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Electron-electron interaction corrections to the thermal conductivity in disordered conductors

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We evaluate the electron-electron interaction corrections to the electronic thermal conductivity in a disordered conductor in the diffusive regime. We use a diagrammatic many-body method analogous to that of Altshuler and Aronov for the electrical conductivity. We derive results in one, two, and three dimensions for both the singlet and triplet channels, and in all cases find that the Wiedemann-Franz law is violated.

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I. INTRODUCTION

The effect of the electron-electron interaction on the electrical conductivity of disordered systems has been extensively investigated both theoretically and experimentally over the past two decades.[1,2] There are two main types of corrections to the Drude electrical conductivity that can have similar magnitudes and temperature dependences: weak localization and interaction effects. Weak localization is due to interference between pairs of time-reversed scattering trajectories of electrons from impurities; this effect can occur in a noninteracting system. Interaction effects are due to the increased effective electron-electron interaction strength due to incomplete screening by diffusively moving electrons. Experimentally the two effects can be distinguished by applying a magnetic field; weak localization is suppressed while interaction effects are not.

In comparison there has been relatively little work done on the thermal conductivity, either theoretically or experimentally. This is largely because thermal conductivity is hard to measure in low-dimensional systems at low temperature, and it is difficult to separate the electronic and lattice contributions. It is therefore doubly hard to observe the disorder-driven corrections to the electronic thermal conductivity. Moreover, there have been theoretical predictions[3,4] that the Wiedemann-Franz law holds, which allows one to deduce the thermal conductivity, κ, directly from the electrical conductivity, σ,

$$\frac{\kappa}{\sigma} = \frac{\pi^2}{3}\left(\frac{k_B}{e}\right)^2 T = L_0 T, \tag{1}$$

where k_B is Boltzmann's constant, e is the electronic charge, T is the temperature, and L_0 is known as the Lorenz number. Clearly it is important to establish whether the Wiedemann-Franz law is valid in interacting disordered systems; if it is, there is nothing to be gained from measuring the thermal conductivity in addition to the electrical conductivity.

In this paper we evaluate the interaction corrections to the thermal conductivity in one, two, and three dimensions in both the singlet and triplet channels. We find that the Wiedemann-Franz law is violated in all cases. Our results are presented in detail in Table I. We have not included a result for the singlet channel in one dimension since this is just one third of the corresponding triplet channel result with F replaced by an effective singlet channel interaction $F_s = \kappa_3^2 a^2 \ln(D\kappa_3^2/T)$, as explained in the table caption. Each term consists of two pieces: the first piece leads to violation of the Wiedemann-Franz law; the second piece does not. The exception is the triplet channel in two dimensions, which does not possess a violating piece. Note that in each case both terms have the same parametric dependences; it is only the constant prefactors that are different.

Our calculation is the exact analog of the original Altshuler-Aronov calculation for electrical conductivity.[5] There have been several previous attempts at this calculation;[4,6,7] however, these are in disagreement with each other, and we believe all of them to be incorrect. The fact that it has taken so long (over 20 years) to get the correct result for the thermal conductivity is due to three inherent difficulties in the problem. First, the heat-current operator is not uniquely defined, and some choices of definition will be renormalized by the electron-electron interaction.[6,8] Second, the heat-current operator has opposite electron-hole parity to the charge-current operator, which leads to the presence of an extra "heat-drag" diagram for the thermal conductivity, which is vanishingly small for the electrical conductivity. (We use the term "heat-drag" by analogy to the Coulomb drag effect between two layers of electrons; this effect is described by the same diagram.[9]) Third, spurious ultraviolet divergences occur in the diagrammatic approach to thermal conductivity; these can be understood and evaluated within the framework of divergent series theory[10] (their origin is due to illegal series expansions in the derivation of the perturbation theory).

