the bath correlator, and, for the spin-boson model, $X = \sum_n c_n (b_n^\dagger + b_n)$. The time-dependent operators are in the interaction picture, i.e.,

$$\Xi(t) = e^{i(H_S + H_B)t} \Xi e^{-i(H_S + H_B)t}, \quad (14)$$

for any operator Ξ .

The GME in Eq. (10) can be written in the matrix form

$$\langle \dot{\sigma}(t) \rangle = R * \langle \sigma \rangle + \underline{k}. \quad (15)$$

Here σ denotes the vector $(\sigma_x, \sigma_y, \sigma_z)^T$ and convolution is denoted $A * B \equiv \int_0^t dt' A(t')B(t-t')$. When the system Hamiltonian is chosen as in Eq. (5),¹⁹ and the system part of the system-bath interaction Hamiltonian is $S = \sigma_z$, then we have²⁰

$$R(t) = \begin{pmatrix} -\frac{E^2}{\Delta^2}\Gamma_1(t) & -\epsilon\delta(t) + \frac{E}{\Delta}K_y^+(t) & 0 \\ \epsilon\delta(t) - \frac{E}{\Delta}K_y^+(t) & -\Gamma_y(t) & -\Delta\delta(t) \\ 0 & \Delta\delta(t) & 0 \end{pmatrix}, \quad (16)$$

$$\underline{k}(t) = \left(-\frac{E}{\Delta}k^-(t), -k_y^-(t), 0 \right)^T, \quad (17)$$

with

$$E = \sqrt{\epsilon^2 + \Delta^2}. \quad (18)$$

We have introduced the functions

$$\Gamma_1(t) = \frac{4\Delta^2}{E^2} \cos(Et)C'(t), \quad (19)$$

$$\Gamma_y(t) = \frac{4\Delta^2}{E^2} \left(1 + \frac{\epsilon^2}{\Delta^2} \cos(Et) \right) C'(t), \quad (20)$$

$$K_y^+(t) = \frac{4\epsilon\Delta}{E^2} \sin(Et)C'(t), \quad (21)$$

$$k^-(t) = \frac{4\Delta^2}{E^2} \int_0^t dt' \sin(Et')C''(t'), \quad (22)$$

$$k_y^-(t) = \frac{4\epsilon\Delta}{E^2} \int_0^t dt' [1 - \cos(Et')]C''(t'). \quad (23)$$

Equation (15) can be solved in the Laplace domain. Defining the Laplace transform as

$$f(s) = \int_0^\infty e^{-st}f(t)dt, \quad (24)$$

the solutions are, for the “standard” initial conditions $\langle \sigma(t=0) \rangle = (0, 0, z_0 = 1)^T$,

$$\langle \sigma_x(s) \rangle = \frac{1}{s + \frac{E^2}{\Delta^2}\Gamma_1(s)} \left[\left(\epsilon - \frac{E}{\Delta}K_y^+(s) \right) \frac{N(s)}{D(s)} - \frac{E}{\Delta}k^-(s) \right], \quad (25)$$

$$\langle \sigma_y(s) \rangle = -\frac{N(s)}{D(s)}, \quad (26)$$

$$\langle \sigma_z(s) \rangle = -\frac{\Delta N(s)}{s D(s)} + \frac{z_0}{s}, \quad (27)$$

$$N(s) = \frac{E}{\Delta} \left(\epsilon - \frac{E}{\Delta}K_y^+(s) \right) k^-(s) + \left(\frac{\Delta}{s} z_0 + k_y^-(s) \right) \left(s + \frac{E^2}{\Delta^2}\Gamma_1(s) \right), \quad (28)$$

$$D(s) = \left(s + \Gamma_y(s) + \frac{\Delta^2}{s} \right) \left(s + \frac{E^2}{\Delta^2}\Gamma_1(s) \right) + \left(\epsilon - \frac{E}{\Delta}K_y^+(s) \right)^2. \quad (29)$$

To go further, we need an explicit expression for the bath correlator $C(t)$. For the spin-boson model, the well-known formula is

$$C(t) = \int_0^\infty d\omega J(\omega) [\coth(\beta\omega/2)\cos(\omega t) + i \sin(\omega t)]. \quad (30)$$

For the Ohmic case, Eq. (9), Eq. (30) becomes

$$C(t) = -\frac{\alpha}{\beta^2} \text{Re} \psi' \left(\frac{1 - i\omega_c t}{\beta\omega_c} \right) - \frac{\alpha\omega_c^2}{2(i + \omega_c t)^2}, \quad (31)$$

where ψ' is the derivative of the digamma function.²¹

III. MARKOVIAN LIMIT

For discussing the exact solution, it is instructive to understand the structure of the solution in a Markov approximation. This approximation is obtained by replacing all the kernels Γ_1 , Γ_y , K_y^+ , k^- , and k_y^- by their forms near $s=0$. For all except k^- , this means replacing them by constants; k^- has a $1/s$ divergence at small s . Then the solutions Eqs. (27) are rational functions of s . If the poles of these rational functions are located at positions s_k in the complex s plane, with residues $r_k/2\pi i$, then the inverse Laplace transform can be written $\langle \sigma_\mu(t) \rangle = \sum_k r_k^\mu \exp(s_k t)$. We indicate here that while the residues do depend on the label $\mu=x, y, z$, the pole positions do not, as is shown by the form of Eq. (27).

