Theory of quantum grid turbulence in superfluid ^{3}He-*B*

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A theory is developed to describe grid turbulence in a superfluid in the case where the normal fluid is held stationary, as would be the case for superfluid ^{3}He-*B* in which the normal fluid is very viscous. The theory is a straightforward development of earlier work, reviewed by Vinen and Niemela [J. Low Temp. Phys. **128**, 167 (2002)], and it shows that on large length scales the turbulence is strongly damped by mutual friction. A comparison is made with recent work by Volovik and his colleagues [G. E. Volovik, JETP Lett. **78**, 533 (2003); G. E. Volovik, J. Low Temp. Phys. (to be published); and L'vov, Nazarenko, and Volovik, JETP Lett. (to be published)], which was developed while our work was in progress.

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I. INTRODUCTION

Superfluid, or quantum, turbulence continues to attract both experimental and theoretical study. A special form of superfluid turbulence that accompanies a thermal counterflow in superfluid ^{4}He at temperatures above about 1 K was discovered in the 1950s;[1] it takes the form of a self-sustaining tangle of vortex lines in the superfluid component, the tangle being maintained by the force of mutual friction that acts between the lines and the normal fluid. This form of turbulence has no classical analogue. Much more recently, there has been study of the analogue of classical grid turbulence in which the turbulence is produced by steady motion of a grid through the fluid. In the classical case such turbulence is approximately homogeneous and isotropic. At high Reynolds number it is characterized by a Richardson cascade in which energy flows in a cascade from large eddies, containing most of the turbulent energy, to small eddies in which it is dissipated by viscosity. In the steady state and over the range of eddy size that is not significantly affected by viscosity (the *inertial range*), the energy is distributed among different eddy sizes according to the Kolmogorov spectrum[2]

$$E(k) = C\epsilon^{2/3}k^{-5/3}, \tag{1}$$

where $E(k)dk$ is the energy per unit mass associated with wave numbers with magnitudes in the range dk (we are working in terms of a spatial Fourier transform of the velocity field). The Kolmogorov constant C is of order unity, and ϵ is the rate at which energy per unit mass flows down the cascade (energy being conserved in the inertial regime). The Kolmogorov spectrum does not provide an exact description of the turbulence, but it is sufficient for our present purposes.

Extensive experimental study of fully developed grid turbulence in superfluid ^{4}He above 1 K has now been published,[3] along with theoretical discussion of the results.[3–6] It turns out that the quantum and classical cases are very similar. Turbulence in the superfluid component must again involve a tangle of quantized vortices, with a characteristic spacing that we denote by ℓ. For $k\ell \ll 1$ there is a *quasiclassical* inertial range in which the two fluids have the same turbulent velocity field, this matching of the two velocity fields ensuring that there is negligible dissipation from mutual friction. Flow of the superfluid on a scale greater than ℓ is achieved by partial polarization of the vortex tangle. The magnitude of the normal-fluid viscosity is, accidentally, such that eddies in the normal fluid suffer negligible viscous dissipation as long as they are larger than ℓ. For eddies with size of order, or less than, ℓ the velocity fields in the two fluids are no longer matched because that in the superfluid component is strongly dominated by the quantization of circulation, so that there is then dissipation by mutual friction, accompanied by viscous dissipation in the normal fluid. Thus the basic structure of grid turbulence is the same in both the quantum and classical cases: an inertial range with a single velocity field, described by the Kolmogorov spectrum, for $k\ell \ll 1$, and dissipation for $k\ell \gg 1$, dissipation in the quantum case being due to a combination of mutual friction and normal-fluid viscosity. The existence of the quasiclassical Kolmogorov spectrum is strongly supported by experiment,[3,7] although there is as yet no formal theoretical demonstration that the superfluid component must behave classically on scales greater than ℓ.

This simple form of quantum grid turbulence owes its existence to a very small normal-fluid viscosity. Recent attention has turned to possible forms of turbulence in superfluid ^{3}He-*B*, which is probably similar in the present context to superfluid ^{4}He, except that the normal fluid is extremely viscous. Flow through a grid could therefore lead to turbulence in the superfluid component, but hardly in the normal component. The normal fluid must therefore remain essentially at rest, and a force of mutual friction must act on the superfluid component on all length scales. In this paper we aim to discuss the possible character of grid turbulence in this case. Discussions relevant to our subject have recently been published by Volovik and his collaborators,[8–10] and we shall compare their conclusions with our own. In order to facilitate comparison our mathematical formulation will, where appropriate, be similar to that used by Volovok.

In Sec. II we discuss briefly the ideas underlying grid turbulence in a classical fluid because important aspects of these ideas carry over to the quantum case. Quantum grid turbulence can be modified by the presence of a stationary normal fluid in two ways: it can introduce a damping of the superfluid turbulence through the action of mutual friction;

and it can lead to the generation of extra turbulence in the manner that operates in a counterflow heat current. We discuss these matters in Sec. III: the form of the mutual friction is discussed first (Sec. III A); then the effect of the damping in a way that is only semi-quantitative but which emphasizes the essential physics (Sec. III B); and then in a way that leads to more quantitative predictions (Secs. III C and III D). In Sec. III E we show that the generation of extra turbulence is likely to be unimportant, and in Sec. III F we summarize our conclusions. In Sec. IV we compare our results with those of Volovik; earlier papers by Volovik[8,9] led to conclusions very different from our own, although this difference seems to have disappeared in the most recent paper.[10]

In order to simplify the discussion we shall assume that the grid turbulence exists in a steady state. That is, we assume that energy is injected continuously into the largest eddies, as a rate equal to the ultimate rate of dissipation of energy, so that the characteristics of the cascade, such as its energy spectrum, are independent of time. We believe that the essential physics remains the same if, as is often the case, the turbulence is decaying; indeed, this belief underlies the existing discussions of grid turbulence in superfluid ^{4}He.