Before we proceed to the details of our calculation, we present a short history of the field. We will first consider effects that do not require interaction, such as weak localization and the Anderson transition. Chester and Thellung[3] used an exact eigenstate approach to show that the Wiedemann-Franz law should hold in a noninteracting disordered system, independent of the strength of impurity scattering. Strinati and Castellani[11] used a Ward identity construction to argue that the Wiedemann-Franz law holds all the way to the Anderson transition. Kearney and Butcher[12] used the exact eigenstates approach of Chester and Thellung to deduce the weak localization correction to thermal conductivity. This effect was later seen in the experiment of Bayot *et al.*,[13] which measured the electrical and thermal magnetoconductance, which were found to obey the Wiedemann-Franz law. Enderby and Barnes[14] later pointed out that the Wiedemann-

TABLE I. Interaction corrections to the electronic thermal conductivity in one, two, and three dimensions in both the singlet and triplet channels. In the above equations T is temperature, D is the diffusion constant, F is the effective interaction in the triplet channel, and κ_2 is the inverse screening length in two dimensions. Note that $\hbar=1$ and $k_B=1$ in these calculations. We do not include a result for the singlet channel in one dimension as it is just one third of that for the triplet channel with F replaced by the effective interaction parameter for the singlet channel, $F_s \approx \kappa_3^2 a^2 \ln(D\kappa_3^2/T)$, where κ_3 is the inverse screening length in three dimensions, and a is the wire width. The inverse screening lengths in two and three dimensions are given by $\kappa_3^2 = 8\pi N(0)e^2$ and $\kappa_2 = 4\pi N(0)e^2$, where $N(0)$ is the single-spin electronic density of states in the appropriate dimension.

Dimension	Singlet term	Triplet term
3	$\frac{105\zeta(5/2)-16\pi^2\zeta(1/2)}{288\sqrt{2}\pi^{3/2}}\frac{T^{3/2}}{D^{1/2}}$	$\frac{15[(4+7F)\sqrt{1+F}-(4+9F)]\zeta(5/2)-16\pi^2[2-(2-F)\sqrt{1+F}]\zeta(1/2)}{96\sqrt{2}\pi^{3/2}F\sqrt{1+F}}\frac{T^{3/2}}{D^{1/2}}$
2	$\frac{T}{12}\left[\ln\left(\frac{D\kappa_2^2}{T}\right)-2\ln\left(\frac{1}{T\tau}\right)\right]$	$-\frac{T}{2}\left[1-\frac{1}{F}\ln(1+F)\right]\ln\left(\frac{1}{T\tau}\right)$
1		$\frac{3\{[(4+3F)\sqrt{1+F}-(4+5F)]-4\pi[(2+F)-2\sqrt{1+F}]\}\zeta(3/2)}{8\sqrt{2\pi}F}T^{1/2}D^{1/2}$

Franz law is not obeyed at the Anderson transition—the previous treatments all use the Sommerfeld expansion, which is not valid close to the Anderson transition. However, the only effect is to decrease the constant in the Lorenz number from $\pi^2/3 \approx 3.29$ to 2.17 for a conductivity exponent $\nu=1$. The overall conclusion for the noninteracting effects is that there is no extra information in the thermal conductivity that is not present in the electrical conductivity. In the weak localization regime the Wiedemann-Franz law holds; close to the Anderson transition it is only modified by a simple numerical factor, so all critical properties are correctly predicted by it.

It would therefore seem that the only possibility of interesting behavior lies with the interaction corrections to thermal conductivity. Castellani *et al.*[4] predicted that the Wiedemann-Franz law would hold even for the interacting disordered system. These authors evaluated the dynamic energy-energy correlation function in the interacting system using a skeleton graph analysis, and derived the thermal conductivity from this. Since their method of calculation is somewhat different from ours, it is not obvious to us why they obtain a different result. Livanov *et al.*[6] later directly calculated the interaction corrections using a quantum kinetic equation approach, and predicted that the Wiedemann-Franz law is violated in all dimensions. They appear to have evaluated some interaction contributions correctly, but to have missed other contributions of the same order of magnitude; for example, in two dimensions, they predict a logarithmic increase in κ/T at low temperature rather than the correct logarithmic decrease. We believe that the results obtained initially from their diagrammatic formalism are correct, and that they have then erroneously thrown away some terms, believing them to be smaller in magnitude. Arfi[7] performed the equivalent calculation in the Matsubara formalism and again found violation of the Wiedemann-Franz law. However, this calculation has errors relating to all three of the difficulties referred to previously: (a) the Matsubara form of heat current is used, but the heat current is then erroneously renormalized by interaction; (b) the "heat drag" or Aslamazov-Larkin-like diagram is omitted; (c) the final results are parametrically larger than those for electrical conductivity by factors of $1/(T\tau)^2$, where τ is the elastic scattering time, which appears to be due to incorrect treatment of spuriously divergent terms, as will be described later. The aim of the present work is therefore to resolve the discrepancies between previous calculations and to present a consistent calculation of the interaction correction to thermal conductivity.