As is well known,¹⁰ there are four poles at positions

$$s_1 = 0, \quad s_2 = -\Gamma_1^0, \quad s_{3,4} = -\Gamma_2^0 \pm i\tilde{E}. \quad (32)$$

The first pole describes the long-time asymptote of the solution (stationary state), the second the purely exponential, “ T_1 ”-type decay (relaxation), and the last two (complex conjugate paired) describe an exponentially decaying sinusoidal part, the “ T_2 ”-type decay of coherent oscillations. The expressions for the constants in Eq. (32) are, to lowest order in α , given by

$$\Gamma_1^0 = T_1^{-1} = \frac{\alpha\pi\Delta^2}{E} \coth(\beta E/2), \quad (33)$$

$$\Gamma_2^0 = T_2^{-1} = \frac{1}{2}\Gamma_1^0 + \frac{2\alpha\pi\epsilon^2}{E^2} k_B T, \quad (34)$$

and²²

$$\tilde{E} = E + \delta E, \quad \delta E = \delta E^{\text{Lamb}} + \delta E^{\text{Stark}}, \quad (35)$$

$$\delta E^{\text{Lamb}} = \frac{\alpha\Delta^2}{E} \left(C - \ln \frac{\omega_c}{E} \right), \quad (36)$$

$$\delta E^{\text{Stark}} = \frac{\alpha \Delta^2}{E} [\text{Re } \psi(iE\beta/2\pi) - \ln(E\beta/2\pi)], \quad (37)$$

where we have dropped terms of order E/ω_c and higher, C is the Euler constant, and ψ is the digamma function.²¹ These

expressions are straightforwardly derivable, and agree with the literature,¹⁰ except for the energy shift due to vacuum fluctuations, δE^{Lamb} [which contains in general $\ln(\omega_c/E)$ and not $\ln(\omega_c/\Delta)$].

The residues of these poles are, in the limit $\alpha \rightarrow 0$,

$$\begin{aligned} r_1^x = x_\infty = -(\Delta/E)\tanh(\beta E/2), \quad r_1^y = y_\infty = 0, \quad r_1^z = z_\infty = -(\epsilon/E)\tanh(\beta E/2), \\ r_2^x = \epsilon\Delta/E^2 - x_\infty, \quad r_2^y = 0, \quad r_2^z = \epsilon^2/E^2 - z_\infty, \\ r_{3,4}^x = -\epsilon\Delta/2E^2, \quad r_{3,4}^y = -\Delta/2E, \quad r_{3,4}^z = \Delta^2/2E^2. \end{aligned} \quad (38)$$

We note that this Markovian theory satisfies the expected fundamental relation²³

$$\Gamma_2^0 = \Gamma_1^0/2 + (2\epsilon^2/E^2) \int_{-\infty}^{\infty} dt \langle X(t)X \rangle_B \quad (\text{Korringa relation}); \quad (39)$$

also, to lowest order in α , the asymptotic values of $\langle \sigma_\mu(t \rightarrow \infty) \rangle$ go to the Boltzmann equilibrium distribution of the system, e.g., $z_\infty = -(\epsilon/E)\tanh(\beta E/2)$, unlike in the popular “noninteracting blip” approximation.¹⁰

IV. BRANCH CUTS AT $T=0$

We now return to the exact solution, examining it in detail at vanishing temperature $T=0$. In this case, Eq. (31) becomes

$$\begin{aligned} C'_{T=0}(t) &= \frac{\alpha \omega_c^2}{2} \frac{1 - \omega_c^2 t^2}{(1 + \omega_c^2 t^2)^2}, \\ C''_{T=0}(t) &= \alpha \omega_c^2 \frac{\omega_c t}{(1 + \omega_c^2 t^2)^2}, \end{aligned} \quad (40)$$

and the Laplace transform of C is

$$\begin{aligned} C'_{T=0}(s) &= \alpha s/2 [-\cos(\tilde{s})\text{Ci}(\tilde{s}) - \sin(\tilde{s})\text{si}(\tilde{s})], \\ C''_{T=0}(s) &= -i\alpha/2 [-\omega_c + s \sin(\tilde{s})\text{Ci}(\tilde{s}) - s \cos(\tilde{s})\text{si}(\tilde{s})], \end{aligned} \quad (41)$$

where $\tilde{s} = s/\omega_c$.²⁴ There is an important feature of this correlation function that makes the Markov solution qualitatively incomplete: while the sine integral $\text{si}(s)$ is an analytic function of s , the cosine integral $\text{Ci}(s)$ behaves like $\ln(s)$ for $s \rightarrow 0$.²¹ This means that $C(s)$ is nonanalytic at $s=0$ —it has a branch point there. Thus, the exact solutions $\langle \sigma_\mu(s) \rangle$ have extra analytic structure not present in the Markov approximation, and the real-time dynamics $\langle \sigma_\mu(t) \rangle$ has qualitatively different features in addition to the pole contributions we have just discussed.

The $s=0$ branch point in $C(s)$ leads the kernels $\Gamma_1(s)$, $K_y^+(s)$, and $k^-(s)$ to have branch points at $s = \pm iE$; the kernels

$\Gamma_y(s)$ and $k_y^-(s)$ have three branch points, at $s=0$ and $s = \pm iE$. Thus, the solutions to the GME $\langle \sigma_{x,y,z}(s) \rangle$ also have three branch points in the complex plane. We find by numerical study that the exact solutions still have four poles as before, which, for small α , have nearly (but not exactly) the same pole positions and residues as in the Markov approximation.

