Holonomic quantum computing based on the Stark effect

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We propose a spin manipulation technique based entirely on electric fields applied to acceptor states in *p*-type semiconductors with spin-orbit coupling. While interesting on its own, the technique can also be used to implement fault-resilient holonomic quantum computing. We explicitly compute adiabatic transformation matrix (holonomy) of the degenerate states and comment on the feasibility of the scheme as an experimental technique.

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The physical realization of quantum computing rests on the ability to reversibly manipulate two level systems called qubits. While the promise of high computational power is certainly a tantalizing one, the intrinsic challenges associated with decoherence, adiabatic evolution, control, and noise errors in quantum gate operations are still to be mastered.

One ingenious way to overcome quantum noise errors is the use of non-Abelian holonomic (geometric) quantum computation schemes.[1] Through the slow tuning of some external parameters such as applied magnetic or electric fields, the qubit evolves adiabatically (with constant energy) around a path that changes its eigenstate from an initial to a final state. Generically, this quantum evolution is free of dynamical factors and is geometric in nature, depending only on the path in parameter space. Geometric holonomy could constitute a fault-tolerant way to perform quantum computation.[2,3] However, it is fair to say that its actual robustness against different kinds of noise is still under investigation, being the subject of a debate.[5] Although some experimental systems that would exhibit such behavior have been proposed,[4,6] holonomic quantum computing overall still lacks the variety of concrete application proposals that conventional quantum computation enjoys.

In this paper we propose using electric fields to manipulate the spin of acceptor states in semiconductors with spin-orbit coupling such as Ge, GaAs, and Si. The acceptor impurity ion will bind a *p* hole from the spin $3/2$ valence band of the semiconductor[7] and the full Hamiltonian of the impurity system in an electric field is given by the linear or quadratic Stark effect.[8] Spin-orbit coupling is essential to the existence of the Stark effect in these semiconductors. There are two doubly degenerate Kramer states for any value of the electric field and slowly rotating the electric field induces SU(2) rotations in the degenerate eigenstates of each energy level. When the electric field swaps over a cycle and returns to its initial orientation the holonomy matrix is dependent on the geometry of the swap only. Consistent with prior theoretical analysis, we take these holonomies to represent quantum gate transformations.[9] Previous work involving holonomies in semiconductors is found in Ref. 10.

We begin by a short introduction of the main idea of holonomic quantum computing. We then introduce the Hamiltonian of the acceptor states in an electric field and show how any SU(2) holonomy can be obtained by changing the field's orientation adiabatically, hence providing the basis for a set of gate operations. We discuss the viability of such a scheme and the experimental challenges involved. We close by proposing an alternative scheme, using external uniaxial strain, which can be used to achieve spin manipulation in the absence of any external field. The holonomic qubits discussed here are based on a principle similar to the dissipationless spin current in hole doped semiconductors.[11] The individual qubits can therefore be coupled to each other by a quantum bus architecture based on the dissipationless spin current, offering exciting new possibilities towards the realization of an all solid-state holonomic quantum computer.

In holonomic quantum computing, quantum information is encoded in an *n*-fold degenerate Hilbert space of a Hamiltonian H_λ dependent on some external "control" parameters (fields) λ.[9] Upon a cyclical change of these parameters around a loop C during time T such that $\lambda_{\text{in}}=\lambda_{\text{out}}$, the system will evolve between the initial state (n vector) $|\psi\rangle_{\text{in}}$ into $|\psi\rangle_{\text{out}}=e^{i\epsilon_0 T}\Gamma(C)|\psi\rangle_{\text{in}}$, where ϵ_0 is the initial eigenvalue $H_{\lambda_{\text{in}}}|\psi\rangle_{\text{in}}=\epsilon_0|\psi\rangle_{\text{in}}$. The first factor is just the dynamical phase, and will be omitted, while the second factor is the non-Abelian Wilczeck-Zee[12] curvature connection (matrix)