II. GRID TURBULENCE IN A CLASSICAL FLUID

We start with some discussion of classical grid turbulence at high Reynolds number, which will allow us to formulate some general ideas that we can apply later to the quantum case. Flow of the classical fluid obeys the Navier-Stokes equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \boldsymbol{u}, \tag{2}$$

where ν is the kinematic viscosity. We suppose that the grid produces eddies on a length scale R and with characteristic velocity U, at a high Reynolds number, $\mathrm{Re}=UR/\nu$. Owing to the action of the nonlinear term in the Navier-Stokes equation, energy is transferred from the scale R to smaller scales, which we denote by r. This process is believed to occur in a cascade, transfer taking place in steps with gradually decreasing r. A large Reynolds number, means that the nonlinear term in the Navier-Stokes equation is more important than the viscous term, so that this transfer takes place initially with conservation of energy; i.e., we have an *inertial regime*. Eventually, energy passes to eddies sufficiently small that viscous dissipation can operate and the cascade is terminated.

We see from the form of the Navier-Stokes equation that the characteristic time associated with the nonlinear transfer of energy from eddies of size r and characteristic velocity u_r is the "turnover time" $\tau_r = r/u_r$. It follows that eddies of size r lose energy per unit mass at a rate of order

$$\epsilon = \frac{u_r^2}{\tau_r} = \frac{u_r^3}{r}. \tag{3}$$

In a steady-state inertial cascade, this rate of loss (transfer) of energy must be the same for all r. It follows that within the inertial regime

$$u_r = \epsilon^{1/3} r^{1/3}, \tag{4}$$

apart from a constant of order unity. We note that $u_r^2/2$, which is the energy per unit mass associated with eddies of size r, can be obtained from the Kolmogorov spectrum [Eq. (1)] by integrating $E(k)$ over a range of k of order r^{-1} centered on $k=r^{-1}$, which leads to $u_r^2 \sim \epsilon^{2/3} r^{2/3}$, in agreement with Eq. (4).

This steady-state inertial cascade can exist only as long as it is not disrupted by viscous dissipation. Now the time taken for viscous dissipation to damp out motion on the length scale r is given by

$$\tau_d = \frac{r^2}{\nu}. \tag{5}$$

It follows that the inertial-range cascade can exist only as long as

$$\tau_d \gg \tau_r. \tag{6}$$

This condition can be written

$$\mathrm{Re}_r \gg 1, \tag{7}$$

where the "scale-dependent Reynolds number" Re_r is given by

$$\mathrm{Re}_r = \frac{r u_r}{\nu}. \tag{8}$$

Viscous dissipation sets in when $\mathrm{Re}_r \sim 1$, i.e., when

$$r = r_d = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}. \tag{9}$$

The scale r_d is the Kolmogorov dissipation length at which viscous dissipation sets in, and the inertial cascade is terminated.

This argument does not allow us to understand the precise way in which the Kolmogorov spectrum is modified as the scale of the turbulence approaches the Kolmogorov dissipation length. A more complete argument can be based on the idea that the flow of energy toward larger wave numbers can be described as a diffusion of energy in k space. It can be shown that this diffusion must obey a nonlinear diffusion equation, which can be written, for the case of homogeneous, isotropic turbulence, in the form (for a recent application of this idea see)[11]

$$\frac{1}{k^2}\frac{d}{dk}\left(k^2 D(k)\frac{dE'}{dk}\right) = \nu k^2 E'(k), \tag{10}$$

where $E'(k)d^3k$ is the turbulent energy associated with wave vectors in the range d^3k, and the diffusion coefficient $D(k)$ is given by

$$D(k) = (E'(k))^{1/2} k^{9/2}. \tag{11}$$

Such a diffusion equation does not have a rigorous basis and must be used with caution, but it helps to understand the processes that are occurring. We note that the $E(k)$ of Eq. (1) is related to $E'(k)$ by the relation $E(k)=4\pi k^2 E'(k)$. It is easy to show analytically that, in the limit $k \ll r_d^{-1}$, Eq. (10) yields the Kolmogorov spectrum (1), where $\epsilon = 4\pi k^2 D(k) dE'/dk$ is the constant rate at which energy flows down the inertial-range cascade; but a full numerical solution yields also the form of the cutoff as k approaches r_d^{-1}.

III. SUPERFLUID GRID TURBULENCE WITH A STATIONARY NORMAL FLUID

We first discuss the form of the mutual friction (Sec. III A) and then formulate an argument that is analogous to that leading to Eq. (9), but taking into account the mutual friction acting on the turbulence in the superfluid component (Sec. III B). Our conclusions are confirmed and elaborated, first in Sec. III C, where we present an analysis based on an analog of the diffusion equation (10), and then in Sec. III D, where we discuss the behavior on very small length scales.

A. The force of mutual friction

Let us consider a coarse-grained average superfluid velocity u, obtained by averaging over small volumes δV that are significantly larger than ℓ^3, where ℓ is the mean spacing between the vortex lines in the turbulent superfluid. We make the reasonable assumption the vortex lines within δV move with an average velocity that is close to u, but we make no assumption about the configuration of vortex lines within δV. (In this respect our approach is different from that of Volovik, as we discuss later.) The resulting force of mutual friction per unit volume can then be written as

$$f = \alpha \rho_s \kappa L u, \tag{12}$$

where κ is the quantum of circulation ($h/2m_3$ for superfluid ^{3}He-B), and $L = \ell^{-2}$ is the length of vortex line per unit volume. The force f is antiparallel to u; for the sake of simplicity we have neglected any component of the mutual friction perpendicular to u (it is not obvious how to calculate this transverse component in the case when the vortex lines are more or less random). The value of the dimensionless parameter α allows for any averaging over different vortex orientations within δV.

It is important to understand the nature of the frictional force f. We shall assume that in the steady state to which we are confining our attention, the length L of vortex line per unit volume is essentially constant, apart from small fluctuations, the effect of which we shall mention later. There is little doubt that this assumption is justified in the case when most of the vorticity in concentrated at the largest wave numbers (smallest eddies), so that the configurations of vortex line within the volume ℓ^3 are almost random, the large-scale motion being produced by a relatively small polarization of the random tangle. The random tangle evolves on length scales and time scales that are much smaller than those involved in the evolution of turbulence on scales larger than ℓ, so that fluctuations affecting the large scale motion are relatively small. Although the vorticity is indeed usually concentrated at the largest wave numbers, there are, as we shall see later, circumstances where this condition is violated; in this case we must reexamine our assumptions, as explained in Sec. III C.