Thus, the structure of the solutions in the complex s plane is as shown in Fig. 1(a). The locations of the branch cuts are chosen for computational convenience, as discussed shortly. Given this branch-cut structure, the inverse Laplace transform (the Bromwich integral) is evaluated by closing the contour as shown. Thus, the exact inverse Laplace transform can be expressed as ($t > 0$)

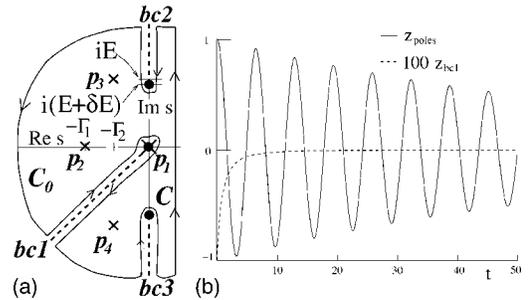


FIG. 1. (a) Structure of the solutions $\langle \sigma_\mu(s) \rangle$ in the complex s plane. The four poles p_1 , p_2 , p_3 , and p_4 are indicated by crosses; the three branch points at $s=0$, $\pm iE$ are indicated by solid circles, and the three branch cuts chosen, $bc1$, $bc2$, and $bc3$, are indicated by dashed lines. The inverse Laplace transform requires an integration along the contour C parallel to the imaginary axis. This integral may be evaluated by closing with a contour in the left half-plane (C_0 , the Bromwich contour), which lies at infinity except for looping back around each of the branch cuts. (b) $z_{\text{poles}}(t)$ and $z_{bc1}(t)$ for the unbiased case, $\epsilon=0$, $\Delta=1$, $\omega_c=30$, $T=0$, and $\alpha=0.01$. t is in units of $1/E$ (i.e., $E=1$).

$$\begin{aligned}
\langle \sigma_\mu(t) \rangle &= \frac{1}{2\pi i} \int_C ds e^{st} \langle \sigma_\mu(s) \rangle \\
&= \frac{1}{2\pi i} \oint_{C_0} ds e^{st} \langle \sigma_\mu(s) \rangle - \frac{1}{2\pi i} \sum_{k=1}^3 q_k \int_{p_k}^\infty dx e^{q_k x t} \\
&\quad \times [\langle \sigma_\mu(q_k x + \eta_k) \rangle - \langle \sigma_\mu(q_k x - \eta_k) \rangle]. \quad (42)
\end{aligned}$$

Here $q_k = e^{i\theta_k}$ and $\eta_k = \eta e^{i(\theta_k - \pi/2)}$, with η an infinitesimal positive real number. That is, η_k is an infinitesimal displacement perpendicular to the direction of branch cut k . For the cut choices we have made, $\theta_1 = 5\pi/4$, $\theta_2 = \pi/2$, $\theta_3 = 3\pi/2$, $p_1 = 0$, and $p_2 = p_3 = E$. The closed-contour integral in the expression can be written as a sum over the four poles, and so gives complex exponential contributions to the solution as in the Markovian case. The extra terms, the sum over the three branch cuts, are new and give qualitatively different features. The contributions of the second and third branch cuts are complex conjugates of each other, so we will be discussing them together.

The contribution of these cuts to the solution is independent of the detailed positioning of the branch cuts, so long as they are not moved across a pole; the choice of the direction of bc1 is a computational convenience—the apparently most natural choice of this cut direction, along the negative real axis, passes it essentially on top of the Γ_1 pole, making the evaluation of the branch-cut integral numerically inconvenient. As a check, we find that the results we discuss now are indeed independent of the cut direction.

We will do a detailed study of these branch-cut contributions for $\langle \sigma_z(t) \rangle \equiv z(t)$. We will use the following notation for the branch-cut terms in Eq. (42); for “branch cut 1” (bc1),

$$z_{bc1}(t) = -\frac{1}{2\pi i} q_1 \int_{p_1}^\infty dx e^{q_1 x t} [\langle \sigma_z(q_1 x + \eta_1) \rangle - \langle \sigma_z(q_1 x - \eta_1) \rangle], \quad (43)$$

and for two complex-conjugate cuts denoted together as “branch cut 2” (bc2),

$$z_{bc2} = -\frac{1}{2\pi i} \sum_{k=2}^3 q_k \int_{p_k}^\infty dx e^{q_k x t} [\langle \sigma_z(q_k x + \eta_k) \rangle - \langle \sigma_z(q_k x - \eta_k) \rangle]. \quad (44)$$

A. Unbiased case

For the unbiased spin-boson case, $\epsilon=0$, an essentially analytic calculation can be done for all contributions; we find that these agree, as expected, with the weak-coupling limit of the calculations presented in Ref. 7. [For a discussion of nonexponential effects for large α (near $\alpha = \frac{1}{2}$), see Ref. 25.] In this case, there is no bc2 contribution, $z_{bc2}(t) = 0$ for all t . The bc1 contribution can be obtained analytically to leading order in α : $z(t) = z_{poles}(t) + z_{bc1}(t)$,

$$z_{bc1}(t) = -\alpha \{1 - \Delta t [\text{Ci}(\Delta t) \sin(\Delta t) - \text{si}(\Delta t) \cos(\Delta t)]\}. \quad (45)$$

This function, plotted along with the pole contribution in Fig. 1(b) for the choice of parameters shown, has the following features: $z_{bc1}(t)$ is negative for all t , it is monotonically increasing, and its long-time behavior is $z_{bc1}(t) \sim -2\alpha/(\Delta t)^2$. Also, $z_{bc1}(t=0) = -\alpha$.

