Macroscopic conductivity tensor of a three-dimensional composite with a one- or two-dimensional microstructure

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Exact linear relations are found among different elements of the macroscopic conductivity tensor of a three-dimensional, two-constituent composite medium with a columnar microstructure, without any further assumptions about the forms of the constituent conductivities: Those can be arbitrary nonscalar, nonsymmetric, and nonreal (i.e., complex valued) tensors. These relations enable all the elements of the macroscopic conductivity tensor of such a system to be obtained, from a knowledge of the macroscopic conductivity tensor components only in the plane perpendicular to the columnar axis. Exact linear relations are also found among different elements of the macroscopic resistivity tensor of such systems. Again, these relations enable all the elements of the macroscopic resistivity tensor of such a system to be obtained, from a knowledge of the macroscopic resistivity tensor components only in the plane perpendicular to the columnar axis. We also present simple exact expressions for all elements of the macroscopic conductivity tensor of a three-dimensional composite medium with a parallel slabs or laminar microstructure and an arbitrary number of constituents, again without making any assumptions about the forms of the constituent conductivities, which can be arbitrary nonscalar, nonsymmetric, and nonreal tensors. The latter results were obtained previously, but their great generality and extreme simplicity were not realized by most physicists.

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I. INTRODUCTION

In a composite medium, the relevant heterogeneity is always on a length scale that is much greater than any microscopic length, e.g., atomic size, mean free path, Fermi wavelength. Therefore the "microphysical properties" are those of a continuous medium, and are characterized by "local values" of material parameters such as electrical conductivity and electrical permittivity. On length scales that are even much greater than these heterogeneity length scales, the response of the medium can be characterized by "macroscopic" or "bulk effective" values of those same physical parameters. To be more precise, the macroscopic or bulk effective conductivity tensor $\hat{\sigma}_e$ is defined as providing the linear relation between the volume averaged electric field $\langle \mathbf{E} \rangle$ and current density $\langle \mathbf{J} \rangle$,

$$\langle \mathbf{J} \rangle = \hat{\sigma}_e \cdot \langle \mathbf{E} \rangle. \tag{1.1}$$

Since $\hat{\sigma}_e$ is independent of the precise boundary conditions imposed on the electrical potential, we can choose those conditions such that the average electric field is the unit vector $\mathbf{e}_\alpha$ along r_α. In that case the local potential, electric field, and current density are denoted by $\phi^{(\alpha)}(\mathbf{r})$, $\mathbf{E}^{(\alpha)}(\mathbf{r})$, and $\mathbf{J}^{(\alpha)}(\mathbf{r})$, and we can write

$$\sigma^{(e)}_{\alpha\beta} = \langle J^{(\beta)}_\alpha \rangle = \sum_{\gamma=x,y,z} \langle \sigma_{\alpha\gamma} E^{(\beta)}_\gamma \rangle, \tag{1.2}$$

where $\langle\ \rangle$ denotes averaging over the entire volume of the composite medium. This definition is valid for the general case where no restrictions are imposed on the form of the local conductivity tensor $\hat{\sigma}(\mathbf{r})$, i.e., it can be nonscalar and nonsymmetric, and its elements can have arbitrary complex values.

Perhaps the simplest possible microstructures for a composite are the "parallel cylinders" and the "parallel slabs" or "lowest-order laminate" microgeometries. In the case of these microgeometries there exist very simple exact expressions for the macroscopic conductivity and macroscopic resistivity, respectively, if all the constituents are characterized by scalar conductivities. The conductivity along the columnar axis x, which is parallel to all the interfaces of the parallel cylinder microstructure, is given by the arithmetic volume average of the constituent scalar conductivities (p_i is the volume fraction of the constituent characterized by conductivity σ_i)

$$\sigma_{xx}^{(e)} = \langle \sigma \rangle = \sum_i p_i \sigma_i. \tag{1.3}$$

This result also holds for parallel slab microstructure, with scalar conductivities in all constituents, if x is any axis parallel to those flat slabs. For the latter type of systems, there is also another exact result: The resistivity along the direction z perpendicular to the parallel slabs is given by the arithmetic or volume average of the constituent scalar resistivities (ρ_i denotes those resistivities)

$$\rho_{zz}^{(e)} = \langle \rho \rangle = \sum_i p_i \rho_i. \tag{1.4}$$

These exact results also provide upper bounds for the principal values of conductivity/resistivity of any composite made of the same constituents, when all the conductivities and resistivities are real and positive—see, e.g., Ref. 1.

The parallel slab microstructure has received considerable attention because it forms the basic building block for higher rank laminates, which are stacks of parallel slabs made of lower-rank laminates.[2,3] For this reason it was natural to try and generalize the exact results of Eqs. (1.3) and (1.4) to the case where the constituent conductivities/resistivities are not simple scalar quantities. Such generalizations were indeed found—see Refs. 3 and 4. However, the simplicity and generality of these results have not been emphasized: In Ref. 3 the derivation appears to assume that the constituent conductivity tensors are all symmetric. In Ref. 4 no such assumption is made, but it is difficult for a nonprofessional mathematician to appreciate the extreme simplicity of the results. Therefore, in Sec. II below, we present those results once again—they are in the form of simple explicit expressions for all the components of $\hat{\sigma}_e$, and are obtained without making any assumptions about the constituent conductivities. These expressions are always valid, even when the constituent conductivity tensors are nonsymmetric and have complex-valued components. We also find some exact and quite simple linear relations among the various components of the macroscopic resistivity tensor $\hat{\rho}_e = 1/\hat{\sigma}_e$.

The parallel cylinder microstructure has also received some attention recently, when it was found to exhibit surprising behavior in the presence of a magnetic field that is strong enough to make the Hall-to-Ohmic resistivity ratio greater than 1 in at least one of the constituents.[5–20] Here, too, some exact results were found earlier in special cases of columnar microstructures made of two constituents—see Refs. 3 and 14. In Sec. III below we generalize those results to two-constituent columnar composites where the constituent conductivity tensors are entirely general, i.e., nonsymmetric and complex. Our main results in that section are the following. (a) One can solve separately the two-dimensional conductivity problem in any planar section that is perpendicular to the columnar axis of such a composite, and use that solution to compute the planar components of the 3×3 conductivity tensor $\hat{\sigma}_e$. (b) In the case of a two-constituent columnar composite, all the other components of that tensor are simple linear functions of its planar components and of (all) the components of the two constituent-conductivity-tensors $\hat{\sigma}_1, \hat{\sigma}_2$. The forms of these functions are quite explicit, and are independent of any microstructural details, apart from the columnar aspect. All dependence on details of the microstructure comes in via the planar components of $\hat{\sigma}_e$.

II. ONE-DIMENSIONAL OR PARALLEL-SLAB MICROSTRUCTURES

A one-dimensional (1D) microstructure of a three-dimensional (3D) composite medium means a stack of parallel slabs, which we take to be perpendicular to the z axis. Each slab is characterized by a conductivity tensor $\hat{\sigma}_i$, which relates the local electric current density $\mathbf{J}(\mathbf{r})$ to the local electric field $\mathbf{E}(\mathbf{r})$ in the usual linear fashion for $\mathbf{r}$ inside one of these slabs

$$\mathbf{J}(\mathbf{r}) = \hat{\sigma}_i \cdot \mathbf{E}(\mathbf{r}) \text{ for } \mathbf{r} \in V_i.$$

This simple microstructure has often been considered for the case where all the $\hat{\sigma}_i$ are scalar tensors, or where z is a principal axis of all the $\hat{\sigma}_i$'s.[1] In that case, the macroscopic resistivity along the z axis, $\rho_{zz}^{(e)} = 1/\sigma_{zz}^{(e)}$, is the simple arithmetic average of the zz constituent resistivities $\rho_{zz}^{(i)} = 1/\sigma_{zz}^{(i)}$ along that same axis,

$$\rho_{zz}^{(e)} = \frac{1}{\sigma_{zz}^{(e)}} = \sum_i \frac{p_i}{\sigma_{zz}^{(i)}} = \sum_i p_i \rho_{zz}^{(i)} = \left\langle \frac{1}{\sigma_{zz}} \right\rangle. \tag{2.1}$$

In contrast with those discussions, which are very straightforward and lead to the simple result of Eq. (2.1), we make *no assumptions* regarding the nature of the constituent conductivity tensors $\hat{\sigma}_i$. Thus, not only is $\hat{\sigma}_i$ allowed to be nonscalar, it is also allowed to be nonsymmetric, as when an external magnetic field is applied to the system. Nevertheless, we will obtain exact expressions for all the components of the macroscopic conductivity tensor $\hat{\sigma}_e$ that are comparably simple to Eq. (2.1).