The remainder of the paper is organized as follows: in Sec. II we derive the Drude contribution to electronic thermal conductivity; in Sec. III we present an outline of the calculation of the interaction corrections to thermal conductivity; finally in Sec. IV we analyse our results and draw conclusions.

II. DRUDE THERMAL CONDUCTIVITY AND THE WIEDEMANN-FRANZ LAW

Before we proceed to the calculation of the interaction corrections, we demonstrate the simplicity and elegance of the Matsubara approach to evaluating thermal conductivity by deriving the Drude result. The thermal conductivity is obtained from the imaginary time heat response kernel, $Q_{hh}(i\Omega_p)$, by analytic continuation from positive Bose Matsubara frequencies, $\Omega_p = 2\pi T p$,

$$\kappa T = \lim_{\Omega\to 0}\left[\frac{Q_{hh}(i\Omega_p)}{\Omega_p}\right]_{i\Omega_p\to\Omega+i0}. \tag{2}$$

The Feynman diagram for the Drude thermal conductivity is shown in Fig. 1. The solid lines are disordered electron Green functions

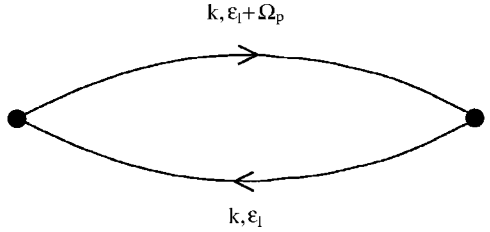

FIG. 1. Feynman diagram for the Drude contribution to the heat response kernel. The black circles represent the Matsubara heat current vertex given in Eq. (4).

$$G(k,i\varepsilon_l)=\frac{1}{i\varepsilon_l-\xi_k+(i/2\tau)\mathrm{sgn}(\varepsilon_l)}, \qquad (3)$$

where $\varepsilon_l=2\pi T(l+1/2)$ is a Fermi Matsubara frequency, $\xi_k=k^2/2m-\mu$ is the electronic excitation spectrum, and τ is the elastic scattering time. The black dots represent heat-current vertices, which are given by

$$\mathbf{j}_h(\mathbf{k},\varepsilon_l,\varepsilon_l+\Omega_p)=\frac{\mathbf{k}}{2m}i(2\varepsilon_l+\Omega_p) \qquad (4)$$

in the Matsubara frequency representation that we use. The heat current kernel is then given by

$$Q_{hh}^{\alpha\beta}(i\Omega_p)=2T\sum_{\epsilon_l}\sum_{k}\frac{k_\alpha}{m}\frac{k_\beta}{m}[i(\epsilon_l+\Omega_p/2)]^2 \times G(k,i\epsilon_l)G(k,i\epsilon_l+i\Omega_p). \qquad (5)$$

Performing the $\mathbf{k}$ integral, we only obtain a nonzero result if the frequencies $\epsilon_l+\Omega_p$ and ϵ_l have opposite sign, which means that the ϵ_l is restricted to the range $-\Omega_p<\epsilon_l<0$. Upon changing the sign of ϵ_l we obtain

$$Q_{hh}^{\alpha\beta}(i\Omega_p)=-4\pi N(0)D\delta_{\alpha\beta}T\sum_{0<\epsilon_l<\Omega_p}(\epsilon_l-\Omega_p/2)^2, \qquad (6)$$

where $N(0)$ is the single-spin electronic density of states at the Fermi surface. We see that the response function is isotropic, so we drop the spatial indices. We can then perform the ϵ_l sum to obtain

$$Q_{hh}=-16\pi^3N(0)DT^3\sum_{l=0}^{p-1}(l+1/2-p/2)^2 =-\frac{4\pi^3}{3}N(0)DT^3(p^3-p), \qquad (7)$$

and we finally extract κ using Eq. (2) to get the Drude result

$$\kappa_0=\frac{2\pi^2}{3}N(0)DT=\frac{\pi^2 n\tau T}{3}, \qquad (8)$$

where n is the electron number density and we have used the Einstein relation $2N(0)D=n\tau/m$. From the corresponding Drude formula for electrical conductivity, $\sigma_0=ne^2\tau/m$, we see that the Wiedemann-Franz law is obeyed (note that $\hbar=1$ and $k_B=1$ in our calculations).