Let us survey, then, the peculiar features that this branch cut contribution introduces into the time response $z(t)$. Visualizing the $\epsilon=0$ spin-boson model as a symmetric double-well system coupled to its environment, the bc1 piece being negative means that, if the system is initially in the left well, it will, in the course of coherently tunneling back and forth, spend more time in the *right* well. This effect becomes strongest at long time, much longer than T_2 , for in this regime the pole contributions are exponentially small, while the bc1 contribution decays like a power law. Experimentally it may be hard to see the effect in this regime (on account of finite-temperature effects, for example), so it is important to note that this memory effect appears already at early times, indicating that already in the first couple of coherent oscillations, there will be an excess amplitude in the right-well excursions as compared with the left-well excursions, by an amount proportional to α . We judge, on the basis of a variety of evidence,²⁶ that the Born approximation should be reliable up to α 's of order 1–2%; thus, experiments that look at coherent oscillations accurately at the percent level (which, it seems, will ultimately be necessary for performing quantum computation) could readily see this bc1 effect.

We note several other interesting features of our solution for $\epsilon=0$. Taking into account the non-Markovian effects, we can do a more precise calculation of the pole positions and residues (only poles 3 and 4 contribute). We find, for $T=0$, $\Gamma_2 \equiv -\text{Re}(s_3) = \Gamma_2^0 r$, where, as before, $\Gamma_2^0 = \alpha\pi\Delta/2$, and the renormalization factor r is given by $r = (1 - \alpha)/(\kappa^2 + \alpha^2\pi^2) < 1$, with $\kappa = 1 - 2\alpha[1/2 + C - \ln(\omega_c/\Delta)]$. Further, $\text{Im}(s_3) = E + \delta E^{\text{Lamb}} \tilde{r}$, with $\tilde{r} = \{\kappa - \alpha\pi^2[C - \ln(\omega_c/\Delta)]/2\}/[\kappa^2 + (\alpha\pi)^2]$. These expressions are obtained as systematic expansions in the small parameters Γ_2/E and $\delta E/E$, and they match a direct numerical evaluation of the pole positions very well up to α 's of a few percent. For the corresponding pole residues, we find the simple result in leading order $r_3 + r_4 = 1 + \alpha + O(\alpha^2)$. This would be impossible in a Markovian theory, in which $z(t=0) = r_3 + r_4$, so that $r_3 + r_4$ would be exactly 1 to all orders in α . In fact, this excess pole residue is exactly what is needed to cancel out the initial value of the bc1 contribution to $z(t)$. We note that our results for the residues differ from the weak-coupling expressions in the literature¹⁰ (we are not aware of prior reports on the renormalization factors r and \tilde{r}).

B. Biased case

For the biased model ($\epsilon \neq 0$), the bc2 contributions become nonzero; we find that they give other peculiar nonexponential corrections to the solution $z(t)$, very different from the bc1 contribution. The previous “NIBA” calculations of Ref. 7 are inapplicable in this case, giving our results here

particular value. We can do a nearly analytic evaluation of the bc2 contribution to Eq. (42): Using Eq. (27) and expanding to lowest order in α , we find for the integrand of the sum of the $k=2$ and 3 terms of Eq. (42),

$$z_{bc2}(s=i\omega) \approx \frac{2\Delta^2}{\omega} \frac{b^-(\omega)}{[E^2 - \omega^2 + b^+(\omega)]^2 + b^-(\omega)^2}. \quad (46)$$

Here $b(i\omega \pm \eta) \equiv b^+(\omega) \pm ib^-(\omega)$, $b(s) \equiv \alpha[d(s) + n(s)(s^2 + E^2)/\Delta]$, where $d(s)$ and $n(s)$ are given by $N(s) = \Delta + \alpha n(s)$ [see Eq. (28)] and $D(s) = s^2 + E^2 + \alpha d(s)$ [see Eq. (29)]. Since $b^-(\omega) = 0$ for $|\omega| \leq E$, it is reasonable to expect that b^- will grow linearly as one passes onto the branch cut; and, in fact, we find from numerical study that a good ansatz is $b^-(\omega) = (E - \omega)\tilde{b}^-(\omega)$, with $\tilde{b}^-(\omega)$ being a weakly varying, real function of ω/E . With this, for ω of order E , Eq. (46) simplifies to

$$z_{bc2}(s=i\omega) \approx -\frac{\Delta^2 \tilde{b}^-(E)}{2E^3} \frac{1}{\omega - E}. \quad (47)$$

We find that Eq. (47) should be valid for $\omega > E + b^+(E)/2E$. Using Eq. (47), we can do the branch cut integral, which gives [for $t \leq 1/(\alpha E x_0)$ —see the Appendix for an alternative approach]

$$z_{bc2}(t) \approx \alpha x_1 \log(x_0 \alpha E t) \cos(Et + \phi). \quad (48)$$

Here ϕ is a constant phase shift that we have not determined explicitly (but see the Appendix), and the dimensionless constants x_0 and x_1 are

$$x_0 = |b^+(E)|/2\alpha E^2 = |\delta E|/\alpha E, \quad (49)$$

$$x_1 = \Delta^2 \tilde{b}^-(E)/2\alpha E^3. \quad (50)$$

Since $b^\pm \propto \alpha$, these constants are independent of α . The last expression for x_0 comes from an evaluation of $b^+(E)$: it is directly related to the energy renormalization in the Markov approximation, $b^+(E) = 2E\delta E^{\text{Lamb}}$.