A. Explicit results for $\hat{\sigma}_e$

We first note that, because the system of slabs is invariant under arbitrary translations along x or y, then under boundary conditions that would result in a uniform electric field for a uniform value of the local conductivity tensor, the field and current density $\mathbf{E}(z)$ and $\mathbf{J}(z)$ will be independent of x and y. From $\nabla \times \mathbf{E} = \mathbf{0}$ it then follows that E_x and E_y are both uniform everywhere, while from $\nabla \cdot \mathbf{J} = 0$ it follows that J_z is also uniform everywhere. In a slab characterized by the conductivity tensor $\hat{\sigma}_i$, we can write

$$J_z = \sigma_{zx}^{(i)} E_x + \sigma_{zy}^{(i)} E_y + \sigma_{zz}^{(i)} (E_z)_i,$$

where $(E_z)_i$ denotes the z component of electric field in the $\hat{\sigma}_i$ material. Recalling that E_x, E_y, and J_z are uniform everywhere, we conclude that E_z, though nonuniform, has the same constant value in all the slabs of type i. The same will be true for J_x and J_y. In this case we can obtain exact explicit expressions for all components of the macroscopic conductivity tensor $\hat{\sigma}_e$, for a parallel slab microstructure with any number of different constituent conductivity tensors $\hat{\sigma}_i$.

In the most general case, when $\hat{\sigma}_e$ as well as the constituent tensors $\hat{\sigma}_i$ are nonsymmetric second-rank tensors, the macroscopic or bulk effective conductivity tensor is given by[1] (note that we adopt the unconventional convention that $\mathbf{E}=\boldsymbol{\nabla}\phi$)

$$\sigma_{\alpha\beta}^{(e)} \equiv \langle J_\alpha^{(\beta)}\rangle = \sum_\gamma \langle \sigma_{\alpha\gamma}E_\gamma^{(\beta)}\rangle = \sum_\gamma \left\langle \sigma_{\alpha\gamma}\frac{\partial\phi^{(\beta)}}{\partial r_\gamma}\right\rangle. \quad (2.2)$$

As already explained in the Introduction, the superscript (β) in $\phi^{(\beta)}$, etc., identifies quantities obtained under boundary conditions that lead to a uniform electric field that is a unit vector along r_β in the case of a uniform conductivity tensor. Under such boundary conditions, the volume average field in the actual composite structure also has that same value $\langle\mathbf{E}^{(\beta)}\rangle=\mathbf{e}_\beta$.

Consider first the component $\sigma_{zz}^{(e)}$, which is given by

$$\sigma_{zz}^{(e)} = \langle J_z^{(z)}\rangle \equiv \sum_{\alpha=x,y,z}\langle\sigma_{z\alpha}E_\alpha^{(z)}\rangle = \langle\sigma_{zz}E_z^{(z)}\rangle, \quad (2.3)$$

where the last result follows from the fact that $E_x^{(z)}\equiv E_y^{(z)}\equiv 0$ everywhere. Because $J_z^{(z)}$ is in fact uniform everywhere, we can omit the volume averaging in Eq. (2.3), writing instead

$$\sigma_{zz}^{(e)} = J_z^{(z)} = \sigma_{zz}^{(i)}(E_z^{(z)})_i, \quad (2.4)$$

where $(E_z^{(z)})_i$ is the uniform value of $E_z^{(z)}$ in the slabs of type i. Recalling that the average of $E_z^{(z)}$ is 1, since $\langle\mathbf{E}^{(z)}\rangle=\mathbf{e}_z$, we can write

$$\frac{1}{\sigma_{zz}^{(e)}} = \frac{\langle E_z^{(z)}\rangle}{\langle J_z^{(z)}\rangle} = \sum_i \frac{p_i}{\sigma_{zz}^{(i)}} = \left\langle\frac{1}{\sigma_{zz}}\right\rangle, \quad (2.5)$$

where p_i are the constituent volume fractions.

This result is reminiscent of the well-known result of Eq. (2.1) for the conductivity of a stack of parallel slabs of isotropic conductors, or anisotropic conductors for which z is a principal axis. However, we have obtained this result *without making any assumptions about the constituent conductivity tensors*. Indeed, in the general case where $\hat{\sigma}_i$ is nonscalar and even nonsymmetric, $1/\sigma_{zz}^{(i)}$ is not equal to the resistivity component $\rho_{zz}^{(i)}$. Likewise, $1/\sigma_{zz}^{(e)}$ is usually not equal to $\rho_{zz}^{(e)}$.

Now consider elements of $\hat{\sigma}_e$ of the form $\sigma_{za}^{(e)}$, $a=x,y$. These are given by

$$\sigma_{za}^{(e)} = \langle J_z^{(a)}\rangle = \sum_{\alpha=x,y,z}\langle\sigma_{z\alpha}E_\alpha^{(a)}\rangle = \langle\sigma_{za}\rangle + \langle\sigma_{zz}E_z^{(a)}\rangle,$$

since $E_b^{(a)}=\delta_{ab}$ for $a=x,y$ and $b=x,y$. Since $J_z^{(a)}$ is uniform everywhere, we can again omit the volume averages in these expressions,

$$\sigma_{za}^{(e)} = J_z^{(a)} = \sigma_{za}^{(i)} + \sigma_{zz}^{(i)}(E_z^{(a)})_i. \quad (2.6)$$

Finally, using Eq. (2.5), we can write

$$\frac{\sigma_{za}^{(e)}}{\sigma_{zz}^{(e)}} = \frac{J_z^{(a)}}{J_z^{(z)}} = \sum_i p_i\frac{\sigma_{za}^{(e)}}{\sigma_{zz}^{(i)}} = \sum_i p_i\left[\frac{\sigma_{za}^{(i)}}{\sigma_{zz}^{(i)}} + (E_z^{(a)})_i\right] = \left\langle\frac{\sigma_{za}}{\sigma_{zz}}\right\rangle, \quad (2.7)$$

where we used the fact that $\langle E_z^{(a)}\rangle=0$ for $a=x,y$ to get the final result.

Elements of the form $\sigma_{az}^{(e)}$, $a=x,y$ are calculated as follows:

$$\sigma_{az}^{(e)} = \langle J_a^{(z)}\rangle = \sum_{\alpha=x,y,z}\langle\sigma_{a\alpha}E_\alpha^{(z)}\rangle = \langle\sigma_{az}E_z^{(z)}\rangle,$$

because $E_x^{(z)}\equiv E_y^{(z)}\equiv 0$. Using Eq. (2.4) we then get

$$\frac{\sigma_{az}^{(e)}}{\sigma_{zz}^{(e)}} = \frac{\langle J_a^{(z)}\rangle}{J_z^{(z)}} = \sum_i p_i\frac{\sigma_{az}^{(i)}(E_z^{(z)})_i}{\sigma_{zz}^{(e)}} = \sum_i p_i\frac{\sigma_{az}^{(i)}}{\sigma_{zz}^{(i)}} = \left\langle\frac{\sigma_{az}}{\sigma_{zz}}\right\rangle. \quad (2.8)$$

Finally, elements of the form $\sigma_{ab}^{(e)}$, $a=x,y$ and $b=x,y$, are calculated as follows:

$$\sigma_{ab}^{(e)} = \langle J_a^{(b)}\rangle = \sum_{\alpha=x,y,z}\langle\sigma_{a\alpha}E_\alpha^{(b)}\rangle = \langle\sigma_{ab}\rangle + \langle\sigma_{az}E_z^{(b)}\rangle,$$

because $E_b^{(b)}=1$, and the only other nonvanishing component of $\mathbf{E}^{(b)}$ is $E_z^{(b)}$. Extracting $(E_z^{(b)})_i$ from Eq. (2.6), and using this value in the last equation, we can now write

$$\begin{aligned}\sigma_{ab}^{(e)} &= \langle\sigma_{ab}\rangle + \sum_i p_i\sigma_{az}^{(i)}\frac{\sigma_{zb}^{(e)}-\sigma_{zb}^{(i)}}{\sigma_{zz}^{(i)}} \\ &= \langle\sigma_{ab}\rangle + \left\langle\frac{\sigma_{az}}{\sigma_{zz}}\right\rangle\sigma_{zb}^{(e)} - \left\langle\frac{\sigma_{az}\sigma_{zb}}{\sigma_{zz}}\right\rangle. \end{aligned} \quad (2.9)$$

This can also be rewritten in the following alternative forms:

$$\sigma_{ab}^{(e)} - \frac{\sigma_{az}^{(e)}\sigma_{zb}^{(e)}}{\sigma_{zz}^{(e)}} = \left\langle\sigma_{ab} - \frac{\sigma_{az}\sigma_{zb}}{\sigma_{zz}}\right\rangle, \quad (2.10)$$

$$\langle\sigma_{ab}\rangle - \sigma_{ab}^{(e)} = \left\langle\frac{\sigma_{az}}{\sigma_{zz}}(\sigma_{zb}-\sigma_{zb}^{(e)})\right\rangle \quad (2.11)$$

$$= \left\langle\frac{(\sigma_{az}-\sigma_{az}^{(e)})(\sigma_{zb}-\sigma_{zb}^{(e)})}{\sigma_{zz}}\right\rangle. \quad (2.12)$$

Note that if any one of the two bracketed differences in the numerator of the last line were omitted, i.e., replaced by 1, then the right-hand side of Eq. (2.12) would be 0—this follows from Eqs. (2.5), (2.7), and (2.8).