$$\Gamma(C)=\mathbf{P}e^{\oint_C A^\mu d\lambda_\mu}, \quad A^\mu_{ij}=\left\langle \psi^j(\lambda)\left|\frac{\partial}{\partial\lambda_\mu}\right|\psi^j(\lambda)\right\rangle, \qquad (1)$$

where $i, j=1,\ldots,n$ and where $\mathbf{P}$ represents the path ordering due to the fact that the gauge connection A^μ is now a matrix acting on the degenerate space of Hamiltonian eigenstates (μ denotes the different control parameters). The degenerate Hilbert space of the Hamiltonian encodes the quantum information where the eigenstates are the codewords while the nontrivial holonomies associated with it represent the unitary transformations or "computations" over the code. Zanardi and Rasetti[1] showed that this prescription is sufficient to implement quantum computation on single qubit holonomic gates. In subsequent papers[4,6,9,13,14] several schemes for realizing holonomic computation have been proposed. The schemes involve geometric manipulation of trapped ions,[14] charge pumping within Josephson junction networks,[4] and Josephson charge qubits.[6] Controlled manipulation of U(1) holonomies (Berry phases) using nuclear magnetic resonance on a system of weakly coupled H^1 and C^{13} nuclei has been experimentally achieved with great accuracy by Jones *et al.*[15]

TABLE I. Values of the coefficients in the linear and quadratic Stark Hamiltonians (Ref. 8).

	α	β	δ	χ	$\bar{r}$(Å)
Ge	1	−0.3	−0.36	0.7×10^{-3}	91
Si	1	−0.2	−0.42	1×10^{-2}	34.4

It would be of great advantage to have a conventional solid-state system where holonomic computation can be implemented by using only electric fields. In this paper we are concerned only with single qubit holonomy, leaving multiqubit ones for a later publication.[16] We look at *p*-type cubic symmetry semiconductors such as Ge, GaAs, and Si. The strong spin-orbit coupling in these systems breaks up the valence bands into two doubly degenerate bands of spin 3/2 with helicity $\pm1/2$ and $\pm3/2$. The double degeneracy is nothing else than Kramers degeneracy and is guaranteed by T invariance. This is maintained even when acceptor impurities (B, Al, Ga, In) are introduced in the semiconductors. The holes that bind to these impurities will maintain a certain symmetry subgroup of the original cubic symmetry of the valence bands they came from. Let us now consider the effect of an applied external electric field E on the acceptor-bound hole state. For large electric fields, the field distortion near the impurity ion can be safely neglected and the acceptor-hole state has the cubic symmetry of the crystal $T_d\times I$, giving rise to a quadratic Stark effect[8]

$$H_{E^2}=-\frac{p_0^2}{\varepsilon_i}\left\{\alpha E^2 I+\beta\left[E_x^2S_x^2+E_y^2S_y^2+E_z^2S_z^2-\frac{5}{4}E^2I\right]+\frac{2}{\sqrt{3}}\delta(E_yE_z\{S_y,S_z\}+E_zE_x\{S_z,S_x\}+E_xE_y\{S_x,S_y\})\right\}, \tag{2}$$

where ε_i is the ionization energy, $p_0=e\bar{r}$ the dipole moment ($\bar{r}$ being the mean radius of the ground state), and $\vec{S}$ are the spin-3/2 matrices, describing the valence band states, which essentially have $P_{3/2}$ character. We have also defined $\{A,B\}=(AB+BA)/2$. Readers familiar with semiconductor theory will recognize in the form of H_{E^2} the Luttinger Hamiltonian structure, with the substitution $\vec{k}\to\vec{E}$. This is no coincidence since the symmetry group of both Hamiltonians is the same. For small applied electric field, we must take into consideration the local field of the ions, thereby the symmetry lowers from $T_d\times I$ to T_d and giving rise to a linear Stark effect[8]

$$H_E=\frac{2p\chi}{\sqrt{3}}(E_x\{S_y,S_z\}+E_y\{S_z,S_x\}+E_z\{S_x,S_y\}), \tag{3}$$

where $p=ea_B$ with a_B the Bohr radius. The constants $\alpha,\beta,\delta,\chi,\bar{r}$ are given in Table I, although the estimates for χ in the literature vary considerably ($\chi=0.26$ according to Kopf and Lassman[17] so the value in Table I should be taken as a lower limit). We want to mention that the donor and acceptor Hamiltonians and physics are essentially different, with the donors undergoing only a quadratic Stark shift as opposed to the acceptor combination of the above linear and quadratic shifts.