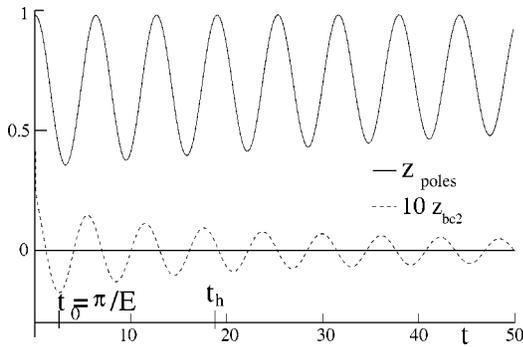


FIG. 2. $z_{poles}(t)$ and $z_{bc2}(t)$ for the biased case, illustrating the prompt loss of coherence produced by bc2. Here $E=1$, $\epsilon/\Delta = -1.38$, $\omega_c=30$, $T=0$, and $\alpha=0.01$. For these parameters, the time scale for the prompt loss of coherence [using Eq. (51)] is $t_h = 18.98$. t_h is the time at which the envelope of z_{bc2} falls to half its value at $t_0 = \pi/E$. This time scale is much shorter than the regular exponential decay of coherence in z_{poles} ; for our parameters, $T_2 = 204.6$.

In Fig. 2, we show a direct numerical evaluation of $z_{bc2}(t)$. One can see the decay of the oscillatory part, which is logarithmic according to Eq. (48). Even though the decay is very nonexponential, it is reasonable to attempt to characterize this decay by a time scale. Equation (48) obviously does not work at $t=0$, since it is logarithmically divergent. This is not surprising, since our calculation has neglected cutoff effects (dependence on ω_c), so Eq. (48) is not expected to be correct for $t < 1/\omega_c$. However, if we consider “early” time to be the first half-period of the coherent oscillation, $t_0 = \pi/E$, then Eq. (48) should be valid and we can use it to characterize the decay by determining the time t_h at which $z_{bc2}(t)$ decreases to half its early-time value, i.e., $z_{bc2}(t_h) = \frac{1}{2} z_{bc2}(t_0)$. We obtain

$$t_h = \frac{1}{E} \sqrt{\frac{\pi E}{|\delta E|}} \propto \frac{1}{E} \frac{1}{\sqrt{\alpha}}. \quad (51)$$

Surprisingly, $t_h \propto 1/\sqrt{\alpha}$ depends nonanalytically on α . This explains the effect that is evident in Fig. 2: for small α , $t_h \ll T_2$, that is, on the scale of T_2 , there is a very rapid loss of coherence as contributed by bc2. This phenomenon may be called a *prompt loss of coherence*, as it would appear experimentally as a fast initial loss of coherence [from 100% to $(1-c\alpha)100\%$, c being some constant near unity], followed by a much slower, exponential decay of coherence on the regular T_2 time scale.

We make a few final remarks about the bc2 calculation. The absolute size of the bc2 contribution reaches a maximum near the value of ϵ/Δ used in Fig. 2; the relative size of this contribution continues to increase as $|\epsilon|/\Delta$ increases, so that it eventually becomes much larger than the pole contribution [but all contributions to $z(t)$ go to zero as $|\epsilon|/\Delta \rightarrow \infty$]. When $|\epsilon| \approx \Delta$, we find that, because of the prompt loss of coherence, there is a *deficit* in the total pole contribution, that is, $\sum_k r_k = 1 - O(\alpha) < 1$. Even in the absence of an explicit branch-cut computation, this deficit signals the prompt loss of coherence, in that it indicates that the exponentially decaying contributions to $z(t)$ do not account for all the coherence near $t=0$. Note that this is opposite to the unbiased case, where, as a result of the bc1 part, there is an *excess* pole contribution.

V. NEXT-ORDER CALCULATION: STEADY-STATE SOLUTION

Finally, we present the result of a calculation of the corrections to order α to the steady-state (long-time) solution of the GME. To this order, as we will now show, the spin does not go to the Gibbs distribution of the uncoupled system (i.e., at $T=0$, the ground-state density matrix of the isolated spin). However, the result is consistent with the Gibbs distribution of the *coupled* system, giving good evidence for a strong form of ergodicity, even at $T=0$.

These apparently simple corrections, reported below, require an enormous additional calculation, in that they require an evaluation of the next order of the Born series. That is, we must take the expansion of the self-energy superoperator Σ to fourth order in L_{SB} . The formal expression for $\Sigma^{(4)}$ is simple enough to generate: it is well known that the full Born

series is generated by repeated substitution of the following propagator identity into the exact expression for Σ (Eq. 3.4.7a of Ref. 16):

$$e^{-iQLt} = e^{-iQL_0t} - i \int_0^t dt_1 e^{-iQL(t-t_1)} QL_{SB} e^{-iQL_0t_1}. \quad (52)$$

Here

$$L_0 = L_S + L_B. \quad (53)$$

This generates the superoperator expression for $\Sigma^{(4)}$,

$$\Sigma^{(4)}(t) = (-i)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 P \text{Tr}_B L_{SB} e^{-iQL_0(t-t_1)} QL_{SB} e^{-iQL_0(t_1-t_2)} QL_{SB} e^{-iQL_0t_2} L_{SB} \rho_B. \quad (54)$$