The expressions obtained for $\hat{\sigma}_e$ in this section have been known at least since 1979.[4] They were also derived more recently by Milton in a fashion that is valid for the general case, where those tensors are nonsymmetric.[3] However, the latter aspect, which is important whenever a magnetic field is present, because antisymmetric parts then appear in the conductivity and resistivity tensors due to the Hall effect, has not been emphasized in those previous derivations: In Ref. 4 the results for $\hat{\sigma}_e$ are not presented in a form that is very transparent for most physicists, while the derivation in Ref. 3 seems at first sight to be limited to the symmetric case. Our derivation, though not essentially different from that of Ref. 3, emphasizes the general validity of the results for arbitrary constituent conductivity tensors, and presents the results for $\hat{\sigma}_e$ in a simple explicit form. It also emphasizes some of the basic physical consequences of the laminar nature of the microstructure—for example, the fact that all components of both **E** and **J** have constant values in each type of slab, irrespective of the properties of the different constituent conductivity tensors.

The fact that Eq. (2.7) turns into Eq. (2.8) when we replace all the conductivity tensors by their transposes is a special case of the following general theorem: If, in a heterogeneous medium, the local conductivity tensor $\hat{\sigma}(\mathbf{r})$ is replaced at every point by its transpose $\hat{\sigma}^t(\mathbf{r})$, then the macroscopic conductivity tensor $\hat{\sigma}_e$ also gets replaced by its transpose $\hat{\sigma}_e^t$:

$$\hat{\sigma}_e[\hat{\sigma}^t] = \hat{\sigma}_e^t[\hat{\sigma}]. \tag{2.13}$$

Taking the inverse of both sides of this equation we get a similar result for the macroscopic resistivity tensor $\hat{\rho}_e$ as functional of the transposed local resistivity tensor $\hat{\rho}^t(\mathbf{r})$:

$$\hat{\rho}_e[\hat{\rho}^t] = \hat{\rho}_e^t[\hat{\rho}]. \tag{2.14}$$

These theorems are not widely known, and we have not found them mentioned in any book or article. In fact, it is possible that Eqs. (2.13) and (2.14) constitute a previously unknown theorem that we have discovered. For this reason, we present a proof of Eq. (2.13) in the Appendix. We note, in passing, that this theorem can also be related to Onsager's theorem on the dependence of conductivity on an applied magnetic field **H**:[21]

$$\hat{\sigma}(-\mathbf{H}) = \hat{\sigma}^t(\mathbf{H}). \tag{2.15}$$

Since this relation holds for the local conductivity tensor $\hat{\sigma}(\mathbf{r})$, as well as for the macroscopic conductivity tensor $\hat{\sigma}_e$, Eq. (2.13) follows as an inevitable conclusion. Notwithstanding this remark, the proof, which is provided in the Appendix, shows that Eqs. (2.13) and (2.14) are independent of any symmetry considerations that characterize kinetic phenomena and microphysics, such as invariance under time reversal. Such symmetry considerations lie at the basis of Onsager's theorem.[21]

B. Exact relations among elements of $\hat{\rho}_e$

If, instead of focusing on the conductivity tensors $\hat{\sigma}_i$ and $\hat{\sigma}_e$, we focus on the resistivity tensors $\hat{\rho}_i$ and $\hat{\rho}_e$, we can also find some exact linear relations among the different components of $\hat{\rho}_e$. Although the existence of exact relations among the elements of $\hat{\rho}_e$ can be expected from the exact results of Sec. II A for $\hat{\sigma}_e$, it is not obvious that there would exist *linear relations* of this type. These relations are not as numerous as the ones we found for $\hat{\sigma}_e$; therefore they do not enable us, in general, to compute the components of $\hat{\rho}_e$. However, their simplicity should make them useful sometimes. We start by recalling that, because E_x and E_y are uniform everywhere and E_z is uniform in all slabs of the same type, J_x and J_y will also have uniform values in all slabs of the same type.

In this section we will denote by $\mathbf{J}^{(J\alpha)}$ and $\mathbf{E}^{(J\alpha)}$ the local current density and electric field that result when the boundary conditions are such that the volume-averaged current density is equal to the unit vector $\mathbf{e}_\alpha$. We can then write the following expression for an arbitrary component of the macroscopic resistivity tensor $\hat{\rho}_e$:

$$\rho_{\alpha\beta}^{(e)} = \langle E_\alpha^{(J\beta)}\rangle = \sum_{\gamma=x,y,z} \langle \rho_{\alpha\gamma} J_\gamma^{(J\beta)}\rangle. \tag{2.16}$$

Because $J_z^{(Jx)} \equiv J_z^{(Jy)} \equiv 0$ everywhere and E_x and E_y are uniform everywhere, while J_x and J_y have uniform values $(J_x)_i$, $(J_y)_i$ in all the slabs of type i, we get from this

$$\rho_{ab}^{(e)} = E_a^{(Jb)} = \sum_{c=x,y} \rho_{ac}^{(i)} (J_c^{(Jb)})_i, \quad a = x,y, \quad b = x,y.$$

Note that the volume averaging has been omitted here. If we fix i, then these are two linear algebraic equations for $(J_x^{(Jb)})_i$, $(J_y^{(Jb)})_i$, which can be solved to yield

$$(J_a^{(Jb)})_i = \sum_{c=x,y} [(\hat{\rho}_{2\mathrm{D}}^{(i)})^{-1}]_{ac} \rho_{cb}^{(e)},$$

where $\hat{\rho}_{2\mathrm{D}}^{(i)}$ is the 2×2 submatrix of the x,y components of the full 3D resistivity matrix $\hat{\rho}_i$. This solution can be used in the expression for $\rho_{za}^{(e)}$ to get

$$\begin{aligned}\rho_{za}^{(e)} &= \langle E_z^{(Ja)}\rangle = \sum_{b=x,y} \langle \rho_{zb} J_b^{(Ja)}\rangle \\ &= \sum_{b=x,y}\sum_{c=x,y} \langle \rho_{zb} (\hat{\rho}_{2\mathrm{D}}^{-1})_{bc}\rangle \rho_{ca}^{(e)}, \quad a = x,y.\end{aligned} \tag{2.17}$$

Applying Eq. (2.14) to the last result, we can immediately get a similar expression for $\rho_{az}^{(e)}$:

$$\begin{aligned}\rho_{az}^{(e)} &= (\hat{\rho}_e^t[\hat{\rho}])_{za} = (\hat{\rho}_e[\hat{\rho}^t])_{za} \\ &= \sum_{b=x,y}\sum_{c=x,y} \langle (\hat{\rho}^t)_{zb} [(\hat{\rho}_{2\mathrm{D}}^t)^{-1}]_{bc}\rangle (\hat{\rho}_e[\hat{\rho}^t])_{ca} \\ &= \sum_{b=x,y}\sum_{c=x,y} \rho_{ac}^{(e)} \langle (\hat{\rho}_{2\mathrm{D}}^{-1})_{cb} \rho_{bz}\rangle.\end{aligned} \tag{2.18}$$

We note that, in the general case when at least some of the $\hat{\rho}_i$ are nonsymmetric, these relations are independent of Eq. (2.17).