Although for some field E the acceptor Hamiltonian will be a weighted sum of linear and quadratic Stark effects, we prefer, without any loss of generality, to work in either of the two regimes and not in the intermediate one. Each of the Hamiltonians above has two doubly degenerate Hilbert spaces, roughly corresponding to values of the z component of the spin S_z being either $\pm1/2$ or $\pm3/2$ (this would be exactly true if the Hamiltonians were isotropic). The "control" parameters are the components of the electric field $\vec{E}$. We must now show we can achieve "quantum computations over the code." These are represented by SU(2) holonomies over each degenerate Hilbert space (equivalently, we must show we can move within an energy subspace by adiabatically changing $\vec{E}$).

Such holonomies do indeed exist in our system. We now give an explicit generic procedure to calculate them. While we could just brute-force diagonalize the Hamiltonians above and treat each of them separately, we prefer to use a more elegant approach that reveals more of the Hilbert space structure. This was developed in Ref. 18 in the context of the SO(5) theory of high-T_c superconductivity, and extended by Ref. 19 to the case of hole band in semi-conductors. Out of the spin-3/2 matrices S_x, S_y, S_x we can define the new 4×4 matrices $\Gamma^1=(2/\sqrt{3})\{S_y,S_z\}$, $\Gamma^2=(2/\sqrt{3})\{S_z,S_x\}$, $\Gamma^3=(2/\sqrt{3})\{S_y,S_x\}$, $\Gamma^4=(1/\sqrt{3})(S_x^2-S_y^2)$, $\Gamma^5=S_z^2-\frac{5}{4}I_{4\times4}$, which satisfy the SO(5) Clifford algebra $\Gamma^a\Gamma^b+\Gamma^b\Gamma^a=2\delta_{ab}I_{4\times4}$. Explicitly,

$$\Gamma^i=\begin{pmatrix}0 & i\sigma^i\\ -i\sigma^i & 0\end{pmatrix},\quad \Gamma^4=\begin{pmatrix}0 & I\\ I & 0\end{pmatrix},\quad \Gamma^5=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},$$

where σ^i, $i=1,2,3$ are the usual Pauli matrices and I is the identity matrix (2×2 in this case). The Hamiltonians H_E and H_{E^2} can now be cast into a new clean form

$$H_E=d_E^0I+d_E^a\Gamma^a,\quad H_{E^2}=d_{E^2}^0I+d_{E^2}^a\Gamma^a, \tag{4}$$

where $a=1,\ldots,5$, I is the 4×4 identity matrix and

$$d_E^0=d_E^4=d_E^5=0,\quad d_E^1=p\chi E_x,\quad d_E^2=p\chi E_y,\quad d_E^3=p\chi E_z,$$

$$d_{E^2}^0=-\frac{p_0^2}{\varepsilon_i}\alpha E^2,\quad d_{E^2}^1=-\frac{p_0^2}{\varepsilon_i}\delta E_zE_y,\quad d_{E^2}^2=-\frac{p_0^2}{\varepsilon_i}\delta E_zE_x,$$

$$d_{E^2}^3=-\frac{p_0^2}{\varepsilon_i}\delta E_xE_y,\quad d_{E^2}^4=-\frac{p_0^2}{\varepsilon_i}\frac{\sqrt{3}}{2}\beta(E_x^2-E_y^2),$$

$$d_{E^2}^5=-\frac{p_0^2}{\varepsilon_i}\frac{1}{2}\beta(2E_z^2-E_x^2-E_y^2). \tag{5}$$