If the fluctuations are indeed small, the whole factor multiplying u in Eq. (12) is essentially constant, so that the frictional force is linear in u. This important point was first discussed very soon after it was recognized that mutual friction is due to vortex lines.[1,12] We are interested in the effect of the mutual friction on turbulent motion on different length scales. We can therefore usefully carry out a Fourier analysis of the velocity field, so that a Fourier component u_k relates to motion on the length scale k^{-1}. We see immediately that the effect of mutual friction on this Fourier component is described by the linear equation

$$\dot{u}_k = -\gamma u_k, \tag{13}$$

where $\gamma = \alpha\kappa L$. It follows that the mutual friction causes u_k to decay with a time constant given by

$$\tau_d' = \frac{1}{\gamma}, \tag{14}$$

which is independent of k. In other words, mutual friction gives rise to the decay of an eddy of size r at a rate that is independent of r. The corresponding dissipation per unit mass per unit wave number, given by

$$\epsilon_k^{(f)} = \gamma u_k^2, \tag{15}$$

is also independent of length scale, although it is proportional to u_k^2. We can easily convert Eq. (15) to give a rate of dissipation of energy per unit mass in an eddy of size r

$$\epsilon_r^{(f)} = \gamma u_r^2. \tag{16}$$

In some of Volovik's papers he seems to have argued that Eq. (16) should have a fundamentally different form. We have therefore included in the Appendix an argument confirming our own view and based on a Fourier transform of the equation of motion used by Volovik himself.

B. Comparison of characteristic times

We shall now proceed with an argument that parallels that in Sec. II. Let us assume that in the absence of dissipation there is an inertial-range cascade in the superfluid component, with a Kolmogorov spectrum. Taking into account dissipation due to mutual friction, we suggest, in analogy to Eq. (6), that the inertial-range cascade will still exist provided that

$$\tau_d' \gg \tau_r, \tag{17}$$

where $\tau_r = r/u_r$. This condition can be written in terms of an effective scale-dependent Reynolds number as

$$\mathrm{Re}'_r = \frac{u_r}{\gamma r} \gg 1. \tag{18}$$

Substituting for u_r from the Kolmogorov spectrum (4), we find that

$$\mathrm{Re}'_r = \frac{\epsilon^{1/3}}{\gamma r^{2/3}}. \tag{19}$$

Thus we arrive at the rather curious result that the effective

Reynolds number *decreases* with increasing r. It appears therefore that dissipation will be important in eddies *larger* than the size given by

$$r = r'_d = \left(\frac{\epsilon}{\gamma^3}\right)^{1/2}. \qquad (20)$$

Therefore we conclude that an inertial-range cascade (no significant dissipation) can exist only on *length scales that are significantly less than* r'_d.

A grid moving in a classical fluid with velocity U_0 produces eddies at an early stage with a characteristic velocity U (proportional to, but a little less than, U_0) and size of order the mesh size M of the grid, although there is subsequently some flow of energy into larger eddies as well as the flow into smaller eddies. Flow into the larger eddies saturates because eddies cannot be larger than the size of the channel in which the grid moves. In superfluid ^{4}He a similar situation obtains, although the large eddies must in this case be formed from arrays of vortex lines that are partly polarized. Eventually an inertial-range Kolmogorov spectrum is established for wave numbers smaller than the inverse of the Kolmogorov dissipation length in the classical case or the inverse of the vortex-line spacing ℓ in superfluid ^{4}He. Our analysis so far suggests strongly that, for the case of a stationary normal fluid, such an inertial-range Kolmogorov spectrum cannot be established if the largest eddies have a linear size greater than r'_d. If, however, the width of the channel in which the grid moves is such that the largest eddies have a linear size less significantly less than r'_d, then it seems likely that an inertial-range Kolmogorov spectrum can still be established, the mutual friction having only a small effect. In Sec. III C we confirm and extend these conclusions by solving the appropriate diffusion equation for the flow of energy in k space. If an inertial-range Kolmogorov spectrum can indeed exist over a range of wave numbers, even in the presence of a stationary normal fluid, then we must ask how the turbulent energy, flowing at rate ϵ down the inertial-range cascade, is eventually dissipated; we address this question in Sec. III D.

C. Use of the k-space diffusion equation

In Sec. II we outlined a treatment of the Kolmogorov spectrum in a classical fluid and the cutoff resulting from the action of viscosity, based on a nonlinear diffusion equation that aimed to describe the diffusive flow of turbulent energy toward higher wave numbers. We can treat the case of superfluid turbulence in the presence of a stationary normal fluid by a similar method, as we now describe. As in the classical case, we warn that this treatment is not rigorous.

The appropriate diffusion equation, analogous to Eq. (10), has the form

$$\frac{1}{k^2}\frac{d}{dk}\left(k^2 D(k)\frac{dE'(k)}{dk}\right) = \frac{E'(k)}{\tau'_d} = \gamma E'(k), \qquad (21)$$

as we see immediately from Eq. (15). We have assumed that the turbulence is isotropic, so that E' is a function of k only. If there were no mutual friction the solution of this equation

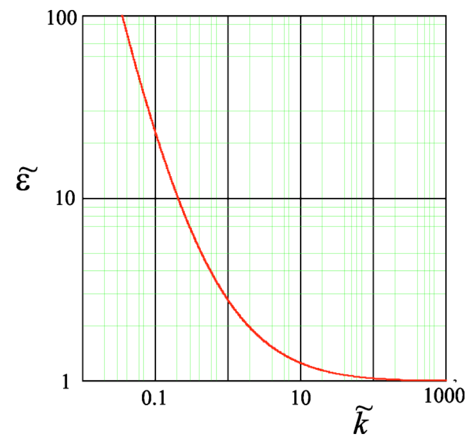

FIG. 1. (Color online) The solution of Eq. (25) with the boundary conditions explained in the text.