This expression can be simplified with the use of the operator identities

$$QL_0 = L_0, \quad PL_{SB}Q = PL_{SB}. \quad (55)$$

Only one factor of the projection superoperator $Q = 1 - P$ survives,

$$\Sigma^{(4)}(t) = (-i)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \text{Tr}_B L_{SB} e^{-iL_0(t-t_1)} L_{SB} e^{-iL_0(t_1-t_2)} QL_{SB} e^{-iL_0t_2} L_{SB} \rho_B. \quad (56)$$

Note also that the projector P has been dropped from the expression; since it is immediately followed by a trace over the bath, it acts as the identity. We can also write $\Sigma^{(4)}$ in several equivalent convenient forms using the identity

$$e^{-iL_0t} L_{SB} = L_{V(-t)} e^{-iL_0t}, \quad V(t) = e^{iL_0t} H_{SB}. \quad (57)$$

This gives the following two equivalent forms for the self-energy:

$$\Sigma^{(4)}(t) = (-i)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 \text{Tr}_B L_{V(0)} L_{V(t_1-t)} QL_{V(t_2-t)} L_{V(-t)} \rho_B e^{-iL_S t}, \quad (58)$$

$$\Sigma^{(4)}(t) = (-i)^3 \int_0^t dt_1 \int_0^{t_1} dt_2 e^{-iL_S t} \text{Tr}_B L_{V(t)} L_{V(t_1)} QL_{V(t_2)} L_{V(0)} \rho_B. \quad (59)$$

Equation (58) can be used to evaluate corrections to the last term of Eq. (10); we must add to Eq. (12) a term of the form

$$I_{\mu\nu}^{(4)}(t) = \frac{i}{2} \text{Tr}_S \sigma_\mu \Sigma^{(4)}(t) \sigma_\nu. \quad (60)$$

The bath part of these traces requires the fourth-order bath correlator, which using Wick's theorem is, at $T=0$,

$$\begin{aligned} \text{Tr}_B [X(t_1)X(t_2)X(t_3)X(t_4)\rho_B] &= \frac{\alpha^2 \omega_c^4}{4} \left[\frac{1}{[i + \omega_c(t_3 - t_2)]^2 [i + \omega_c(t_4 - t_1)]^2} + \frac{1}{[i + \omega_c(t_3 - t_1)]^2 [i + \omega_c(t_4 - t_2)]^2} \right. \\ &\quad \left. + \frac{1}{[i + \omega_c(t_2 - t_1)]^2 [i + \omega_c(t_4 - t_3)]^2} \right]. \end{aligned} \quad (61)$$

Equations (58)–(61) are the starting point of our next-order calculation of the $s=0$ residue, which gives the long-time asymptote of the density matrix. Every detail of this calculation is presented in the accompanying MATHEMATICA notebook. It can be understood why computer algebra is necessary for the completion of this calculation if one considers the complexity of the above expressions when written out in

ordinary operator form. The four nested commutators generated by the Liouvillian superoperator produces thousands of distinct terms, which all need to be integrated and studied in the limit of $\omega_c/E \rightarrow \infty$.

To illustrate the complexity of this calculation, we give in Appendix B one example of a relatively “simple” intermediate result [the integral form for $I_{xx}^{(4)}(t)$] that is obtained in the

MATHEMATICA notebook (see EPAPS supplement, Ref. 27).

Given the enormity of the calculation, the final result is very simple,

$$x_\infty = -\frac{\Delta}{E} + \alpha \left[-\frac{\Delta^3}{E^3} + \left(C - \ln \frac{\omega_c}{E} \right) \left(\frac{\Delta^3}{E^3} - \frac{2\Delta}{E} \right) \right], \quad (62)$$

$$z_\infty = -\frac{\epsilon}{E} + \alpha \frac{\epsilon \Delta^2}{E^3} \left(C - 1 - \ln \frac{\omega_c}{E} \right). \quad (63)$$

Recall that $y_\infty=0$ exactly in the spin-boson model. In this expression, all terms that vanish in the limit of $\omega_c/E \rightarrow \infty$ have been dropped. Note that as in the δE calculation above, we see a mild (logarithmic) divergence with the ultraviolet cutoff; all physical quantities that we have calculated at this order have no divergence more severe than this. These results differ with the $O(\alpha)$ limit results reported in Sec. 21.5.2 of Ref. 10; we can offer no explanation for this. There is no obvious way of treating the logarithmic divergences in x_∞ and z_∞ by introduction of a renormalized Δ and ϵ , except in the unbiased case. Nevertheless, the expressions given are perfectly physical ($x_\infty^2 + z_\infty^2 < 1$) within the expected limits [$\omega_c \gg E$, and $\alpha < 1/\ln(\omega_c/E)$].

After obtaining the above results, we separately calculated the equilibrium density matrix, i.e.,

$$\langle \sigma_\mu \rangle_{eq} = \frac{\text{Tr} \sigma_\mu e^{-\beta H}}{Z} = -\frac{2}{\beta} \frac{\partial}{\partial c_\mu} \ln Z, \quad (64)$$

in the limit $T \rightarrow 0$ and for large ω_c . Here $Z = \text{Tr} e^{-\beta H}$, $c_x = \Delta$, and $c_z = \epsilon$. We find that

$$\lim_{\beta \rightarrow \infty} \frac{1}{\beta} \ln Z = \frac{1}{2} (E + \delta E^{\text{Lamb}} + \alpha \omega_c), \quad (65)$$

with δE^{Lamb} from Eq. (36). Then it is a simple calculation to show that the equilibrium and steady-state solutions actually coincide, i.e.,

$$x_\infty = \langle \sigma_x \rangle_{eq}, \quad z_\infty = \langle \sigma_z \rangle_{eq}. \quad (66)$$

While this result is natural, it should not be considered obvious; it provides a rigorous demonstration that, to order α , the system is ergodic in a strong sense.