We brought the linear and quadratric Stark Hamiltonians to the same symbolic form. We now manipulate them together and only substitute for the values of d^a at the end of the calculation. The deep physical reason as to why the two apparently different Hamiltonians are actually very similar is the unbroken T invariance of the system that leads to Kramers degeneracy. The eigenvalues are $\epsilon_\pm=d^0\pm d$, where d

$=\sqrt{d^a d^a}$ and they depend on the electric field. For the linear Stark effect explicit substitution shows us that the split is independent on the direction of the electric field $\vec{E}$ while for the quadratic case, it strongly depends on its orientation. The Clifford matrices Γ^a have two eigenvalues, each twofold degenerate (this is obvious from the form of Γ^5). The gauge connection then represents SU(2) adiabatic changes on the twofold degenerate subbands $\pm$ hence the total gauge group is $SU(2)_- \times SU(2)_+ = SO(4)$. Identical to Ref. 19, we define the projection operators into the two energy subspaces $H = \epsilon_+ P^+ + \epsilon_- P^-$:

$$P^+ = \frac{1}{2}\left(1 + \frac{d^a}{d}\Gamma^a\right), \quad P^- = \frac{1}{2}\left(1 - \frac{d^a}{d}\Gamma^a\right). \tag{6}$$

Adiabatic rotation of the field $\vec{E}$ implies moving within one of the subspaces of energy ε. We can define a covariant gauge field strength

$$A_a = i\left[\frac{\partial P^+}{\partial d^a}, P^+\right] = i\left[\frac{\partial P^-}{\partial d^a}, P^-\right] = -\frac{1}{2d^2} d^b \Gamma^{ab}, \tag{7}$$

where $\Gamma^{ab} = (1/2i)[\Gamma^a, \Gamma^b]$ are the generators of the SO(5) algebra. The field A_a lives in the space of the d^a's but our control parameters are the electric field components E_i. We do a "coordinate transformation" and obtain

$$A^i = \frac{\partial d^a}{\partial E_i} A_a = -\frac{1}{2d^2} d^b \frac{\partial d^a}{\partial E_i} \Gamma^{ab}. \tag{8}$$

which gives us a holonomy computation when the electric field is varied between $\vec{E}_{\text{initial}} = \vec{E}_{\text{final}}$ over an arbitrary (closed) curve C

$$\Gamma(C) = \mathbf{P} \exp\left(\oint A^i dE_i\right) = \mathbf{P} \exp\left(-\oint \frac{1}{2d^2} d^b \frac{\partial d^a}{\partial E_i} \Gamma^{ab} dE_i\right). \tag{9}$$

Let us, without any loss of generality, momentarily focus on the ϵ_+ subspace. By choosing specific rotations (specific contours C) of the field E_i we can change an initial state $|\psi\rangle_{\text{in}} = (1,0,0,0)$ into the degenerate state within the same energy level, i.e., $|\psi\rangle_{\text{out}} = \Gamma(C)|\psi\rangle_{\text{in}} = (0,1,0,0)$. In fact, in the general case, starting from an arbitrary $|\psi\rangle_{\text{in}}$ we can reach, through carefully choosing the contour C, any other eigenstate within the degenerate subspace by electric field manipulation. In a physical intuitive picture, the spin within the ϵ_+ subspace will follow the electric field as it tries to stay within the energy subspace. By choosing an appropriate pair of loops C_1 and C_2, two noncommuting holonomies $\Gamma_1 = \Gamma(C_1)$ and $\Gamma_2 = \Gamma(C_2)$ can be enacted. By combining them in all possible ways, the whole unitary group SU(2) can be obtained. We show this explicitly in the case of the quadratic Stark effect, the more experimentally viable of the two. In this way, every unitary transformation over the computational space can be realized in terms of holonomies. We have achieved spin manipulation with electric fields and showed that holonomic computation is possible in semiconductors with spin-orbit coupling.

In general, due to the non-Abelian nature of A_i, the path ordered integral has to be done numerically, over infinitesimal segments in parameter space and taking into account that different components of A do not commute with each other. While this is more of a nuisance than an intellectual challenge, it is comforting to know that for certain curves the expression can be simplified and path ordering can be easily implemented while still maintaining the full capability to transform the eigenstates into one another. We give such examples for both the linear and the quadratic Stark effect below.