If we specialize Eq. (2.16) to the case $\alpha = x,y$, $\beta = z$, recalling that $J_z^{(Jz)} = 1$ and that E_x, E_y are uniform everywhere, while J_x, J_y are uniform in all slabs of the same type i, we get

$$\rho_{az}^{(e)} = E_a^{(Jz)} = \rho_{az}^{(i)} + \sum_{b=x,y} \rho_{ab}^{(i)} (J_b^{(Jz)})_i, \quad a = x,y.$$

Note that we have omitted the volume averaging here. For every i, this constitutes two coupled linear equations for $(J_x^{(Jz)})_i$, $(J_y^{(Jz)})_i$, which can be solved to yield

$$(J_a^{(Jz)})_i = \sum_{b=x,y} [(\hat{\rho}_{2\mathrm{D}}^{(i)})^{-1}]_{ab} (\rho_{bz}^{(e)} - \rho_{bz}^{(i)}).$$

This result can be used in the expression for $\rho_{zz}^{(e)}$

$$\rho_{zz}^{(e)} = \langle E_z^{(Jz)} \rangle = \langle \rho_{zz} \rangle + \sum_{b=x,y} \langle \rho_{zb} J_b^{(Jz)} \rangle$$

to get

$$\begin{aligned}\langle \rho_{zz} \rangle - \rho_{zz}^{(e)} &= \sum_{b=x,y} \sum_{c=x,y} \langle \rho_{zb} (\hat{\rho}_{2\mathrm{D}}^{-1})_{bc} (\rho_{cz} - \rho_{cz}^{(e)}) \rangle, \\ &= \sum_{b=x,y} \sum_{c=x,y} \langle (\rho_{zb} - \rho_{zb}^{(e)}) (\hat{\rho}_{2\mathrm{D}}^{-1})_{bc} (\rho_{cz} - \rho_{cz}^{(e)}) \rangle.\end{aligned} \tag{2.19}$$

The last line results from the fact that $\langle J_a^{(Jz)} \rangle = 0$, $a = x, y$. Therefore, the last line vanishes if the first bracket $(\rho_{zb} - \rho_{zb}^{(e)})$ is omitted along with the sum over b:

$$\sum_{c=x,y} \langle (\hat{\rho}_{2\mathrm{D}}^{-1})_{bc} (\rho_{cz} - \rho_{cz}^{(e)}) \rangle = 0. \tag{2.20}$$

Applying Eq. (2.14) to this result shows that the last line of Eq. (2.19) also vanishes if the last bracket $(\rho_{cz} - \rho_{cz}^{(e)})$ is omitted, along with the sum over c:

$$\sum_{b=x,y} \langle (\rho_{zb} - \rho_{zb}^{(e)}) (\hat{\rho}_{2\mathrm{D}}^{-1})_{bc} \rangle = 0. \tag{2.21}$$

III. TWO-DIMENSIONAL OR COLUMNAR MICROSTRUCTURES

A two-dimensional (2D) columnar microgeometry of a three-dimensional (3D) composite medium means that there exists a fixed direction, which we take to be the x axis, such that all planar sections perpendicular to that direction are identical. Thus, the interfaces between different constituents are surfaces that can be constructed by attaching a straight line, parallel to x, at every point of every curve in that section.

In a columnar microstructure, where the macroscopic dimensions are much greater than any of the heterogeneity length scales, and the boundary conditions would result in uniform values of $\mathbf{E}$ and $\mathbf{J}$ if there were no microstructure (i.e., when the local conductivity tensor has a uniform value everywhere), the local values $\mathbf{E}(y,z)$ and $\mathbf{J}(y,z)$ of the electric field and current density, like those of the local conductivity tensor $\hat{\sigma}(y,z)$, are uniform along the columnar axis x. Furthermore, from the fact that $\nabla \times \mathbf{E} = \mathbf{0}$, it follows that the columnar component of the local electric field in fact has a uniform value everywhere $E_x \equiv \text{const}$. These properties are examples of a fortunate situation wherein, in spite of the heterogeneous microstructure, certain physical fields nevertheless have values that are independent of some of the coordinates in each constituent, or even have uniform values throughout the entire system volume. Such occurrences were noted long ago and used, in the past, to obtain various types of exact results.[22–24] Those results and others are described and reviewed in Chap. 5 of Ref. 3. In the case under consideration here, the field properties decribed above are a consequence of the columnar symmetry of the microstructure, and they lead to some exact results for the macroscopic response. These results were obtained before in special cases.[14] We now proceed to obtain them in the most general case, where no assumptions are made on the form of the constituent conductivity tensors. In this way, exact relations are found among the components of the macroscopic conductivity tensor $\hat{\sigma}_e$ for the case of a two-constituent composite. These relations are very general, and hold regardless of the specific form of the constituent conductivity tensors.

A. Relations among elements of $\hat{\sigma}_e$

First of all, if we apply boundary conditions such that $E_x \equiv 0$, then the electric potential $\phi(y,z)$ is also independent of x and is the solution of a 2D conductivity problem involving the 2×2 submatrix $\hat{\sigma}_{2\mathrm{D}}$ of y and z components of $\hat{\sigma}$:

$$0 = \nabla_{2\mathrm{D}} \cdot \hat{\sigma}_{2\mathrm{D}}(y,z) \cdot \nabla_{2\mathrm{D}} \phi(y,z), \quad \nabla_{2\mathrm{D}} \equiv \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \tag{3.1}$$

Once this equation has been solved, the 2×2 submatrix $\hat{\sigma}_{2\mathrm{D}}^{(e)}$ of y and z components of the full macroscopic conductivity tensor $\hat{\sigma}_e$ can be calculated in the usual fashion, using Eq. (1.2):

$$\sigma_{ab}^{(e)} = \langle J_a^{(b)} \rangle = \sum_{c=y,z} \left\langle \sigma_{ac} \frac{\partial \phi^{(b)}}{\partial r_c} \right\rangle, \quad a = y, z. \tag{3.2}$$

In the case of a two-constituent composite, the local conductivity tensor $\hat{\sigma}(\mathbf{r})$ can be written in terms of the constituent conductivity tensors $\hat{\sigma}_1, \hat{\sigma}_2$ and the characteristic or indicator function $\theta_1(\mathbf{r})$, which equals 1 when $\hat{\sigma}(\mathbf{r}) = \hat{\sigma}_1$ and 0 otherwise

$$\begin{aligned}\hat{\sigma}(\mathbf{r}) &= \hat{\sigma}_1 \theta_1(\mathbf{r}) + \hat{\sigma}_2 [1 - \theta_1(\mathbf{r})] \\ &= \hat{\sigma}_2 - \delta\hat{\sigma} \theta_1(\mathbf{r}), \quad \delta\hat{\sigma} \equiv \hat{\sigma}_2 - \hat{\sigma}_1.\end{aligned}$$

Using this language, we can write the following expression for all the components of $\hat{\sigma}_e$:

$$\delta\sigma_{\alpha\beta}^{(e)} \equiv \sigma_{\alpha\beta}^{(2)} - \sigma_{\alpha\beta}^{(e)} = \sum_{\gamma=x,y,z} \delta\sigma_{\alpha\gamma} \left\langle \theta_1 \frac{\partial \phi^{(\beta)}}{\partial r_\gamma} \right\rangle. \tag{3.3}$$

When b denotes a direction that is perpendicular to x, we note that $\partial \phi^{(b)} / \partial x \equiv E_x^{(b)} \equiv 0$ (note that we again adopt the unconventional convention according to which $\mathbf{E} = \nabla \phi$), since the volume-averaged electric field satisfies $\langle \mathbf{E}^{(\beta)} \rangle = \mathbf{e}_\beta$. Therefore the (y,z)-plane components of $\hat{\sigma}_e$ depend only on the four different volume averages $\langle \theta_1 \partial \phi^{(b)} / \partial r_a \rangle$, $a = y, z$, $b = y, z$. These can be written in terms of the components of the 2×2 (y,z)-plane submatrices $\delta\hat{\sigma}_{2\mathrm{D}}$, $\delta\hat{\sigma}_{2\mathrm{D}}^{(e)}$ of the full 3×3 matrices $\delta\hat{\sigma}$ and $\delta\hat{\sigma}_e$:

$$\left\langle \theta_1 \frac{\partial \phi^{(b)}}{\partial r_a} \right\rangle = [(\delta\hat{\sigma}_{2\mathrm{D}})^{-1} \delta\hat{\sigma}_{2\mathrm{D}}^{(e)}]_{ab},$$

and they can then be used to find $\delta\sigma_{xy}^{(e)}$ and $\delta\sigma_{xz}^{(e)}$:

$$\delta\sigma_{xb}^{(e)} = \sum_{c=y,z} \delta\sigma_{xc}\left\langle \theta_1 \frac{\partial\phi^{(b)}}{\partial r_c}\right\rangle = \sum_{c=y,z} \delta\sigma_{xc}[(\delta\hat{\sigma}_{2D})^{-1}\cdot\delta\hat{\sigma}_{2D}^{(e)}]_{cb}, \quad b=y,z. \tag{3.4}$$

We have thus been able to find $\sigma_{xy}^{(e)}$ and $\sigma_{xz}^{(e)}$ directly and exactly from the results for the y,z components of $\hat{\sigma}_e$.