is easily shown to be of the Kolmogorov form

$$E'(k) = \left(\frac{3}{44\pi}\right)^{2/3}\epsilon^{2/3}k^{-11/3} = 0.078\epsilon^{2/3}k^{-11/3}, \qquad (22)$$

where ϵ is the rate of flow of energy per unit mass of superfluid down the cascade. [We are assuming that Eq. (21), together with Eq. (11) for the diffusion coefficient, holds in the presence of the damping due to mutual friction. The validity of these equations depends probably on the idea that the inertial interactions responsible for the transfer of energy in a cascade are local in k-space, and we see no reason why mutual friction should invalidate this idea.] We write the solution of Eq. (21) in the form

$$E'(k) = 0.078(\epsilon(k))^{2/3}k^{-11/3}, \qquad (23)$$

where $\epsilon(k)$ is now a function of k, a decrease in $\epsilon(k)$ with increasing k being due to the loss of energy through mutual friction. We introduce dimensionless variables $\tilde{\epsilon}(\tilde{k})$ and $\tilde{k}$, defined by

$$\tilde{\epsilon}(\tilde{k}) = \frac{\epsilon(k)}{\epsilon(\infty)}; \quad \tilde{k} = (\epsilon(\infty))^{1/2}\gamma^{-3/2}k, \qquad (24)$$

where $\epsilon(\infty)$ is the value of $\epsilon(k)$ in the limit $k \to \infty$, and we find that $\tilde{\epsilon}(\tilde{k})$ obeys the differential equation

$$\frac{d^2\tilde{\epsilon}(\tilde{k})}{d\tilde{k}^2} - \left(\frac{4.50}{\tilde{k}}\right)\frac{d\tilde{\epsilon}(\tilde{k})}{d\tilde{k}} = 5.38\tilde{k}^{-8/3}(\tilde{\epsilon}(\tilde{k}))^{2/3}. \qquad (25)$$

This equation can be solved numerically by the fourth-order Runge-Kutta method, and we require that the solution does not diverge as $\tilde{k} \to \infty$. A satisfactory solution exists only if $\epsilon(\infty)$ is nonzero, and we make the reasonable assumption that as $\tilde{k} \to \infty$, $d\tilde{\epsilon}(\tilde{k})/d\tilde{k} \to 0$. The required solution is shown in Fig. 1.

To understand the significance of this solution, suppose that turbulent energy is injected at the length scale R, where this scale is of order the width of the channel in which the grid moves. Eddies larger than R cannot be generated, and therefore the minimum value of k is $k_0 = R^{-1}$. In Fig. 2 we

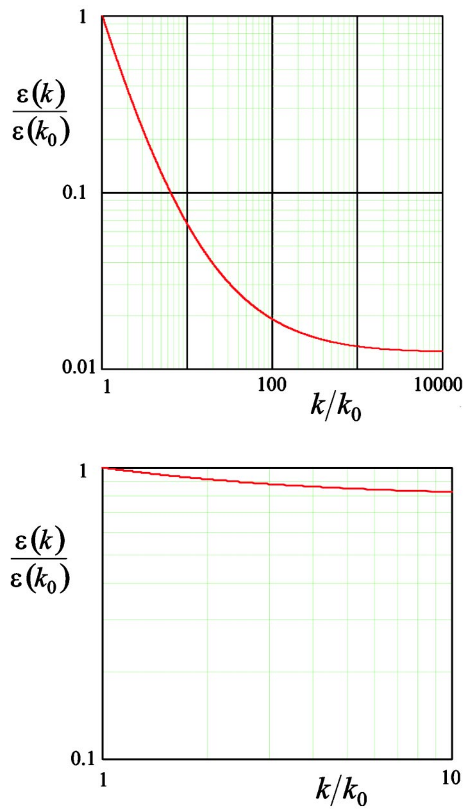

FIG. 2. (Color online) $\epsilon(k)/\epsilon(k_0)$ plotted against k/k_0 for two values of $Q=R\gamma^{3/2}(\epsilon(\infty))^{-1/2}$. Upper graph: $Q=25$; lower graph: $Q=0.1$.

plot the value of $\epsilon(k)/\epsilon(k_0)$ against k/k_0, derived from Fig. 1, for two extreme cases: $R \gg (\epsilon(\infty)/\gamma^3)^{1/2}$; and $R \ll (\epsilon(\infty)/\gamma^3)^{1/2}$. A Kolmogorov spectrum would correspond to $\epsilon(k)/\epsilon(k_0)=1$ [i.e., $d\epsilon(k)/dk=0$ for all k], and in both cases the energy input at k_0 is of order U^3/R. We see that in the former case there is a large, k-dependent reduction in $\epsilon(k)$, corresponding to a large dissipative reduction in $E(k)$ below the Kolmogorov value; while in the latter case the Kolmogorov spectrum is hardly changed. This is in accord with our conclusions in Sec. III A. We note, however, that in the former case $\epsilon(k)$ tends to a small, but finite constant when k is sufficiently large, so that the Kolmogorov form still holds at these high values of k ($\tilde{k} \gg 1$).

We must, however, make a comment about the distribution of vorticity with wave number. In Fig. 3 we plot $\tilde{k}^{1/3}\tilde{\epsilon}^{2/3}$, which is proportional to the *spectrum* of the square of the vorticity, against $\tilde{k}$. The total mean-square vorticity associated with a range of length scales is obtained by integrating this spectrum over the corresponding range of wave numbers. It is then easily seen that for some values of k and some conditions (small k_0, large mutual friction) the total root-mean-square vorticity (roughly the vortex line density) associated with wave numbers less than k can be comparable with that associated with wave numbers greater than k (remember that k cannot exceed ℓ^{-1}, as explained in Sec. III D). Our treatment of the effect of mutual friction at wave number k (including that in the Appendix) is probably valid only if

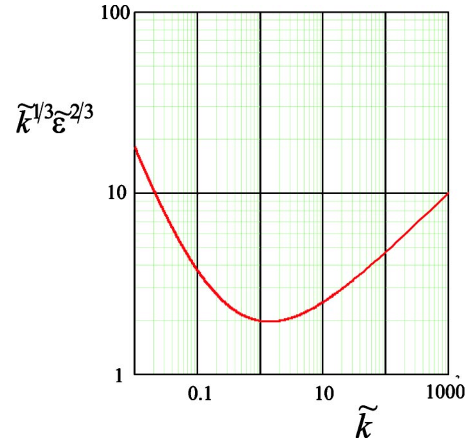

FIG. 3. (Color online) Variation of vorticity with wave number.

the root-mean-square vorticity is concentrated at wave numbers significantly greater than k. Otherwise, mutual friction as it affects the quasiclassical eddies at wave number k may undergo large fluctuations that reflect the time evolution of eddies larger than k^{-1} (including their decay by mutual friction), these fluctuations involving not only the line density but also the orientation of polarized line configurations (remember that the force of mutual friction is anisotropic with respect to this orientation relative to the flow). This situation requires further study. Qualitatively, the time constant τ_d' of Eq. (14) may still be relevant, but there may be complicated coupling between motion on different length scales (see Appendix).