For the linear Stark effect, again working in the ϵ_+ energy subspace the expression for the holonomy $\Gamma_E(C)$ becomes particularly simple:

$$\Gamma_E(C) = \mathbf{P} \exp\left(-\frac{1}{2}\oint \frac{1}{E^2} \epsilon_{ijk} \sigma_k E_j dE_i\right), \tag{10}$$

where the σ_k are the three Pauli matrices. In polar coordinates $\vec{E} = (E \sin\theta\cos\phi, E \sin\theta \sin\phi, E\cos\theta)$ for contours C which keep constant the absolute value of the electric field, we find that spherical triangles between the points A $(\theta = 0, \phi = \phi_1)$, B $(\theta = \pi/2, \phi = \phi_1)$, and C $(\theta = \pi/2, \phi = \phi_2)$ are particularly easy to path order. Since we are changing only one angle at a time achieving this technologically should be easier than trying to implement variations in both angles (although, as Zee points out, there is a bit of confusion on how to go "around the corners"[20]).

For the case of the quadratic Stark effect things are more complicated. While finding a nice form for the holonomy factors in the general case is almost impossible due to the anisotropy in the Hamiltonian H_{E^2}, we can look at the idealized spherical symmetric situation for which $\beta = \delta/\sqrt{3}$. This does not introduce large errors, as the anisotropy in these materials, although significant, is still small enough so that the spherical approximation works well. In this case we find, in units of $-p_0^2/\varepsilon_i$: $d^a_{E^2}\Gamma^a = \beta(\vec{E}\cdot\vec{S})^2 - 5/4\beta E^2 I_{4\times 4}$. The holonomy structure resides exclusively in the first term. In fact, with the electric field replaced by a magnetic field, this is exactly the Hamiltonian studied by Zee[20] in explaining a pioneering experiment by Tycko.[23] The gauge field in polar coordinates is $A_\phi = \cos\theta\sigma_3/2 - \sin\theta\sigma_1$ and $A_\theta = \sigma_2$. For spherical triangles starting at $\theta = 0$ going to some value θ on an arc of fixed ϕ_1 (which we can choose to be zero for convenience) then going at fixed θ on an arc to some nonzero ϕ and then back to the north pole along constant longitude, the holonomy reads $\Gamma_{E^2}(C) = W_1^{-1} V W$[20] where

$$W_1^{-1} = \exp[-i\theta(\cos\phi\sigma_2 - \sin\phi\sigma_1)], \quad W = \exp(i\sigma_2\theta),$$

$$V = \exp\left(-i\frac{\phi}{2}\sigma_3\right)\exp\left(i\frac{\phi}{2}(\cos\theta\sigma_3 - 2\sin\theta\sigma_1)\right). \tag{11}$$

Also, for the Zee connection of constant but different θ_i, $i = 1,2$ we have $\Gamma_i = \Gamma(\theta_i) = \exp[i2\pi(\cos\theta_i\sigma_3 - \sin\theta_i\sigma_1)]$. For generic $\theta_1 \neq \theta_2$ we have $[\Gamma_1, \Gamma_2] = i\{\exp[-i2\pi(\cos\theta_2 + \sin\theta_1)] + \exp[i2\pi(\cos\theta_1 - \sin\theta_2)] - \exp[i2\pi(\cos\theta_2 - \sin\theta_1)] - \exp[-i2\pi(\cos\theta_1 + \sin\theta_2)]\}\sigma_2$ thereby generating other unitary transformations.

We now turn to the problem of the feasibility of the

TABLE II. Ionization energies in meV for different acceptor impurities (B, Al, Ga) in Si and Ge.