If we specialize Eq. (3.3) for the case where $\beta=x$, and note that $\partial\phi^{(x)}/\partial x \equiv E_x^{(x)} \equiv 1$ and that $\langle\theta_1\rangle = p_1$ (p_1 is the volume fraction of the $\hat{\sigma}_1$ constituent), we get

$$\delta\sigma_{\alpha x}^{(e)} = p_1\delta\sigma_{\alpha x} + \sum_{b=y,z}\delta\sigma_{\alpha b}\left\langle\theta_1\frac{\partial\phi^{(x)}}{\partial r_b}\right\rangle, \tag{3.5}$$

which can be rewritten as follows:

$$\delta\sigma_{\alpha x}^{(e)} - p_1\delta\sigma_{\alpha x} \equiv \langle\sigma_{\alpha x}\rangle - \sigma_{\alpha x}^{(e)} = \sum_{b=y,z}\delta\sigma_{\alpha b}\left\langle\theta_1\frac{\partial\phi^{(x)}}{\partial r_b}\right\rangle.$$

The two equations that this represents when $\alpha=y,z$ can be solved for $\langle\theta_1\partial\phi^{(x)}/\partial r_a\rangle$, $a=y,z$, in terms of $\sigma_{bx}^{(e)}$, $b=y,z$:

$$\left\langle\theta_1\frac{\partial\phi^{(x)}}{\partial r_a}\right\rangle = \sum_{b=y,z}[(\delta\hat{\sigma}_{2D})^{-1}]_{ab}(\langle\sigma_{bx}\rangle - \sigma_{bx}^{(e)}). \tag{3.6}$$

This can be used to provide an explicit expression for $\sigma_{xx}^{(e)}$ in terms of $\sigma_{yx}^{(e)}$, $\sigma_{zx}^{(e)}$:

$$\langle\sigma_{xx}\rangle - \sigma_{xx}^{(e)} = \sum_{a=y,z}\delta\sigma_{xa}\left\langle\theta_1\frac{\partial\phi^{(x)}}{\partial r_a}\right\rangle = \sum_{a=y,z}\sum_{b=y,z}\delta\sigma_{xa}[(\delta\hat{\sigma}_{2D})^{-1}]_{ab}(\langle\sigma_{bx}\rangle - \sigma_{bx}^{(e)}). \tag{3.7}$$

From Eq. (3.4) we can also get the following result:

$$\delta\sigma_{xb}^{(e)} - p_1\delta\sigma_{xb} \equiv \langle\sigma_{xb}\rangle - \sigma_{xb}^{(e)} = \sum_c \delta\sigma_{xc}[(\delta\hat{\sigma}_{2D})^{-1}\cdot(\hat{\sigma}_{2D}^{(2)} - p_1\delta\hat{\sigma}_{2D} - \hat{\sigma}_{2D}^{(e)})]_{cb} = \sum_c \delta\sigma_{xc}[(\delta\hat{\sigma}_{2D})^{-1}\cdot(\langle\hat{\sigma}_{2D}\rangle - \hat{\sigma}_{2D}^{(e)})]_{cb}. \tag{3.8}$$

Equations (3.4) [or (3.8)] and (3.7) can be described, in words, as giving the top row of $\hat{\sigma}_e$ in terms of its bottom two rows by a linear transformation. This is enough to determine $\sigma_{xy}^{(e)}$ and $\sigma_{xz}^{(e)}$ from the (y,z)-plane elements of $\hat{\sigma}_e$, but we are still unable to calculate the left column of elements of $\hat{\sigma}_e$, namely $\sigma_{\alpha x}^{(e)}$, $\alpha=x,y,z$.

Using the theorem of Eq. (2.13) on Eqs. (3.4) and (3.8) we now get

$$(\delta\hat{\sigma}_e^t)_{xb} = \delta\sigma_{bx}^{(e)} = \sum_c(\delta\hat{\sigma}^t)_{xc}[(\delta\hat{\sigma}_{2D}^t)^{-1}\cdot(\delta\hat{\sigma}_{2D}^{(e)})^t]_{cb} = \sum_c[\delta\hat{\sigma}_{2D}^{(e)}\cdot(\delta\hat{\sigma}_{2D})^{-1}]_{bc}\delta\sigma_{cx}, \tag{3.9}$$

$$\delta\sigma_{bx}^{(e)} - p_1\delta\sigma_{bx} \equiv \langle\sigma_{bx}\rangle - \sigma_{bx}^{(e)} = \sum_c[(\langle\hat{\sigma}_{2D}\rangle - \hat{\sigma}_{2D}^{(e)})\cdot(\delta\hat{\sigma}_{2D})^{-1}]_{bc}\delta\sigma_{cx}. \tag{3.10}$$

Using the last equation to substitute for $\langle\sigma_{bx}\rangle - \sigma_{bx}^{(e)}$ in Eq. (3.7), we get

$$\langle\sigma_{xx}\rangle - \sigma_{xx}^{(e)} = \sum_{a,b=y,z}\delta\sigma_{xa}[(\delta\hat{\sigma}_{2D})^{-1}\cdot(\langle\hat{\sigma}_{2D}\rangle - \hat{\sigma}_{2D}^{(e)})\cdot(\delta\hat{\sigma}_{2D})^{-1}]_{ab}\delta\sigma_{bx}, \tag{3.11}$$

which can also be written as

$$\delta\sigma_{xx}^{(e)} = \sum_{a,b=y,z}\delta\sigma_{xa}[(\delta\hat{\sigma}_{2D})^{-1}\cdot\delta\hat{\sigma}_{2D}^{(e)}\cdot(\delta\sigma_{2D})^{-1}]_{ab}\delta\sigma_{bx} + p_1\left(\delta\sigma_{xx} + \sum_{a,b=y,z}\delta\sigma_{xa}[(\delta\sigma_{2D})^{-1}]_{ab}\delta\sigma_{bx}\right). \tag{3.12}$$

These equations express $\sigma_{xx}^{(e)}$ directly as a linear function of the y,z elements of $\hat{\sigma}_e$.

The results of Eqs. (3.9) and (3.10) are similar in form to Eqs. (3.4) and (3.8), respectively, but with the ordering of the matrices and matrix indices reversed. We note that, while the equation pairs (3.4), (3.8) and (3.9), (3.10) are equivalent when all the conductivity tensors σ_i, $i=1,2,e$ are symmetric, in the general case the two sets of equations are different and independent of each other. When supplemented by Eq. (3.11) or (3.12), the resulting five independent equations express the five elements in the first row and column of the 3D macroscopic conductivity tensor $\hat{\sigma}_e$ in terms of the other four elements of $\hat{\sigma}_e$. The latter are just the elements of $\hat{\sigma}_e$ in the (y,z) plane, i.e., the 2D macroscopic conductivity tensor in that plane $\hat{\sigma}_{2D}^{(e)}$.

B. Relations among elements of $\hat{\rho}_e$

In some cases it is more convenient to express the exact relations that follow from the columnar nature of the microstructure in terms of the elements of the macroscopic resistivity tensor $\hat{\rho}_e = 1/\hat{\sigma}_e$. Since the transformation from $\hat{\sigma}_e$ to $\hat{\rho}_e$ is nonlinear, one might have expected that the linear relations found above, among different elements of $\hat{\sigma}_e$, would translate into more complicated nonlinear relations among different elements of $\hat{\rho}_e$. Somewhat surprisingly, the relations among different elements of $\hat{\rho}_e$ are also linear and quite simple in form, as we now proceed to show.

In order to do this we first note that $\hat{\rho}_e$ satisfies relations that are formally analogous to the relations satisfied by $\hat{\sigma}_e$ itself:

$$\langle\mathbf{E}\rangle = \langle\hat{\rho}\cdot\mathbf{J}\rangle = \hat{\rho}_e\cdot\langle\mathbf{J}\rangle \tag{3.13}$$

$$\delta\hat{\rho}_e\cdot\langle\mathbf{J}\rangle = \delta\hat{\rho}\cdot\langle\theta_1\mathbf{J}\rangle, \tag{3.14}$$

$$\delta\hat{\rho}_e \equiv \hat{\rho}_2 - \hat{\rho}_e, \quad \delta\hat{\rho} \equiv \hat{\rho}_2 - \hat{\rho}_1. \tag{3.15}$$

Because $E_x=$const, we can omit the first two volume averages in the x component of Eq. (3.13), and then replace them by averages over the subvolume of only one constituent. In this way we get

$$\sum_{\alpha=x,y,z} \rho^{(e)}_{x\alpha}\langle J_\alpha\rangle = E_x = \sum_{\alpha=x,y,z} \frac{\rho^{(1)}_{x\alpha}}{p_1}\langle \theta_1 J_\alpha\rangle, \tag{3.16}$$

where p_1 is the volume fraction of the $\hat{\rho}_1$ constituent. Solving Eq. (3.14) for $\langle\theta_1\mathbf{J}\rangle$ and substituting in Eq. (3.16) we get

$$\sum_{\alpha=x,y,z} \rho^{(e)}_{x\alpha}\langle J_\alpha\rangle = \sum_{\alpha=x,y,z}\sum_{\gamma=x,y,z} \frac{\rho^{(1)}_{x\gamma}}{p_1}[(\delta\hat{\rho})^{-1}\cdot\delta\hat{\rho}_e]_{\gamma\alpha}\langle J_\alpha\rangle,$$

which must hold whatever the values of $\langle J_\alpha\rangle$. Therefore

$$\begin{aligned} p_1\rho^{(e)}_{x\alpha} &= \sum_{\gamma=x,y,z}\rho^{(1)}_{x\gamma}[(\delta\hat{\rho})^{-1}\cdot\delta\hat{\rho}_e]_{\gamma\alpha} \\ &= [\hat{\rho}_1\cdot(\delta\hat{\rho})^{-1}\cdot\delta\hat{\rho}_e]_{x\alpha}, \quad \alpha=x,y,z. \end{aligned}\tag{3.17}$$

If we now apply Eq. (2.14) to the last result, we get

$$\begin{aligned} p_1\rho^{(e)}_{\alpha x} &= p_1(\hat{\rho}^t_e)_{x\alpha} = [\hat{\rho}^t_1\cdot(\delta\hat{\rho}^t)^{-1}\cdot\delta\hat{\rho}^t_e]_{x\alpha} \\ &= [\delta\hat{\rho}_e\cdot(\delta\hat{\rho})^{-1}\cdot\hat{\rho}_1]_{\alpha x}, \quad \alpha=x,y,z. \end{aligned}\tag{3.18}$$

Equation (3.17) provides three equations that express each element in the top row of the matrix $\hat{\rho}_e$ as a linear function of the other two elements in the same column of that matrix. Similarly, Eq. (3.18) provides three equations that express each element in the left column of that matrix as a linear function of the other two elements in the same row.