We give a few final notes about other quantities that require a calculation of Σ to the $\Sigma^{(4)}$ level. The $O(\alpha)$ corrections to pole positions, given in an earlier section, are unaffected by inclusion of $\Sigma^{(4)}$; however, $O(\alpha)$ corrections to residues of both pole 1 ($s=0$) and pole 2 for σ_x and σ_z are affected by $\Sigma^{(4)}$. $O(\alpha)$ corrections to σ_y residues, and $\sigma_{x,z}$ residues of poles 2 and poles 3 and 4, are determined entirely by $\Sigma^{(2)}$; they do *not* have contributions from $\Sigma^{(4)}$.

VI. DISCUSSION

Naturally, many more regimes could be studied using the present approach. For finite temperature, the time evolution is very different at long times, but it is essentially the same as the $T=0$ evolution when $t < \hbar/kT$. Recently, there has been interest in varying both the system¹³ and bath²⁸ initial conditions, as well as in varying the model of the bath den-

sity of states.²⁸ For all these circumstances, the systematic Born expansion procedure we report here can be done. It is clear on general grounds that the appearance of branch cut contributions will not be restricted to the Ohmic model, however the Ohmic case is special in that the size branch cut contribution is not governed by any small parameter. For any super-Ohmic spectral density of the form $J(\omega) \propto \omega^n$ at low frequencies ($n=1, 2, \dots$), $C(t)$ will have a power-law dependence at long time, and thus $C(s)$ will have a branch point at $s=0$. However, the magnitude of the branch cut contribution in the general case goes like $1/\omega_c^{n-1}$. So, nonexponential contributions to the dynamics vanish in the physical limit in all these other cases.

Our hope is that, using the present and further exact calculations of the weak-coupling behavior of the spin-boson model, a tool will be made available to permit precision experiments to test the validity of the model (which, at present, is only phenomenologically justified) in various physical situations of present interest in quantum information. A fundamentally correct, experimentally verified theory of the system and its environment should ultimately be of great value in finding a satisfactory qubit for the construction of a quantum information processor.

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APPENDIX A: SCALING FORM FOR BRANCH CUT INTEGRALS

By numerical study, we find that the branch cut integrals conform to some simple scaling laws for small α . If we write the bc1 and bc2 integrals as $z_{bc1}(t) = \int_0^\infty dx e^{-q_1 x t} z_{bc1}(s=q_1 x)$ and $z_{bc2}(t) = \text{Re} \int_E^\infty dx e^{ixt} z_{bc2}(s=ix)$, then we find that for small α and for $s \ll \omega_c$, $z_{bc1}(s)$ can be written in a scaling form

$$z_{bc1}(x) = (\alpha/E) f_1(\epsilon/\Delta, x/E). \quad (A1)$$

But for bc2, a very different scaling law applies,

$$z_{bc2}(x) = (1/E) f_2(\epsilon/\Delta, (x-E)/\alpha E). \quad (A2)$$

Here $f_{1,2}$ are dimensionless, ‘‘universal’’ functions that govern the behavior of the branch cut contributions for small α . For bc1, the behavior that the scaling law gives is very simple: Eq. (A1) implies that $z_{bc1}(t) = \alpha \bar{f}_1(\epsilon/\Delta, Et)$, where \bar{f}_1 is the Laplace transform of the scaling function f_1 . We might have expected this behavior from Eq. (45), from which we

can read off the scaling function for $\epsilon=0$. In fact, it appears from numerical studies that f_1 hardly changes as ϵ is varied, except for an overall scale factor; that is, $\bar{f}_1(\epsilon/\Delta, Et) \approx a(\epsilon/\Delta)b(Et)$. We find that the scaling function $a(\tau) > 0$ is peaked at $\tau=0$. So, the memory effect described above for $\epsilon=0$ persists for finite ϵ , but becomes smaller. For $|\epsilon| \approx \Delta$, the bc2 contribution, which we will describe now, becomes dominant over the bc1 one.

Returning to Eq. (A2), if we write the Fourier transform of the scaling function as $\bar{f}_2(\tau) = \int_0^\infty e^{i\tau x} f_2(x) dx$ and consider its polar form $\bar{f}_2(\tau) = r_2(\tau) e^{i\phi_2(\tau)}$, then we obtain

$$z_{bc2}(t) = \alpha r_2(\alpha Et) \cos[Et + \phi_2(\alpha Et)]. \quad (\text{A3})$$

This shows that bc2 contributes an oscillatory part to the solution, whose “ T_2 ” decay is determined by the features of the scaling function r_2 . A few more observations about f_2 (obtained initially from numerical study) reveal some crucial

properties of the r_2 function: (i) $f_2(0)=0$; (ii) $|f_2(x)|$ has a single maximum at $x=x_0$, where x_0 is some constant of order unity; (iii) most important for the present discussion, for $x > x_0$, $f_2(x)$ approaches $1/x$, that is, $f_2(x) \sim x_1/x$, where x_1 is another real constant of order unity. Fact (iii) implies that, for $\tau \rightarrow 0$, $r_2(\tau) \approx x_1 \log(x_0 \tau)$. That is, we conclude that at sufficiently short time [actually for $t \leq 1/(\alpha Ex_0)$, so a relatively long time],

$$z_{bc2}(t) = \alpha x_1 \log(x_0 \alpha Et) \cos(Et + \phi), \quad (\text{A4})$$

as stated in the text.