This diagrammatic technique offers a simple proof that the Wiedemann-Franz law is obeyed for a noninteracting disordered system with arbitrary disorder strength. For any diagrammatic contribution to electrical conductivity, σ there is a corresponding contribution to thermal conductivity, κ. Moreover, the corresponding expressions differ only in the form of the current vertices, yielding a factor $k_\alpha k_\beta e^2/m^2$ for σ, and $k_\alpha k_\beta[i(\epsilon_l+\Omega_p/2)]^2/m^2$ for κ. The only essential difference between κ and σ then lies in the frequency sums; the ratio of these two sums is independent of disorder and leads to the Wiedemann-Franz ratio. In particular, the weak localization correction to thermal conductivity is obtained directly from the Wiedemann-Franz law. Note that this proof relies on the Sommerfeld expansion since we are linearizing our energy integrals about the Fermi surface. Interaction effects can violate the Wiedemann-Franz law since the presence of an interaction line can alter the Matsubara frequencies at the two current vertices.

III. THE INTERACTION CORRECTIONS

In this section we calculate the interaction corrections arising from the singlet and triplet interaction channels[15]—we do not evaluate Cooperonic contributions since these are expected to be small for a system with repulsive interactions. In the singlet channel the dominant contribution arises from small energy and momentum transfers between electrons. This is dominated by the bare Coulomb interaction, which takes the form

$$V_0(q)=\begin{cases}\dfrac{4\pi e^2}{q^2}, & d=3\\[2mm] \dfrac{2\pi e^2}{q}, & d=2\\[2mm] e^2\ln\left(\dfrac{1}{q^2a^2}\right), & d=1\end{cases} \qquad (9)$$

where d is the dimensionality, and a is a measure of the transverse width in a quasi-one-dimensional wire. The disorder-screened singlet interaction then takes the form

$$V_s(q,i\omega_n)=\frac{1}{V_0(q)^{-1}+\Pi(q,i\omega_n)}, \qquad (10)$$

where the polarization operator $\Pi(q,i\omega_n)$ is given by

$$\Pi(q,i\omega_n)=2N(0)\frac{Dq^2}{Dq^2+|\omega_n|}. \qquad (11)$$

If any integrals we obtain involving $V_s(q,i\omega_n)$ are convergent at small momentum, q, we can ignore the $V_0(q)^{-1}$ term in Eq. (10) in comparison to $\Pi(q,i\omega_n)$; $V_s(q,i\omega_n)$ then takes the universal form

$$2N(0)V_s(q,i\omega_n)=\frac{Dq^2+|\omega_n|}{Dq^2}. \qquad (12)$$

In the triplet channel the dominant contribution arises from momentum transfers of the order of $2k_F$, and the unsceened triplet interaction can be treated as a constant. The disorder screened triplet interaction then takes the form

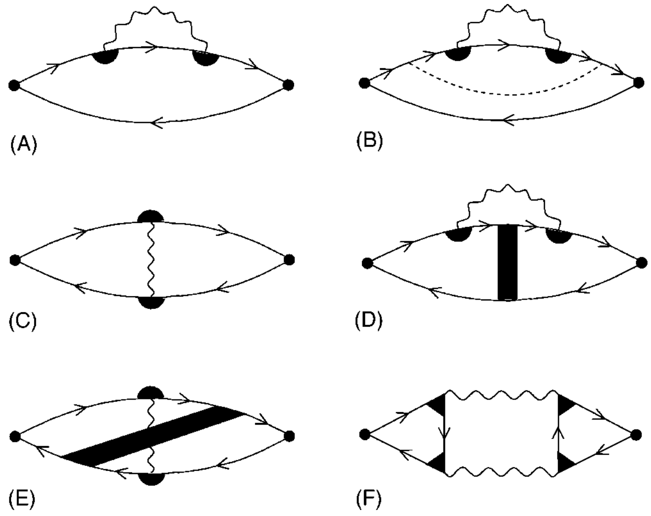

FIG. 2. Feynman diagrams for the leading order interaction corrections to the heat response kernel. Diagrams (A)–(E) are identical to those considered by Alshuler and Aronov for the electrical response; diagram (F) is the heat-drag contribution discussed in the text.