It is straightforward to show that switching the roles of the two constituents $1\leftrightarrow 2$ in Eqs. (3.17) and (3.18) does not lead to independent new relations—in each case the sum of the two equation sets is a set of three identities (recall that under this switch, $\delta\hat{\rho}\equiv\hat{\rho}_2-\hat{\rho}_1\to-\delta\hat{\rho}$ and $\delta\hat{\rho}_e\equiv\hat{\rho}_2-\hat{\rho}_e\to\delta\hat{\rho}_e-\delta\hat{\rho}$). For example, if we add to Eq. (3.17) the equation obtained by switching the roles of the two constituents, we get

$$\begin{aligned} \rho^{(e)}_{x\alpha} &\equiv p_1\rho^{(e)}_{x\alpha} + p_2\rho^{(e)}_{x\alpha} \\ &= [\hat{\rho}_1\cdot(\delta\hat{\rho})^{-1}\cdot\delta\hat{\rho}_e]_{x\alpha} - [\hat{\rho}_2\cdot(\delta\hat{\rho})^{-1}\cdot(\delta\hat{\rho}_e-\delta\hat{\rho})]_{x\alpha} \\ &\equiv \rho^{(2)}_{x\alpha} - \delta\rho^{(e)}_{x\alpha}. \end{aligned}$$

Clearly, the first and last expressions in this chain of equalities are identical: They are always equal, irrespective of the actual value of $\rho^{(e)}_{x\alpha}$, which depends on details of the microstructure.

IV. SOME SPECIAL CASES

A. Normal conductor/perfect insulator mixture

A special case where Eqs. (3.17) and (3.18) lead to simple results is when the No. 2 constituent is a perfect insulator, i.e., $\hat{\rho}_2\to\infty$, but all elements of $\hat{\rho}_e$ remain finite. In that case $\delta\hat{\rho}\cong\hat{\rho}_2$ and $\delta\hat{\rho}_e\cong\hat{\rho}_2$, and we get

$$p_1\rho^{(e)}_{x\alpha} = \rho^{(1)}_{x\alpha}, \quad p_1\rho^{(e)}_{\alpha x} = \rho^{(1)}_{\alpha x}, \quad \alpha=x,y,z. \tag{4.1}$$

These results hold whenever the No. 1 constituent percolates in the (y,z) plane. When that constituent does not percolate in the (y,z) plane, then no macroscopic current can flow in that plane. Consequently the (y,z)-plane components of $\hat{\rho}_e$ diverge and the reduction of Eqs. (3.17) and (3.18) to Eq. (4.1) must be done with care. When this is carried out, one finds that Eq. (4.1) remains valid only for $\alpha=x$. For $\alpha=y,z$ we then find

$$p_1\rho^{(e)}_{xa} = \rho^{(1)}_{xa} - \lim_{\hat{\rho}_2\to\infty}\sum_{b=y,z}\sum_{\beta=x,y,z}\rho^{(1)}_{x\beta}(\hat{\rho}_2^{-1})_{\beta b}\rho^{(e)}_{ba} \text{ for } a=y,z, \tag{4.2}$$

$$p_1\rho^{(e)}_{ax} = \rho^{(1)}_{ax} - \lim_{\hat{\rho}_2\to\infty}\sum_{b=y,z}\sum_{\beta=x,y,z}\rho^{(e)}_{ab}(\hat{\rho}_2^{-1})_{b\beta}\rho^{(1)}_{\beta x} \text{ for } a=y,z. \tag{4.3}$$

In contrast with Eq. (4.1), these last results depend on details of the 2D microstructure and on the form of $\hat{\rho}_2$, which determine the values of the two limits

$$\lim_{\hat{\rho}_2\to\infty}(\hat{\rho}_2^{-1}\cdot\hat{\rho}_e)_{\beta a}, \quad \lim_{\hat{\rho}_2\to\infty}(\hat{\rho}_e\cdot\hat{\rho}_2^{-1})_{a\beta}. \tag{4.4}$$

Applied to this case, Eq. (3.4) leads to

$$\sigma^{(e)}_{xa} = \sum_{c=y,z}\sigma^{(1)}_{xc}[(\hat{\sigma}^{(1)}_{2\mathrm{D}})^{-1}\cdot\hat{\sigma}^{(e)}_{2\mathrm{D}}]_{ca} \text{ for } a=y,z.$$

Applying Eq. (3.9) to this case, we get

$$\sigma^{(e)}_{ax} = \sum_{c=y,z}[\hat{\sigma}^{(e)}_{2\mathrm{D}}\cdot(\hat{\sigma}^{(1)}_{2\mathrm{D}})^{-1}]_{ac}\sigma^{(1)}_{cx} \text{ for } a=y,z.$$

Finally, applying Eq. (3.11) to this case, we get (note that $\hat{\sigma}_2=0$)

$$\begin{aligned} &p_1\sigma^{(1)}_{xx} - \sigma^{(e)}_{xx} \\ &\quad= \sum_{a,b=y,z}\sigma^{(1)}_{xa}[(\hat{\sigma}^{(1)}_{2\mathrm{D}})^{-1}\cdot(p_1\hat{\sigma}^{(1)}_{2\mathrm{D}}-\hat{\sigma}^{(e)}_{2\mathrm{D}})\cdot(\hat{\sigma}^{(1)}_{2\mathrm{D}})^{-1}]_{ab}\sigma^{(1)}_{bx}. \end{aligned}\tag{4.5}$$

From the above three equations it follows, as expected, that all the elements of $\hat{\sigma}_e$ in the first row and the first column are usually nonzero, except when the (y,z)-plane elements happen to vanish, due to nonpercolation of the nonzero conductor $\hat{\sigma}_1$. In that case, $\sigma^{(e)}_{xa}=\sigma^{(e)}_{ax}=0$ for $a=y,z$, and only

$$\sigma^{(e)}_{xx} = p_1\left\{\sigma^{(1)}_{xx} - \sum_{a,b=y,z}\sigma^{(1)}_{xa}[(\hat{\sigma}^{(1)}_{2\mathrm{D}})^{-1}]_{ab}\sigma^{(1)}_{bx}\right\} \tag{4.6}$$

is nonzero. It can be verified that the last equation is also a special case of Eq. (4.1). However, while the form of Eq. (4.1) remains valid for $\rho^{(e)}_{xx}$ even when the y,z elements of $\hat{\sigma}_e$ are nonzero, Eq. (4.6) will revert to Eq. (4.5) in that case.

B. Normal conductor/perfect conductor mixture

Another special case occurs when the No. 2 constituent is a perfect conductor, i.e. $\hat{\rho}_2\to 0$. In that case $\delta\hat{\rho}\cong-\hat{\rho}_1$ and $\delta\hat{\rho}_e\cong-\hat{\rho}_e$, and we get

$$p_1\rho^{(e)}_{x\alpha} = \rho^{(e)}_{x\alpha} \Rightarrow \rho^{(e)}_{x\alpha} = 0 \text{ for any value of } p_1 < 1. \tag{4.7}$$

In order to determine the other elements of $\hat{\rho}_e$, we first note that the (y,z)-plane elements of $\hat{\sigma}_e$ will all be finite if the perfect conductor $\hat{\rho}_2$ does not percolate in that plane. Assuming that the inverse of $\hat{\rho}_2$ an be written simply as $\hat{\sigma}_2 = \lim_{A\to\infty} A\hat{I} + \hat{\mathcal{O}}(1)$, where $\hat{\mathcal{O}}(1)$ is an order 1 3×3 matrix, Eq. (3.4) leads to

$$\sigma^{(e)}_{xb} = \sigma^{(1)}_{xb} \text{ for } b = y,z.$$

Turning to Eq. (3.9) we then find

$$\sigma^{(e)}_{ax} = \sigma^{(1)}_{ax} \text{ for } a = y,z.$$

Although the latter result is not very useful in practice, since a finite electric field component E_x cannot be imposed in the presence of perfectly conducting cylindrical inclusions, we shall see immediately that it does lead to some other useful results. To wit, Eq. (3.11) leads to

$$\sigma^{(e)}_{xx} \cong p_2\sigma^{(2)}_{xx} \to \infty,$$

while all the other elements of $\hat{\sigma}_e$ are finite. Therefore, when we invert $\hat{\sigma}_e$ to get $\hat{\rho}_e$, we find

$$\rho^{(e)}_{x\alpha} = \rho^{(e)}_{\alpha x} = 0 \text{ for } \alpha = x,y,z, \tag{4.8}$$

$$\rho^{(e)}_{ab} = [(\hat{\sigma}^{(e)}_{2\mathrm{D}})^{-1}]_{ab} \text{ for } a = y,z \text{ and } b = y,z. \tag{4.9}$$

Equations (4.1), (4.7), and (4.8) were known before for special forms of $\hat{\rho}_1$—see Refs. 14 and 15. Here we have obtained them in full generality, without any assumptions regarding the form of $\hat{\rho}_1$.