D. Behavior at large wave numbers

We have based our discussion thus far on a course-grained average superfluid velocity, obtained by averaging over small volumes δV that are significantly larger than ℓ^3; therefore, our conclusions apply only on length scales down to values a little in excess of ℓ. Indeed, it is clear that superfluid turbulence of the type described in Secs. III B and III C, which we describe as *quasiclassical*, can exist only on length scales significantly larger than ℓ (wave numbers significantly smaller than ℓ^{-1}). Otherwise the scale of the turbulence becomes comparable with or less than the vortex-line spacing, so that the motion is dominated by the quantization of circulation or, equivalently, by the discrete nature of quantized vortex lines.[4,6] In particular there can no longer be a Kolmogorov spectrum, even when the mutual friction is very small. In this section we discuss how the turbulence can be expected to behave as the wave number k approaches and then exceeds the value ℓ^{-1}.

We shall suppose initially that the mutual friction is not too large, so that the condition $\tilde{k} \gg 1$ holds when $k \sim \ell^{-1}$; in that case, a Kolmogorov spectrum can be expected to hold over a reasonably wide range of wave numbers significantly less than ℓ^{-1}. On the scale ℓ, however, the characteristic velocity can no longer be obtained from the Kolmogorov spectrum, but rather must be given by[6]

$$u_\ell^2 = \frac{\beta\kappa^2}{\ell^2}, \tag{26}$$

where the numerical factor β is of order unity. Now we make the reasonable assumption that the velocity u_r in the Kolmogorov spectrum of Eq. (4) joins smoothly to Eq. (26) when $r=\ell$.[6] Thus we find that

$$\epsilon = \beta^{3/2}\kappa^3\ell^{-4}, \tag{27}$$

where ϵ is now the energy flux in the region of the Kolmogorov spectrum $\tilde{k}>1$. It follows that we can write for the velocity in the quasiclassical inertial range (Kolmogorov spectrum)

$$u_r^2 = \beta\left(\frac{\kappa^2}{\ell^2}\right)\left(\frac{r}{\ell}\right)^{2/3}. \tag{28}$$

We note that if we combine Eq. (20) with Eq. (27) we find

$$r_d' = \left(\frac{\beta^{3/4}}{\alpha^{3/2}}\right)\ell, \tag{29}$$

which does not depend explicitly on ϵ. We showed in Secs. III B and III C that an inertial-range Kolmogorov spectrum can exist only at scales $r<r_d'$. We see now that r must also be greater than ℓ. Therefore the condition that an inertial-range Kolmogorov spectrum can exist over a significant range of length scales is that $r_d'/\ell \gg 1$, and therefore that $\alpha \ll 1$. [Strictly speaking Eq. (26) holds only if the motion of a particular element of vortex line is governed by the local velocity generated by the rest of the line and by other lines, which means that the motion is not seriously perturbed by mutual friction. This is indeed the case if $\alpha \ll 1$, as noted in,[6] so that our analysis is self-consistent.]

The existence of quasi-classical inertial-range behavior (Kolmogorov spectrum) implies that there must be dissipation, equal to the energy flow rate ϵ, at some length scale less than $(\delta V)^{1/3}$. We must now ask how and on what length scale this dissipation occurs, if α is significantly less than unity.

First we estimate the dissipation due to mutual friction on length scales of order ℓ. Per unit mass of helium, and for the case $\alpha \ll 1$, this dissipation must be of order

$$\epsilon_\ell = \alpha\left(\frac{\rho_s}{\rho}\right)\kappa L u_\ell^2 = \alpha\left(\frac{\rho_s}{\rho}\right)\beta\kappa^3\ell^{-4}. \tag{30}$$

We see from Eqs. (27) and (30) that the ratio

$$\frac{\epsilon_\ell}{\epsilon} = \alpha\left(\frac{\rho_s}{\rho}\right)\beta^{-1/2}. \tag{31}$$

Thus the requirement $\alpha \ll 1$ implies that $\epsilon_\ell/\epsilon \ll 1$, so that there is negligible dissipation down to the scale ℓ. Therefore we must search for dissipative processes on smaller length scales.

A situation in which there is no dissipation on length scales down to and including ℓ has already been discussed in connection with dissipation in superfluid ^{4}He at temperatures below 1 K. In the case of turbulence in superfluid ^{4}He in an unbounded volume, two processes that operate on scales less than ℓ have been identified:[6] dissipation during the reconnection of vortex lines and dissipation associated with Kelvin waves with wave number q exceeding ℓ^{-1}, such waves being produced by the close approach, or actual reconnection, of two vortex lines (or two parts of the same line), followed by the transfer of energy within a Kelvin-wave cascade to higher wave numbers. A preliminary discussion of dissipation by Kelvin waves has been given in Refs. 6,13, showing that Kelvin waves can produce the necessary dissipation by mutual friction at a wave number given by

$$q_c\ell \sim \alpha^{-1/2}. \tag{32}$$

For $\alpha \ll 1$, $q_c\ell \gg 1$. In the limit of zero α the Kelvin waves are dissipated by radiation of phonons, at a wave number given by

$$q_c' \sim \left(\frac{c\ell}{\kappa}\right)^{3/4}, \tag{33}$$

where c is the speed of sound in the helium. In the case of superfluid ^{3}He-B the Kelvin waves can also be absorbed by the Caroli-Matricon bound states within the cores of the vortex lines. Neither this absorption process nor that associated directly with actual vortex reconnections involves mutual friction; our paper is concerned only with the effect of mutual friction, and therefore we have not taken these other processes into account.