$$2N(0)V_t(q,i\omega_n)=F\frac{Dq^2+|\omega_n|}{(F+1)Dq^2+|\omega_n|}, \tag{13}$$

where $F=2N(0)V_t^0$ and V_t^0 is the bare interaction in the triplet channel.[16] Note that F includes Fermi liquid corrections in the absence of disorder (it is only unscreened with respect to the disorder). F may be determined from the measured paramagnetic spin susceptibility in the experimental system of interest

$$\chi=\frac{2N(0)}{1+F}. \tag{14}$$

If we first calculate the contribution from the singlet channel, we can then obtain the corresponding result for the triplet channel by replacing V_s by V_t and multiplying by 3—the extra factor of 3 arises because there are 3 times as many modes in the spin-one channel than in the spin-zero channel.

Applying standard perturbation theory we find that the leading order interaction corrections are given by the Feynman diagrams shown in Fig. 2. These diagrams are identical to those considered by Altshuler and Aronov for the electrical conductivity,[5] with the exception of diagram (F), which we call the heat-drag term. The corresponding diagram for electrical conductivity is a factor $(T/E_F)^2$ smaller than the other diagrams due to cancellation of electron and hole charge currents; the heat current has opposite electron-hole parity and thus electron and hole heat currents reinforce each other. Consequently diagram (F) is of the same size as the other diagrams in the case of thermal conductivity. The use of the Matsubara heat current vertex defined in Eq. (4) ensures that no interaction renormalization of vertices is needed, greatly simplifying the calculation with respect to other choices of heat current operator (obviously the final results must be independent of this choice).

Details of the calculation of the contributions from diagrams (A)–(F) to the heat-current response function, $Q_{hh}(i\Omega)$, are presented in the Appendix (note that in the following we suppress the subscripts on Matsubara frequencies $\Omega_p=2\pi Tp$ and $\omega_n=2\pi Tn$ for convenience). It is shown there that $Q_{hh}(0)$ vanishes, as it must for internal consistency, and that κT may be written as the limit of the expression

$$+4N(0)D\frac{T}{\Omega}\sum_{0<\omega\leqslant\Omega}\sum_{q}\omega^3\left[1-\frac{8}{3d}\frac{Dq^2}{(Dq^2+\omega)}\right]\frac{V(q,\omega)}{(Dq^2+\omega)^2} \tag{15}$$

$$-\frac{16}{3d}\pi^2T^2N(0)D\left[T\sum_{\omega>0}+\frac{T}{\Omega}\sum_{0<\omega\leqslant\Omega}\omega\right]\sum_{q}\frac{Dq^2V(q,\omega)}{(Dq^2+\omega)^3} \tag{16}$$

$$\begin{aligned}&+\frac{16}{d}N(0)^2DT\sum_{\omega>0}\sum_{q}\omega^3\left[\frac{2Dq^2V(q,\omega)}{(Dq^2+\omega)}\right.\\&\left.+\omega\frac{\partial V(q,\omega)}{\partial\omega}\right]\frac{Dq^2V(q,\omega)}{(Dq^2+\omega)^4}\end{aligned} \tag{17}$$

as Ω tends to zero. As explained in the Appendix, this statement is to be interpeted in the sense of first continuing $i\Omega_p$ to real frequencies, and then taking the limit of real frequency going to zero. Term (16) is just the interaction contribution to the electrical conductivity multiplied by the factor $\pi^2T^2/3e^2$. In other words, this is the contribution to κT predicted by the Wiedemann-Franz law. The other two terms thus lead to violation of the Wiedemann-Franz law if they yield a nonzero result (as we indeed find that they do). The expressions in terms (15)–(17) may now be evaluated in one, two, and three dimensions for both the singlet and triplet interactions to yield the results listed in Table I. Note that all terms (15)–(17) give results having the same parametric form but with different constant prefactors; we must therefore evaluate all of them to get a correct final result. This statement is not quite true in two dimensions because of the presence of logarithmic terms, and we should keep the most singular terms in this case.