C. $\sigma_{xy}(\mathbf{r}) \equiv \sigma_{xz}(\mathbf{r}) \equiv 0$

Yet another special case is when $\sigma^{(i)}_{xy} = \sigma^{(i)}_{xz} = 0$ in both constituents $i=1,2$. It then follows from Eq. (3.4) that also $\sigma^{(e)}_{xy} = \sigma^{(e)}_{xz} = 0$. From these it follows that $\rho^{(i)}_{xy} = \rho^{(i)}_{xz} = 0$ and $\sigma^{(i)}_{xx} = 1/\rho^{(i)}_{xx}$ for $i=1,2,e$, and that $[(\delta\hat{\rho})^{-1}]_{xy} = [(\delta\hat{\rho})^{-1}]_{xz} = 0$ and $[(\delta\hat{\rho})^{-1}]_{xx} = 1/\delta\rho_{xx}$. It then follows from Eq. (3.17) that

$$p_1\rho^{(e)}_{xx} = \rho^{(1)}_{xx}\frac{\delta\rho^{(e)}_{xx}}{\delta\rho_{xx}} \Rightarrow \sigma^{(e)}_{xx} = p_1\sigma^{(1)}_{xx} + p_2\sigma^{(2)}_{xx} = \langle\sigma_{xx}\rangle. \tag{4.10}$$

Using Eq. (1.2), it is easy to see that the results $\sigma^{(e)}_{xy} = \sigma^{(e)}_{xz} = 0$ and $\sigma^{(e)}_{xx} = \langle\sigma_{xx}\rangle$ remain valid for any number of constituents, if $\sigma^{(i)}_{xy} = \sigma^{(i)}_{xz} = 0$ in all of them.

D. $\sigma_{yx}(\mathbf{r}) \equiv \sigma_{zx}(\mathbf{r}) \equiv 0$

A fourth special case is when $\sigma^{(i)}_{yx} = \sigma^{(i)}_{zx} = 0$ in both constituents $i=1,2$. In this case it is easy to see that $\phi^{(x)} \equiv x$ and consequently that $\mathbf{E}^{(x)} \equiv \mathbf{e}_x$, i.e., $E^{(x)}_x \equiv 1$ and $E^{(x)}_y \equiv E^{(x)}_z \equiv 0$ everywhere. Therefore

$$\sigma^{(e)}_{\alpha x} = \langle J^{(x)}_\alpha\rangle = \sum_{\gamma=x,y,z}\langle\sigma_{\alpha\gamma}E^{(x)}_\gamma\rangle = \langle\sigma_{\alpha x}\rangle = \langle\sigma_{xx}\rangle\delta_{\alpha x},$$

$$\alpha = x,y,z.$$

It follows that $\rho^{(i)}_{yx} = \rho^{(i)}_{zx} = 0$ and $\sigma^{(i)}_{xx} = 1/\rho^{(i)}_{xx}$ for $i=1,2,e$, and that $[(\delta\hat{\rho})^{-1}]_{yx} = [(\delta\hat{\rho})^{-1}]_{zx} = 0$ and $[(\delta\hat{\rho})^{-1}]_{xx} = 1/\delta\rho_{xx}$. It then also follows from Eq. (3.17) that Eq. (4.10) is again valid. For an arbitrary number of constituents, all of which satisfy $\sigma^{(i)}_{yx} = \sigma^{(i)}_{zx} = 0$, Eq. (1.2) again leads to the result that $\sigma^{(e)}_{yx} = \sigma^{(e)}_{zx} = 0$ and $\sigma^{(e)}_{xx} = \langle\sigma_{xx}\rangle$ remain valid.

E. Other special cases

A subcase of the above two is the elementary case where both σ_1 and σ_2 are scalar tensors, then $\hat{\sigma}_e$ is a symmetric tensor and x is one of its principal axes, hence $\sigma^{(e)}_{xy} = \sigma^{(e)}_{yx} = \sigma^{(e)}_{xz} = \sigma^{(e)}_{zx} = 0$. It is well known that the conductivity along the columnar axis is then given by $\sigma^{(e)}_{xx} = p_1\sigma_1 + p_2\sigma_2$, where p_1, $p_2 = 1 - p_1$ are the volume fractions of the two constituents. In this special case too, the result for $\sigma^{(e)}_{xx}$ is easily generalized to an arbitrary number of constituents, leading to Eq. (1.3). A similar generalization of the above exact results to columnar composites made of more than two constituents does not seem possible in the case of constituent conductivity tensors of a more general type than the ones considered here.

A fifth special case is when $\hat{\sigma}_1$ and $\hat{\sigma}_2$ both correspond to isotropic conductors in the presence of a magnetic field that is perpendicular to the columnar axis, and if that field lies along a symmetry axis or principal axis of the planar microstructure, which we denote as z. In this case, the component resistivities have the form

$$\hat{\rho}_i = \begin{pmatrix} \rho^{(i)}_\perp & -\rho^{(i)}_H & 0 \\ \rho^{(i)}_H & \rho^{(i)}_\perp & 0 \\ 0 & 0 & \rho^{(i)}_\parallel \end{pmatrix}, \quad i = 1,2,$$

while the macroscopic resistivity has the form

$$\hat{\rho}_e = \begin{pmatrix} \rho^{(e)}_{xx} & -\rho^{(e)}_H & 0 \\ \rho^{(e)}_H & \rho^{(e)}_{yy} & 0 \\ 0 & 0 & \rho^{(e)}_\parallel \end{pmatrix},$$

where the components $\rho^{(1)}_H$, $\rho^{(2)}_H$, $\rho^{(e)}_H$ denote the constituent and the macroscopic Hall resistivities, respectively.

Due to the symmetry of this problem, the only nontrivial result from Eq. (3.4) is obtained when $b=y$, which leads to

$$\delta\sigma^{(e)}_{xy} = \frac{\delta\sigma_{xy}}{\delta\sigma_{yy}}\delta\sigma^{(e)}_{yy}. \tag{4.11}$$

Again, due to the symmetry of this problem, this result also determines the value of $\delta\sigma^{(e)}_{yx} = -\delta\sigma^{(e)}_{xy}$, and this can be used in Eq. (3.7) to get

$$\langle\sigma_{xx}\rangle - \sigma^{(e)}_{xx} = \frac{\delta\sigma_{xy}}{\delta\sigma_{yy}}(\langle\sigma_{yx}\rangle - \sigma^{(e)}_{yx}). \tag{4.12}$$

These exact linear relations obviously suffice to determine both $\sigma^{(e)}_{xx}$ and $\sigma^{(e)}_{xy} = -\sigma^{(e)}_{yx} \equiv \sigma^{(e)}_H$ from $\sigma^{(e)}_{yy}$, which is determined by solving the 2D problem in the (y,z) plane. When these results are translated into results for the elements of $\hat{\rho}_e = 1/\hat{\sigma}_e$, they reduce to the following linear relations among $\rho^{(e)}_{xx}$, $\rho^{(e)}_{yy}$, and $\rho^{(e)}_{yx} = -\rho^{(e)}_{xy} \equiv \rho^{(e)}_H$:

$$\begin{aligned} 0 &= (\sigma^{(2)}_\perp\delta\sigma_H - \sigma^{(2)}_H\delta\sigma_\perp)\rho^{(e)}_{yy} \\ &\quad + (\langle\sigma_\perp\rangle\delta\sigma_\perp + \langle\sigma_H\rangle\delta\sigma_H)\rho^{(e)}_H - \delta\sigma_H, \end{aligned} \tag{4.13}$$

$$0 = (\sigma_\perp^{(2)}\delta\sigma_H - \sigma_H^{(2)}\delta\sigma_\perp)\rho_H^{(e)} - (\langle\sigma_\perp\rangle\delta\sigma_\perp + \langle\sigma_H\rangle\delta\sigma_H)\rho_{xx}^{(e)} + \delta\sigma_\perp. \tag{4.14}$$

Again, the linear character of these relations is noteworthy, since the transformation from elements of $\hat{\sigma}_e$ to elements of $\hat{\rho}_e$ is nonlinear.