APPENDIX B: $I_{xx}^{(4)}(t)$

As an example of one of many, many intermediate results worked through in the accompanying MATHEMATICA notebook, we give here the expression for $I_{xx}^{(4)}(t)$ [Eq. (60)], in “simplified” form,

$$I_{xx}^{(4)}(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \left[\begin{aligned} & - \frac{\epsilon^2 \cos(Et)}{E^2 \left(-t_1 - \frac{i}{\omega_c}\right)^2 \left(t - t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\epsilon^2 \cos(Et)}{E^2 \left(t_1 - \frac{i}{\omega_c}\right)^2 \left(t - t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\epsilon^2 \cos(Et)}{E^2 \left(-t - \frac{i}{\omega_c}\right)^2 \left(t_1 - t_2 - \frac{i}{\omega_c}\right)^2} \\ & - \frac{\epsilon^2 \cos(Et)}{E^2 \left(t - \frac{i}{\omega_c}\right)^2 \left(t_1 - t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\epsilon^2 \cos(Et)}{E^2 \left(-t_1 - \frac{i}{\omega_c}\right)^2 \left(-t + t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\epsilon^2 \cos(Et)}{E^2 \left(t_1 - \frac{i}{\omega_c}\right)^2 \left(-t + t_2 - \frac{i}{\omega_c}\right)^2} \\ & - \frac{\epsilon^2 \cos(Et)}{E^2 \left(-t - \frac{i}{\omega_c}\right)^2 \left(-t_1 + t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\epsilon^2 \cos(Et)}{E^2 \left(t - \frac{i}{\omega_c}\right)^2 \left(-t_1 + t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 - Et_2)}{2E^2 \left(-t_1 - \frac{i}{\omega_c}\right)^2 \left(t - t_2 - \frac{i}{\omega_c}\right)^2} \\ & - \frac{\Delta^2 \cos(Et - Et_1 - Et_2)}{2E^2 \left(t_1 - \frac{i}{\omega_c}\right)^2 \left(t - t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 - Et_2)}{2E^2 \left(-t - \frac{i}{\omega_c}\right)^2 \left(t_1 - t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 - Et_2)}{2E^2 \left(t - \frac{i}{\omega_c}\right)^2 \left(t_1 - t_2 - \frac{i}{\omega_c}\right)^2} \\ & - \frac{\Delta^2 \cos(Et - Et_1 - Et_2)}{2E^2 \left(-t_1 - \frac{i}{\omega_c}\right)^2 \left(-t + t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 - Et_2)}{2E^2 \left(t_1 - \frac{i}{\omega_c}\right)^2 \left(-t + t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 - Et_2)}{2E^2 \left(-t - \frac{i}{\omega_c}\right)^2 \left(-t_1 + t_2 - \frac{i}{\omega_c}\right)^2} \\ & - \frac{\Delta^2 \cos(Et - Et_1 - Et_2)}{2E^2 \left(t - \frac{i}{\omega_c}\right)^2 \left(-t_1 + t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 + Et_2)}{2E^2 \left(-t_1 - \frac{i}{\omega_c}\right)^2 \left(t - t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 + Et_2)}{2E^2 \left(t_1 - \frac{i}{\omega_c}\right)^2 \left(t - t_2 - \frac{i}{\omega_c}\right)^2} \\ & - \frac{\Delta^2 \cos(Et - Et_1 + Et_2)}{2E^2 \left(-t - \frac{i}{\omega_c}\right)^2 \left(t_1 - t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 + Et_2)}{2E^2 \left(t - \frac{i}{\omega_c}\right)^2 \left(t_1 - t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 + Et_2)}{2E^2 \left(-t_1 - \frac{i}{\omega_c}\right)^2 \left(-t + t_2 - \frac{i}{\omega_c}\right)^2} \\ & - \frac{\Delta^2 \cos(Et - Et_1 + Et_2)}{2E^2 \left(t_1 - \frac{i}{\omega_c}\right)^2 \left(-t + t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 + Et_2)}{2E^2 \left(-t - \frac{i}{\omega_c}\right)^2 \left(-t_1 + t_2 - \frac{i}{\omega_c}\right)^2} - \frac{\Delta^2 \cos(Et - Et_1 + Et_2)}{2E^2 \left(t - \frac{i}{\omega_c}\right)^2 \left(-t_1 + t_2 - \frac{i}{\omega_c}\right)^2} \end{aligned} \right]. \quad (\text{B1})$$

This double integral, and many others, are fully evaluated in the MATHEMATICA notebook, in the large ω_c limit.

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- ²⁶For example, the scaling relations discussed in the Appendix work very well up to $\alpha \sim 0.01$.
- ²⁷See EPAPS document No. E-PRBMDO-70-140447 for the results obtained in the MATHEMATICA notebook. A direct link to this document may be found in the online article's HTML reference section. The document may also be reached via the EPAPS homepage (<http://www.aip.org/pubservs/epaps.html>) or from [ftp.aip.org](ftp://ftp.aip.org) in the directory /epaps/. See the EPAPS homepage for more information.
- ²⁸A. Shnirman, Yu. Makhlin, and G. Schön, *Phys. Scr., T* **102**, 147 (2002).