IV. CONCLUSIONS

We have calculated the interaction corrections to thermal conductivity in the diffusive regime of a disordered conductor. Our main result is that the Wiedemann-Franz law is violated in all dimensions; the predicted interaction results have the expected parametric dependences, but different numerical coefficients. For example, in the singlet channel in two dimensions, the actual logarithmic correction is half that predicted by the Wiedemann-Franz law. Unfortunately the experimental work of Bayot *et al.*[13] is the only work we know of on thermal conductivity in disordered conductors, and this only isolates a weak localization contribution. We hope that further experiments will be performed in this area, and that the interaction correction be observed as well as the weak localization correction. In particular the two-dimensional system would seem to be a promising one to investigate. The disordered graphitic system of Bayot *et al.*[13] showed weak localization effects whose magnitude was roughly 20% of the Drude term in the electronic thermal conductivity at a temperature of 2.9 K. If it were possible to cleanly extract the phonon term, one could look at the electronic term as a function of both temperature and magnetic field, and distinguish weak localization and interaction effects. This would then experimentally settle the question of whether the Wiedemann-Franz law is violated in an interacting disordered system. Another sensible quantity to investigate experimentally would be the thermal Hall conductivity, which arises solely from electrons (although there can be phonon drag effects). In future work we intend to derive the interaction corrections to this quantity.

Note added. Recently, two works have appeared by Raimondi *et al.*[17] and Catelani and Aleiner,[18] which independently confirm our results.

ACKNOWLEDGMENTS

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APPENDIX: DIAGRAMMATIC CONTRIBUTIONS

In this appendix we list the complete set of diagrammatic contributions for archival purposes. This should allow interested researchers to reproduce our results in detail. The total contribution to the heat-current response function, $Q_{hh}(i\Omega)$, from all the diagrams in Fig. 2 is given by

$$
\begin{aligned}
&-8\pi N(0)D\delta_{\alpha\beta}T\sum_{\omega>0}T\sum_{0<\epsilon<\omega}\sum_{q}\left[\epsilon+\frac{\Omega}{2}\right]^2\frac{V(q,\omega)}{(Dq^2+\omega)^2}\quad\text{(A1)}\\
&-8\pi N(0)D\delta_{\alpha\beta}T\sum_{\omega>\Omega}T\sum_{0<\epsilon<\omega-\Omega}\sum_{q}\left[\epsilon+\frac{\Omega}{2}\right]^2\frac{V(q,\omega)}{(Dq^2+\omega)^2}\quad\text{(A2)}\\
&+8\pi N(0)D\delta_{\alpha\beta}T\sum_{0<\omega\leqslant\Omega}T\sum_{0<\epsilon<\omega}\sum_{q}\left[\epsilon-\frac{\Omega}{2}\right]^2\frac{V(q,\omega_n)}{(Dq^2+\omega)^2}\quad\text{(A3) + (B)}\\
&+8\pi N(0)D\delta_{\alpha\beta}T\sum_{\omega>\Omega}T\sum_{0<\epsilon<\Omega}\sum_{q}\left[\epsilon-\frac{\Omega}{2}\right]^2\frac{V(q,\omega)}{(Dq^2+\omega)^2}\quad\text{(A3) + (B)}\\
&+16\pi N(0)D\delta_{\alpha\beta}T\sum_{\omega>\Omega}T\sum_{0<\epsilon<\omega-\Omega}\sum_{q}\left[\epsilon+\frac{\Omega}{2}\right]\left[\epsilon+\frac{\Omega}{2}-\omega\right]\frac{V(q,\omega)}{(Dq^2+\omega)^2}\quad\text{(C1) + (C2)}\\
&+32\pi N(0)DT\sum_{\omega>0}T\sum_{0<\epsilon<\omega}\sum_{q}\left[\epsilon+\frac{\Omega}{2}\right]^2\frac{Dq_\alpha q_\beta V(q,\omega)}{(Dq^2+\omega)^2(Dq^2+\omega+\Omega)}\quad\text{(D1)}\\
&+32\pi N(0)DT\sum_{\omega>\Omega}T\sum_{0<\epsilon<\omega-\Omega}\sum_{q}\left[\epsilon+\frac{\Omega}{2}\right]^2\frac{Dq_\alpha q_\beta V(q,\omega)}{(Dq^2+\omega)^2(Dq^2+\omega-\Omega)}\quad\text{(D2)}\\
&-32\pi N(0)DT\sum_{\omega>\Omega}T\sum_{0<\epsilon<\omega-\Omega}\sum_{q}\left[\epsilon+\frac{\Omega}{2}\right]\left[\epsilon+\frac{\Omega}{2}-\omega\right]\frac{Dq_\alpha q_\beta V(q,\omega)}{(Dq^2+\omega)^2(Dq^2+\omega+\Omega)}\quad\text{(E1)}\\
&-32\pi N(0)DT\sum_{\omega>\Omega}T\sum_{0<\epsilon<\omega-\Omega}\sum_{q}\left[\epsilon+\frac{\Omega}{2}\right]\left[\epsilon+\frac{\Omega}{2}-\omega\right]\frac{Dq_\alpha q_\beta V(q,\omega)}{(Dq^2+\omega)^2(Dq^2+\omega-\Omega)}\quad\text{(E2)}\\
&+16N(0)^2DT\sum_{\omega>0}\sum_{q}\omega^2(\omega+\Omega)^2\frac{Dq_\alpha q_\beta V(q,\omega)V(q,\omega+\Omega)}{(Dq^2+\omega)^2(Dq^2+\omega+\Omega)^2}\quad\text{(F)}
\end{aligned}
\tag{A1}
$$