	B	Al	Ga
Ge	10.4	10.2	10.8
Si	45.0	57.0	65.0

scheme proposed for spin manipulation by the Stark effect. We need that the coherence time of spins of bound holes be larger than the time in which we can adiabatically rotate the electric field. New experiments showed that the coherence time is larger than 1 ms,[21,22] justifying the use of acceptor-bound-hole wave functions as qubits. It is indeed difficult to perform experiments which probe non-Abelian phase factors. The original work of Tycko[23] and subsequently the more complete experiment by Zwanziger, Koenig, and Pines[24] on nuclear magnetic quadrupole resonance proved the existence of the Wilczeck and Zee non-Abelian transport of degenerate states. Instead of rotating the applied fields and keeping the sample fixed, these experiments kept the applied field fixed and rotated the sample, which is an equivalent procedure. The rotation frequencies were of the order of a few kHz (2020 Hz in Zwanziger *et al.*). Imagining an electric-field version of this experiment, the rotation period of the field is already less than the bound hole spin coherence time, but further improvement may be necessary for a realistic measurement. We also need to guarantee, during the field rotation, that the adiabatic approximation is accurately maintained and that the acceptor impurity is not ionized. The ionization energies for acceptor states are of the order $10-60$ meV (see Table II). The splitting between the two levels ϵ_+ and ϵ_- can be computed from our expressions for their energies (using the constants in Table I) and are of the order 10 meV for Ge in a field of 10^6 V/m. Hence the ionization and splitting energies are roughly the same size and much larger than the applied electric field frequecy of rotation. The frequencies required for varying the electric field are hence low enough as to cause neither ionization of the impurity-hole system nor a breakdown of adiabaticity. The variation of dynamical phases over the sample volume, which usually leads to extensive dephasing can be overcome by an electric field variant of the double-sweep spin-echo techniques which refocus inhomogeneities in the dynamical phase but double the effect of the geometric phase.[15]

In a future work,[16] we explore the idea of different qubits communicating via a bus architecture based on the dissipationless spin current.[11] The application of an electric field on *p*-doped cubic semiconductors with spin-orbit coupling generates a spin current with polarization and direction of flow perpendicular to the electric field. The spin conductivity is an invariant of the SU(2) symmetry group of the heavy and light hole bands in the semiconductor and has a nice topological formula.[19] It is identical, up to a coordinate transformation with the field strength A_a in the two bands. The spin current couples with the spins of the bound hole in our system and hence mediates the interqubit interaction. Due to the topological nature of the coupling, the two-qubit gate will be holonomic in nature, without dynamical factors. There is, however, one difficulty: as the electric field is swapped, both the spin of the acceptor and the spin of the spin current tend to follow the electric field. This does not give an independent control parameter, unless we build a quantum computer in the manner of Kane, where the single and two qubit gates are controlled by the separate application of independent electric fields. However, there is a much more elegant and practical way to solve the problem, involving the use of strain to fundamentally control the spin current polarization direction. Strain is a second order perturbation for the acceptor states,[7,8] being around ten times weaker than the electric field term, but it is a first order perturbation for the spin current. It can fundamentally modulate the structure and magnitude of the spin conductance. However, it does not spoil its topological nature, since the strain term still maintains Kramer's degeneracy and hence the SU (2) structure of light and heavy hole bands. Hence strain can be used as an independent control parameter for the interqubit interaction. A detailed exposition of these ideas and calculations are forthcoming.[16]

In conclusion, this paper presents a way to manipulate the spins of acceptor impurity-bound hole states in *p*-type semiconductors with strong spin-orbit coupling using electric fields. Depending on its magnitude, the electric field couples both linearly and quadratically to the spin of the acceptor state through the Stark effect, but although apparently opposite, the two effects still maintain the T invariance of the underlying semiconductor. The spin manipulation is completely geometric and realizes, in a practical solid-state system, the theoretical proposal for holonomic quantum computing. We have obtained an explicit and general form for the holonomy matrix which transforms adiabatically transports degenerate eigenstates. While our analysis is specific to spin $3/2$ it is trivially generalized for any spin, provided the Stark effect is present. We have also briefly analyzed the experimental feasibility of the scheme.

Note added. In the closing stages of this work, we noticed the independent, recent work by Yuri Serebrennikov which presents similar ideas to the ones exposed here.[25]

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