E. Possible generation of extra turbulence by relative motion of normal and superfluid components

So far we have assumed that the relative motion of the two fluids has no effect other than to damp the turbulence in the superfluid. However, we know from our experience of heat flow in superfluid ^{4}He that such relative motion can lead to the generation of turbulence, in the form of a self-sustaining vortex tangle that is more or less random on the scale of the mean-vortex separation. We shall show that this process is likely to have a negligible effect on our conclusions.

We shall suppose that within a large eddy of size r, characteristic velocity u_r, the counterflow of the two fluids leads to the generation of an extra length of vortex line similar to that observed in steady counterflow in an infinitely wide channel. This will lead to an overestimate of the extra length because it neglects both the finite time that is required to generate this line, in comparison to the turnover time, and the fact that the eddy has a finite size, u_r perhaps comparable to the critical velocity known to exist for counterflow turbulence in a finite channel.

It is easily shown that the extra length of line per unit volume is given by

$$L_0 = \left(\frac{2\pi\chi_1\alpha}{\chi_2}\right)^2\left(\frac{u_r}{\kappa}\right)^2 = \left(\frac{2\pi\chi_1\alpha}{\chi_2}\right)^2\frac{\beta}{\ell^2}\left(\frac{r}{\ell}\right)^{2/3}, \tag{34}$$

where $\chi_1 \sim 0.3$ and $\chi_2 \sim 1$ are dimensionless parameters,[1] and where we have made use of Eq. (28). We write $L=\ell^{-2}$, and therefore

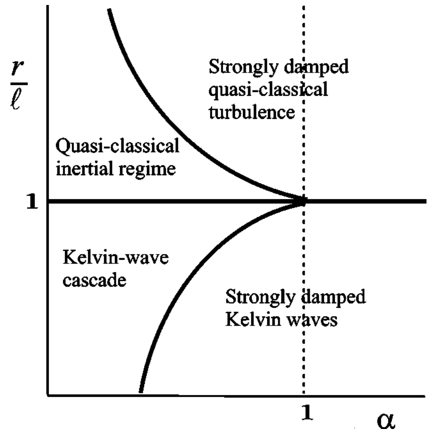

FIG. 4. Different turbulent regimes for different values of r/ℓ and α.

$$\frac{L_0}{L} = \beta\left(\frac{2\pi\chi_1\alpha}{\chi_2}\right)^2\left(\frac{r}{\ell}\right)^{2/3} \sim \alpha^2\left(\frac{r}{\ell}\right)^{2/3}. \qquad (35)$$

We confine our attention to the quasiclassical inertial range, for which r must be less than r'_d, given by Eq. (29). Then the maximum value of the ratio L_0/L becomes

$$\left(\frac{L_0}{L}\right)_{\max} = \beta^{3/2}\left(\frac{2\pi\chi_1}{\chi_2}\right)^2\alpha \sim \alpha. \qquad (36)$$

But the existence of a significant quasiclassical inertial range depends on α being small compared with unity, as we have seen. It follows that within the quasiclassical inertial range $L_0 \ll L$, and that, therefore, the generation of extra line by counterflow can be neglected, at least within this quasiclassical inertial range.

F. Summary of overall behavior

For simplicity we suppose that the Kelvin-wave processes discussed in Sec. III D provide the dominant source of dissipation at length scales less than ℓ. Then our conclusions are summarized concisely in Fig. 4.

In the limit of zero α there is a quasiclassical regime (with a Kolmogorov spectrum) on length scales greater than ℓ and a Kelvin-wave cascade, terminated by phonon radiation, on length scales less than ℓ. As α increases the Kelvin wave cascade starts to be terminated at a larger length scale (wavelength) by mutual friction, and the largest eddies in the quasiclassical regime start to be strongly damped. As α tends toward unity, the range of length scales separating the Kelvin-wave cutoff and the onset of strong damping of the quasiclassical eddies narrows and it disappears near $\alpha=1$. For $\alpha>1$ there can be no turbulent motion that is not strongly damped.

It is possible to construct for our theory a "flow-phase diagram" analogous to that in Fig. 1 of Ref. 8, this earlier diagram being based, as we now see it, on an inadequate theory, as explained in Sec. IV. We have included discussion of such a diagram in another paper recently been submitted for publication.[14]

IV. COMPARISON TO TREATMENT BY VOLOVIK *ET AL.*

Volovik's treatment is based on a coarse-grained averaged equation of motion for the superfluid component, which he writes in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\mu = \boldsymbol{u}\times\boldsymbol{\omega} + \alpha\hat{\boldsymbol{\omega}}\times(\boldsymbol{\omega}\times\boldsymbol{u}), \qquad (37)$$

for the case where, as here, the transverse component of the mutual friction is neglected, and where $\boldsymbol{\omega}$ is the coarse-grained vorticity. This equation is valid only if the vortex lines within the volume δV used in the course-grained averaging are aligned. Our own treatment was not restricted in this way, and, indeed, we have already argued that within δV the vortex lines are likely to be arranged as an approximately random tangle.

The classical Reynolds number for types of flow, in which there is a single length scale R and a single velocity scale U, is simply the ratio of the order of magnitude of the inertial term in the Navier-Stokes equation to that of the (viscous) dissipative term. Applying this idea to Eq. (37), in which the last term relates to dissipation by mutual friction, we see that an effective Reynolds number might be taken as $\mathrm{Re}_{\mathrm{eff}}=\alpha^{-1}$, which is, interestingly, not dependent on velocity. Indeed it was found by Finne *et al.*[15] that $\mathrm{Re}_{\mathrm{eff}}$ does determine whether or not spin-up of superfluid ^{3}He-B is turbulent, although in a context where the vortex density is so low that course-grained averaging is hardly justified; and Volovik tries to emphasize the importance of $\mathrm{Re}_{\mathrm{eff}}$ in more general contexts, including fully developed turbulence. However, for fully developed turbulence there are many length scales, and our analysis of Sec. III B led us to a more complicated scale-dependent Reynolds number, given by Eq. (18), the introduction of which seems to us essential to an understanding of such turbulence.