In the above expression we have listed the contributions according to which diagram they are obtained from; the additional numerical label following the letter refer to different sign configurations of Matsubara frequencies that are possible within the same diagram. For example, in diagram (A), there are three possible sign configurations whose contributions we denote by ((A1)), ((A2)), and ((A3)). To proceed further we next perform the sums over the Fermi Matsubara frequency, ϵ. After some simplification we then obtain the following result for $Q_{hh}(i\Omega)$:

$$\begin{aligned}
& -4N(0)D\delta_{\alpha\beta}T\sum_{\omega>\Omega}\sum_{q}\omega^3\frac{V(q,\omega)}{(Dq^2+\omega)^2}-4N(0)D\delta_{\alpha\beta}T\sum_{0<\omega\leqslant\Omega}\sum_{q}\omega^2\Omega\frac{V(q,\omega)}{(Dq^2+\omega)^2} \quad \text{(ABC)}\\
& +16N(0)DT\sum_{\omega>\Omega}\sum_{q}\omega^2(\omega-\Omega)\frac{Dq_\alpha q_\beta V(q,\omega)}{(Dq^2+\omega)[(Dq^2+\omega)^2-\Omega^2]} \quad \text{(DE)}\\
& +\frac{4}{3}N(0)DT\sum_{\omega>\Omega}\sum_{q}[\Omega^3+12\omega^2\Omega-4\pi^2T^2\Omega]\frac{Dq_\alpha q_\beta V(q,\omega)}{(Dq^2+\omega)^2(Dq^2+\omega+\Omega)} \quad \text{(DE)}\\
& +\frac{4}{3}N(0)DT\sum_{0<\omega\leqslant\Omega}\sum_{q}[4\omega^3+6\omega^2\Omega+3\omega\Omega^2-4\pi^2T^2\omega]\frac{Dq_\alpha q_\beta V(q,\omega)}{(Dq^2+\omega)^2(Dq^2+\omega+\Omega)} \quad \text{(DE)}\\
& +16N(0)^2DT\sum_{\omega>0}\sum_{q}\omega^2(\omega+\Omega)^2\frac{Dq_\alpha q_\beta V(q,\omega)V(q,\omega+\Omega)}{(Dq^2+\omega)^2(Dq^2+\omega+\Omega)^2} \quad \text{(F)}
\end{aligned} \tag{A2}$$