Equation (3.17) with $\alpha=x$ provides a linear relation between $\rho_{xx}^{(e)}$ and $\rho_{yx}^{(e)}\equiv\rho_H^{(e)}$, which is the same as Eq. (4.14), while with $\alpha=y$ it provides a linear relation between $\rho_{xy}^{(e)}\equiv-\rho_H^{(e)}$ and $\rho_{yy}^{(e)}$, which is the same as Eq. (4.13), and with $\alpha=z$ it becomes the trivial identity $0=0$. The relations of Eqs. (4.13) and (4.14) were first derived, from an equation like Eq. (3.17), in Ref. 14, where they appear, in a somewhat different explicit form, as Eqs. (2.9) and (2.8) of that reference. This case has also been discussed in Milton's book.[3]

A sixth special case is the 1D laminar microstructure, which was the subject of Sec. II. In that case, there are two perpendicular columnar axes, which can be chosen at will in the laminar plane. This case will not be discussed any further here.

V. SUMMARY

We obtained exact simple expressions for all components of $\hat{\sigma}_e$ in the case of a parallel slab microstructure or lowest order laminate. No assumptions are necessary regarding the forms of the constituent conductivity tensors, which can be nonscalar, nonsymmetric, and complex valued. The parallel slab microstructure is the basic building block of higher-order laminates.[2,3] For this reason it is hoped that these expressions will have application in the further discussion of such microstructures, especially in the presence of a magnetic field. We also obtained exact relations among different elements of $\hat{\sigma}_e$ in the case of two-constituent columnar composites. Again, no assumptions had to be made regarding the forms of the constituent conductivity tensors. The relations that were obtained suffice to determine all the elements of $\hat{\sigma}_e$ and $\hat{\rho}_e$, once the 2D conductivity problem has been solved in the plane perpendicular to the columnar axis and the corresponding 2D components of $\hat{\sigma}_e$ have been computed. Therefore, the exact relations among different elements of $\hat{\sigma}_e$ and among different elements of $\hat{\rho}_e$, in the case of two-constituent columnar composites, will hopefully also be useful in further studies of such microstructures, especially in the presence of a magnetic field.

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APPENDIX: PROOF OF A THEOREM

Given a heterogeneous medium with local conductivity tensor $\hat{\sigma}(\mathbf{r})$, the elements of the macroscopic conductivity tensor $\hat{\sigma}_e[\hat{\sigma}]$ can be calculated once one knows the local potential field under different boundary conditions—see Eq. (1.2). We note that Eq. (1.2) is valid whatever is the form of $\hat{\sigma}(\mathbf{r})$. This includes matrices with complex elements, and nonsymmetric as well as non-Hermitian matrices. Starting from an expression that is equivalent to that equation, we first obtain a more symmetric expression for the elements of $\hat{\sigma}_e$:

$$\sigma_{\alpha\beta}^{(e)} = \mathbf{e}_\alpha \cdot \frac{1}{V}\int dV\hat{\sigma}\cdot\nabla\phi^{(\beta)} = \frac{1}{V}\int dV\,\nabla\phi_0^{(\alpha)}\cdot\hat{\sigma}\cdot\nabla\phi^{(\beta)}. \tag{A1}$$

Here $\mathbf{e}_\alpha$ is the unit vector along the direction α, while $\phi_0^{(\alpha)}(\mathbf{r})\equiv r_\alpha$ is a linear potential function that satisfies the same boundary conditions as the true potential function $\phi^{(\alpha)}(\mathbf{r})$. These functions have the following properties (∂V denotes the system surface):

$$\mathbf{e}_\alpha = \nabla\phi_0^{(\alpha)} = \langle\nabla\phi^{(\alpha)}\rangle$$

$$\phi^{(\alpha)}(\mathbf{r}) = \phi_0^{(\alpha)}(\mathbf{r}) = r_\alpha \text{ for } \mathbf{r}\in\partial\mathrm{V}.$$

The last integral in Eq. (A1) can be processed by using Green's theorem to transform back and forth from volume to surface integration:

$$\begin{aligned}
\int_V dV\,\nabla\phi_0^{(\alpha)}\cdot\hat{\sigma}\cdot\nabla\phi^{(\beta)}
&= \int_V dV[\nabla\cdot(\phi_0^{(\alpha)}\cdot\hat{\sigma}\cdot\nabla\phi^{(\beta)}) - \phi_0^{(\alpha)}\nabla\cdot(\hat{\sigma}\cdot\nabla\phi^{(\beta)})]\\
&= \oint_{\partial V} d\mathbf{A}\cdot\phi_0^{(\alpha)}\cdot\hat{\sigma}\cdot\nabla\phi^{(\beta)} - \int_V dV\phi_0^{(\alpha)}\nabla\cdot(\hat{\sigma}\cdot\nabla\phi^{(\beta)})\\
&= \oint_{\partial V} d\mathbf{A}\cdot\phi^{(\alpha)}\cdot\hat{\sigma}\cdot\nabla\phi^{(\beta)} - \int_V dV\phi^{(\alpha)}\nabla\cdot(\hat{\sigma}\cdot\nabla\phi^{(\beta)})\\
&= \int_V dV\,\nabla\phi^{(\alpha)}\cdot\hat{\sigma}\cdot\nabla\phi^{(\beta)}.
\end{aligned}$$

In line 4 we were able to replace $\phi_0^{(\alpha)}(\mathbf{r})$ by $\phi^{(\alpha)}(\mathbf{r})$: In the first integral this is due to the fact that these two functions are the same on the system surface ∂V, while in the second integral this is due to the fact that $\phi^{(\alpha)}(\mathbf{r})$ is multiplied by the vanishing expression

$$\nabla\cdot(\hat{\sigma}\cdot\nabla\phi^{(\beta)}) = 0. \tag{A2}$$

In this way we obtain the following symmetric expression for $\sigma_{\alpha\beta}^{(e)}$:

$$\sigma_{\alpha\beta}^{(e)} = \frac{1}{V}\int dV\,\nabla\phi^{(\alpha)}\cdot\hat{\sigma}\cdot\nabla\phi^{(\beta)}. \tag{A3}$$

If we now replace $\hat{\sigma}(\mathbf{r})$ everywhere by its transpose $\hat{\sigma}^t(\mathbf{r})$, then the potential function, denoted by $\phi_t^{(\alpha)}(\mathbf{r})$, satisfies the following partial differential equation and boundary condition:

$$\nabla \cdot (\hat{\sigma}^t \cdot \nabla \phi_t^{(\alpha)}) = 0, \tag{A4}$$

$$\phi_t^{(\alpha)} = r_\alpha \text{ for } \mathbf{r} \in \partial V. \tag{A5}$$

Although the boundary conditions on $\phi^{(\alpha)}$ and $\phi_t^{(\alpha)}$ are the same, these two potential functions satisfy different partial differential equations, namely, Eqs. (A2) and (A4). Therefore these two potential functions are different. Nevertheless, in Eq. (A1) we can also replace $\phi_0^{(\alpha)}$ by $\phi_t^{(\alpha)}$, after following a back-and-forth Green-theorem procedure similar to the one which leads from Eq. (A1) to Eq. (A3). We can then write

$$\begin{aligned}\int dV\, \nabla \phi^{(\alpha)} \cdot \hat{\sigma} \cdot \nabla \phi^{(\beta)} &= \int dV\, \nabla \phi_t^{(\alpha)} \cdot \hat{\sigma} \cdot \nabla \phi^{(\beta)} \\ &= \int dV\, \nabla \phi^{(\beta)} \cdot \hat{\sigma}^t \cdot \nabla \phi_t^{(\alpha)} \\ &= \int dV\, \nabla \phi_t^{(\beta)} \cdot \hat{\sigma}^t \cdot \nabla \phi_t^{(\alpha)},\end{aligned}$$

from which it immediately follows that

$$\sigma_{\alpha\beta}^{(e)}[\hat{\sigma}] = \sigma_{\beta\alpha}^{(e)}[\hat{\sigma}^t],$$

which is the same as Eq. (2.13). We note that this theorem holds without any restrictions on the form of the various $\hat{\sigma}$ matrices. In particular, they do not have to be symmetric or hermitian or real.

We noted earlier (see Sec. II A) the relation of this theorem to Onsager's theorem, which arises from invariance of microscopic kinetic phenomena under time reversal. We stress that, as proven here, Eq. (2.13) does not depend on any symmetries of microphysical processes.

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