The ideas described in earlier sections were first presented at an informal meeting in the Low Temperature Laboratory of the Helsinki University of Technology in Finland toward the end of 2003, and a first draft of our paper was distributed as a preprint in March 2004. At the time of the first informal presentation Volovik[8] had formulated an approach that was very different from that given here. This approach was modified in response to our comments,[9] but the dissipative effect of mutual friction was still different from that discussed here. In an earlier version of our paper we argued that Volovik's formulation was not correct, and that it led to incorrect turbulent energy spectra. Within the last few weeks Volovik, in collaboration with L'vov and Nazarenko,[10] has further modified his approach, and it seems now to agree with our own, the turbulent energy spectrum proposed being similar to that displayed in our Fig. 1. They derive this spectrum from an analytical argument, which leads to a wave number dependence as k^{-3} for small k. The spectrum derived from our numerical studies has the same form. We conclude that our two treatments are now in at least partial agreement, although the formulations remain rather different.

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APPENDIX

We investigate rather formally the effect of mutual friction on superfluid motion described by Eq. (37). In this equation there is course-grained averaging over volumes (b^3) containing a significant number of vortices but within which the vortices are assumed to be completely polarized. We are interested in the effect of the mutual friction term $\alpha\hat{\boldsymbol{\omega}}\times(\boldsymbol{\omega}\times\boldsymbol{u})$; in particular, we are interested in its effect on eddies of size $r \gg b$. The inertial term $\boldsymbol{u}\times\boldsymbol{\omega}$ gives rise to the transfer of energy to smaller eddies in a time of order the turnover time $\tau_r = r/u_r$, where u_r is the characteristic velocity associated with the eddies of size r. The mutual friction gives rise to the decay of these relatively large eddies in a time τ_d'. In Sec. III A we argued that $\tau_d' = 1/(\alpha\kappa L)$. Here we argue that this result follows also from Eq. (37). We assume that the vorticity is concentrated at the highest wave numbers.

To estimate τ_d' we shall investigate solutions of the equation

$$\frac{\partial \boldsymbol{u}(\boldsymbol{r})}{\partial t} = q\hat{\boldsymbol{\omega}}(\boldsymbol{r}) \times [\boldsymbol{\omega}(\boldsymbol{r}) \times \boldsymbol{u}(\boldsymbol{r})], \tag{38}$$

where, following Volovik, we have now replaced α by the more general $q=\alpha/(1-\alpha')$, α' being related to the transverse component of the mutual friction. We introduce a new vector, defined by

$$\boldsymbol{\xi}(\boldsymbol{r}) = |\boldsymbol{\omega}(\boldsymbol{r})|^{1/2}\hat{\boldsymbol{\omega}}(\boldsymbol{r}), \tag{39}$$

so that Eq. (38) can be written

$$\frac{\partial \boldsymbol{u}(\boldsymbol{r})}{\partial t} = q\boldsymbol{\xi}(\boldsymbol{r}) \times (\boldsymbol{\xi}(\boldsymbol{r}) \times \boldsymbol{u}(\boldsymbol{r})). \tag{40}$$

We Fourier analyze the velocity and $\boldsymbol{\xi}$ fields in space, and hence obtain

$$\sum_{\boldsymbol{k}_1} \dot{\boldsymbol{u}}(\boldsymbol{k}_1)\exp(i\boldsymbol{k}_1\cdot\boldsymbol{r}) = q\sum_{\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4} \boldsymbol{\xi}(\boldsymbol{k}_2) \times [\boldsymbol{\xi}(\boldsymbol{k}_3)\times\boldsymbol{u}(\boldsymbol{k}_4)] \times\exp[i(\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)\cdot\boldsymbol{r}]. \tag{41}$$

We multiply Eq. (41) by $\exp(-i\boldsymbol{k}\cdot\boldsymbol{r})$ and integrate over space, so obtaining

$$\dot{\boldsymbol{u}}(\boldsymbol{k}) = q\sum_{\boldsymbol{k}_2,\boldsymbol{k}_3} \boldsymbol{\xi}(\boldsymbol{k}_2) \times (\boldsymbol{\xi}(\boldsymbol{k}_3)\times\boldsymbol{u}(\boldsymbol{k}-\boldsymbol{k}_2-\boldsymbol{k}_3)). \tag{42}$$

Let us neglect all terms in the double summation in Eq. (42) for which $\boldsymbol{k}_2+\boldsymbol{k}_3 \neq 0$; we consider these terms later. Then we have

$$\dot{\boldsymbol{u}}(\boldsymbol{k}) = q\sum_{\boldsymbol{k}_1} \boldsymbol{\xi}(\boldsymbol{k}_1) \times (\boldsymbol{\xi}^*(\boldsymbol{k}_1)\times\boldsymbol{u}(\boldsymbol{k})), \tag{43}$$

where we have used the fact that, because $\boldsymbol{\xi}(\boldsymbol{r})$ is real, $\boldsymbol{\xi}(-\boldsymbol{k}_1)=\boldsymbol{\xi}^*(\boldsymbol{k}_1)$.

Let the vector $\boldsymbol{k}$ relate to eddies of size r. The vectors $\boldsymbol{\xi}(\boldsymbol{k}_1)$ have a significant amplitude only when $k_1 \gg k$ because the vorticity is concentrated at the highest wave numbers.