Note that the contributions from diagrams (A), (B), and (C) do not cancel each other as they do in the corresponding electrical conductivity calculation. At this point we can check that we have chosen a consistent set of diagrams to evaluate by setting the external frequency $\Omega=0$ and checking that $Q_{hh}(0)=0$. This is a very powerful test that should not be omitted—it is very dangerous to merely calculate $[Q_{hh}(i\Omega)-Q_{hh}(0)]/\Omega$, especially when a large number of diagrams are involved. One always runs the risk of missing important physical processes; in fact we only became aware of the presence of the heat-drag term of diagram (F) when this check failed in its absence. Setting $\Omega=0$ here yields

$$\begin{aligned}
Q_{hh}(0) &= 4N(0)DT\sum_{\omega>0}\sum_{q}\left[-\delta_{\alpha\beta}+\frac{4Dq_\alpha q_\beta}{(Dq^2+\omega)}+\frac{4\omega Dq_\alpha q_\beta N(0)V(q,\omega)}{(Dq^2+\omega)^2}\right]\frac{\omega^3V(q,\omega)}{(Dq^2+\omega)^2}\\
&= -2T\sum_{\omega>0}\sum_{q}\omega^2\frac{\partial^2}{\partial q_\alpha\partial q_\beta}\ln[V_0(q)^{-1}+\Pi(q,\Omega)]=0,
\end{aligned} \tag{A3}$$

where in the last step we have used the divergence theorem to convert the q integral into a surface integral with its bounding surface at infinity. Note that this check not only gives us confidence that all relevant diagrams have been included, but also that each diagram has been given the correct combinatorial factors. In the above derivation we have treated $V_0(q)^{-1}$ as though it had no q dependence. This is justified as the terms ignored are smaller in magnitude, and would be canceled by higher-order diagrams. In addition to the above proof that $Q_{hh}(0)=0$, we have also directly evaluated $Q_{hh}(0)$ from Eq. (A3) in one, two, and three dimensions, for both singlet and triplet potentials, and found it to equal to zero.

We can then find κT as the limit of $Q_{hh}(\Omega_p)/\Omega_p$ as Ω_p tends to zero. Now of course this statement does not make sense since Ω_p is discrete—we should first analytically continue to real frequencies and then take the limit. However, it turns out to be possible to avoid performing this analytic continuation, and instead to continue manipulating discrete sums. We can then extract the term that is proportional to Ω_p and discard terms proportional to higher powers of Ω_p. This approach is legitimate provided that any operation we perform on Ω_p would carry over unchanged to the same operation on the corresponding real frequency after analytic continuation. The advantage of this method is simply convenience in calculation—no illegal operations occur, as we have checked by performing the analytic continuation first and then taking the zero-frequency limit. This procedure leads to the expression for κT as the zero-frequency limit of

$$+ 4N(0)D\frac{T}{\Omega}\sum_{0<\omega\leqslant\Omega}\sum_{q}\omega^3\left[1-\frac{8}{3d}\frac{Dq^2}{(Dq^2+\omega)}\right]\frac{V(q,\omega)}{(Dq^2+\omega)^2} \tag{A4}$$

$$-\frac{16}{3d}\pi^2T^2N(0)D\left[T\sum_{\omega>0}+\frac{T}{\Omega}\sum_{0<\omega\leqslant\Omega}\omega\right]\sum_{q}\frac{Dq^2V(q,\omega)}{(Dq^2+\omega)^3} \tag{A5}$$

$$+\frac{16}{d}N(0)^2DT\sum_{\omega>0}\sum_{q}\omega^3\left[\frac{2Dq^2V(q,\omega)}{(Dq^2+\omega)}+\omega\frac{\partial V(q,\omega)}{\partial\omega}\right]\frac{Dq^2V(q,\omega)}{(Dq^2+\omega)^4}. \tag{A6}$$

We can then perform the q integral using the standard identification

$$\sum_{q}=\int\frac{d^dq}{(2\pi)^d} \tag{A7}$$

and the formulas for the $V(q,i\omega)$ given in Eq. (10) and Eq. (13). After performing this q integral we end up with terms that are infinite sums over powers of ω. Provided that the sums are ultraviolet divergent, we can identify them as zeta functions via

$$T\sum_{\omega>0}\omega^k=T(2\pi T)^k\sum_{n>0}n^k=T(2\pi T)^k\zeta(-k). \tag{A8}$$

Any sum that is infrared divergent must be cut off correctly since such a divergence is physical—such a situation occurs in two dimensions where we obtain a logarithmic sum that is cut off at $\omega\sim T$ at low frequency and $\omega\sim 1/\tau$ at high frequency. We have used this particular divergent series trick as it allows a very direct evaluation of results. As a check of its legality we have recalculated the various terms using standard analytic continuation methods and obtained the same results—albeit after a lot more algebraic manipulation.

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