The components of the vectors in Eq. (43) are generally complex numbers. We write

$$\xi_x(\boldsymbol{k}_1) = \overline{\xi_x}(\boldsymbol{k}_1)\exp(i\varphi_x(\boldsymbol{k}_1)), \tag{44}$$

where $\overline{\xi_x}(\boldsymbol{k}_1)$ is real, and similarly for the other components of $\boldsymbol{\xi}(\boldsymbol{k}_1)$. Then

$$\xi_x(\boldsymbol{k}_1)\xi_x^*(\boldsymbol{k}_1) = \overline{\xi_x}(\boldsymbol{k}_1)\overline{\xi_x}(\boldsymbol{k}_1). \tag{45}$$

Now the triple-vector product in Eq. (43) can be written as

$$(\boldsymbol{\xi}(\boldsymbol{k}_1)\cdot\boldsymbol{u}(\boldsymbol{k}))\boldsymbol{\xi}^*(\boldsymbol{k}_1) - (\boldsymbol{\xi}(\boldsymbol{k}_1)\cdot\boldsymbol{\xi}^*(\boldsymbol{k}_1))\boldsymbol{u}(\boldsymbol{k}). \tag{46}$$

Suppose that at some instant $\boldsymbol{u}(\boldsymbol{k})$ has only a z-component. We are interested in only the component of Eq. (43) the direction of $\boldsymbol{u}(\boldsymbol{k})$, since we are concerned with the dissipative effect of the mutual friction. Thus we are interested in the z component of Eq. (46), which is equal to

$$-(\xi_x(\boldsymbol{k}_1)\xi_x^*(\boldsymbol{k}_1) + \xi_y(\boldsymbol{k}_1)\xi_y^*(\boldsymbol{k}_1))u_z(\boldsymbol{k}). \tag{47}$$

Thus, using Eq. (45), we find that

$$\dot{u}_z(\boldsymbol{k}) = -q\sum_{\boldsymbol{k}_1}(\overline{\xi_x^2}(\boldsymbol{k}_1) + \overline{\xi_y^2}(\boldsymbol{k}_1))u_z(\boldsymbol{k}). \tag{48}$$

Let the direction of the vector $\overline{\boldsymbol{\xi}}(\boldsymbol{k}_1)$ be defined by spherical polar angles $\theta(\boldsymbol{k}_1)$, $\phi(\boldsymbol{k}_1)$. Then Eq. (48) can be written

$$\begin{aligned}\dot{u}_z(\boldsymbol{k}) &= -q\sum_{\boldsymbol{k}_1}\overline{\xi^2}(\boldsymbol{k}_1)[\sin^2\theta(\boldsymbol{k}_1)\sin^2\phi(\boldsymbol{k}_1) \\ &\quad + \sin^2\theta(\boldsymbol{k}_1)\cos^2\phi(\boldsymbol{k}_1)]u_z(\boldsymbol{k}) \\ &= -q\sum_{\boldsymbol{k}_1}\overline{\xi^2}(\boldsymbol{k}_1)\sin^2\theta(\boldsymbol{k}_1)u_z(\boldsymbol{k}).\end{aligned} \tag{49}$$

Let us now average Eq. (49) over a time (τ) small compared with the lifetime of the eddies of size r but large compared with the lifetime of the smallest eddies, which account for most of the vorticity. We obtain

$$\langle\dot{u}_z(\boldsymbol{k})\rangle = -q\sum_{\boldsymbol{k}_1}\langle\overline{\xi^2}(\boldsymbol{k}_1)\sin^2\theta(\boldsymbol{k}_1)u_z(\boldsymbol{k})\rangle, \tag{50}$$

where $\langle\ldots\rangle$ denotes averaging over the time τ. Although during this time $\overline{\xi^2}(\boldsymbol{k}_1)$ and $\sin^2\theta(\boldsymbol{k}_1)$ must undergo large fluctuations (because the turnover time at $\boldsymbol{k}_1$ is much less than τ) the fluctuations in $u_z(\boldsymbol{k})$, which relate to the slow motion of an eddy of size r, must surely be small [they must be minimized by the summation in Eq. (50)]. We can therefore neglect the fact that $u_z(\boldsymbol{k})$ fluctuates and write simply

$$\dot{u}_z(\boldsymbol{k}) = -q\sum_{\boldsymbol{k}_1}\langle\overline{\xi^2}(\boldsymbol{k}_1)\sin^2\theta(\boldsymbol{k}_1)\rangle u_z(\boldsymbol{k}). \quad (51)$$

We now make the reasonable assumption that the fluctuations in $\theta(\boldsymbol{k}_1)$ and in $\overline{\xi^2}(\boldsymbol{k}_1)$ are uncorrelated. Then Eq. (51) can be written

$$\dot{u}_z(\boldsymbol{k}) = -q\sum_{\boldsymbol{k}_1}\langle\overline{\xi^2}(\boldsymbol{k}_1)\rangle\langle\sin^2\theta(\boldsymbol{k}_1)\rangle u_z(\boldsymbol{k}). \quad (52)$$

But $\langle\sin^2\theta(\boldsymbol{k}_1)\rangle = \frac{2}{3}$. Therefore

$$\dot{u}_z(\boldsymbol{k}) = -\frac{2}{3}q\sum_{\boldsymbol{k}_1}\langle\overline{\xi^2}(\boldsymbol{k}_1)\rangle u_z(\boldsymbol{k}). \quad (53)$$

From Parseval's theorem we have

$$\sum_{\boldsymbol{k}_1}\langle\overline{\xi^2}(\boldsymbol{k}_1)\rangle = \langle\xi^2(\boldsymbol{r})\rangle. \quad (54)$$

But $\langle\xi^2(\boldsymbol{r})\rangle$ is the mean value of the modulus of the vorticity. As discussed in Ref. 4 this mean value is equal to κL. It follows that

$$\tau_d' = \frac{3}{2q\kappa L}, \quad (55)$$

in essential agreement with Eq. (14).

In writing down Eq. (43) we neglected many terms in the double summation in Eq. (42). These terms serve to couple the velocity $\boldsymbol{u}(\boldsymbol{k})$ to other Fourier components of the velocity field. Physically, this coupling is due to local fluctuations in the vortex line density. The coupling adds to the effect of the inertial term $\boldsymbol{u}\times\boldsymbol{\omega}$, and it must therefore have some effect on any Richardson-Kolmogorov cascade. We guess that it is relatively small, especially for small values of α.

We note that our final result [Eq. (55)] is consistent with the usual expression for the attenuation of second sound[1,16] due to an array of vortex lines with random orientation (in this case the vector $\boldsymbol{k}$ is the wave vector of the second sound).

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