Protected qubits and Chern-Simons theories in Josephson junction arrays

B. Douçot,[1,*] M. V. Feigel'man,[2] L. B. Ioffe,[3,*] and A. S. Ioselevich[2]
[1]*Laboratoire de Physique Théorique et Hautes Énergies, CNRS UMR 7589, Universités Paris 6 et 7, 4, place Jussieu, 75252 Paris Cedex 05, France*
[2]*Landau Institute for Theoretical Physics, Kosygina 2, Moscow, 117940 Russia*
[3]*Department of Physics and Astronomy, Center for Materials Theory, Rutgers University, 136 Frelinghuysen Road, Piscataway, New Jersey 08854, USA*
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We present general symmetry arguments that show the appearance of doubly degenerate states protected from external perturbations in a wide class of Hamiltonians. We construct the simplest spin Hamiltonian belonging to this class and study its properties both analytically and numerically. We find that this model generally has a number of low energy modes which might destroy the protection in the thermodynamic limit. These modes are qualitatively different from the usual gapless excitations as their number scales as the linear size (instead of volume) of the system. We show that the Hamiltonians with this symmetry can be physically implemented in Josephson junction arrays and that in these arrays one can eliminate the low energy modes with a proper boundary condition. We argue that these arrays provide fault tolerant quantum bits. Further we show that the simplest spin model with this symmetry can be mapped to a very special $\mathbb{Z}_2$ Chern-Simons model on the square lattice. We argue that appearance of the low energy modes and the protected degeneracy is a natural property of lattice Chern-Simons theories. Finally, we discuss a general formalism for the construction of discrete Chern-Simons theories on a lattice.

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I. INTRODUCTION

It is generally accepted that a quantum computer would have an enormous advantage over the classical one for the solution of many fundamental and practically important problems.[1–3] However, its practical implementation presents a formidable challenge mostly because of the conflicting requirements posed by scalability and decoupling from the environment. In particular, all scalable designs are based on the solid state devices but these are plagued by a strong decoherence. Quantitatively, in physics it is conventional to measure the decoherence by the quality ratio, Q, that is equal to the product of the decoherence time and a typical energy gap while in computer science one uses the error rate, R, defined as the probability of an error per time required for an individual operation. In order to avoid excitations of higher energy states all operations should be performed slowly on the scale of the inverse energy gap, so $R \gg 1/Q$. The problem posed by the omnipresence of the decoherence in solid state devices is exacerbated by the fact that the error correction codes[4,6] require a small error rate for individual qubits (at the very least 10^{-3} per logical operation that translates into the quality factors larger than 10^4) and lead to a huge increase in the number of qubits, effectively replacing one qubit by a lattice with $L \times L$ qubits with large L.[5] Further, the efficient error correction requires that operations are done *simultaneously* on all $L \times L$ qubits encoding a single error free bit which makes this scheme rather unrealistic. An alternative would be to use individual qubits with a very high quality factor, much greater than 10^8, which would allow one to perform calculations without (or with very little) error correction. If the noise couples linearly to the energy difference between two states of the qubit representing 0 and 1, this would mean that the physical noise should be less than 10^{-8} times than all other energy scales of the device. It is difficult to imagine a solid state physical system that is so well screened from the outside noise, in particular, it is difficult to imagine a Josephson junction, a Cooper box or SET where the motion of stray charges do not result in a significant $1/f$ noise in electric or magnetic fields or in the strength of the Josephson couplings. Further, a significant energy difference between the two states of a qubit results in a phonon emission[7] that limits the quality factor of a typical Josephson device by 10^4. So, the only logical possibility is that the physical noises do not affect the energy difference between lowest energy states in the lowest orders (one or more) in the strength of the noises. This is in principle possible because the effect of the physical noise is always represented by the sum of local physical operators (charge, current, etc.). In the limiting case when the noise does not affect the energy difference between the two states in any finite order, these states form a protected subspace of the Hilbert space.[8] Of course, the formation of such truly protected space becomes possible only for an infinite system. The implementation of these models in solid state (Josephson junction) devices was suggested in Refs. 9–11. From a practical view point it is important to consider simpler but smaller systems which are protected from the noise in the given order, n, i.e., which are not affected by all physical noises in all orders less than n. Clearly, any device in which two levels representing 0 and 1 have a finite energy difference is susceptible to the fluctuations in the physical quantity that sets this energy scale. For instance, in a Cooper box this would be the Josephson, E_J, and a charging energy, E_C, of the individual Josephson junction. Thus, even this limited protection can occur only when the two "working" levels are degenerate.

It is well known that the stable degeneracy of the quantum levels is almost always due to a high degree of the symmetry of the system. Examples are numerous: time inversion invariance ensures the degeneracy of the states with half inte-

ger spin, rotational symmetry results in a degeneracy of the states with nonzero momentum, etc. In order for the degeneracy to be stable with respect to the local noise, one needs that the sufficient symmetry remains even if a part of the system is excluded. The simplest example is provided by the six Josephson junctions connecting four superconducting islands (so that each island is connected with every other).[12] In this mini array all islands are equivalent, so it is symmetric under all transformations of the permutation group S_4. This group has a two-dimensional (2D) representation and thus, pairs of the exactly degenerate states. With the appropriate choice of the parameters one can make these doublets the ground state of the system. The noise acting on one superconducting island reduces the symmetry to the permutation group of the three elements which still has two-dimensional representations. So, this system is protected from the noise in the first order ($n=2$). The goal of this paper is to discuss designs giving the systems that are protected from the noise in the higher orders. Note that systems with higher symmetry groups, such as five junctions connected by ten junctions (group S_5) typically do not have two-dimensional representations, so in these systems one can typically get much higher degeneracy but not higher protection.

Generally, one gets degenerate states if there are two symmetry operations, described by the operators P and Q that commute with the Hamiltonian but do not commute with each other. If $[P,Q]|\Psi\rangle \neq 0$ for any $|\Psi\rangle$, all states are at least doubly degenerate. Local noise term is equivalent to adding other terms in the Hamiltonian which might not commute with these operators thereby lifting the degeneracy. Clearly, in order to preserve the degeneracy one needs to have two *sets* (of n elements each), $\{P_i\}$ and $\{Q_i\}$ of noncommuting operators, so that any given local noise field does not affect some of them; further, preferably, any given local noise should affect at most *one* P_i and Q_i. In this case, the effect of the noise appears when n noise fields act *simultaneously*, i.e., in the nth order in the noise strength. Another important restriction comes from the condition that these symmetry operators should not result in a higher degeneracy of the ideal system. For two operators, P and Q that implies that $[P^2,Q]=0$ and $[P,Q^2]=0$. Indeed, one can construct the degenerate eigenstates of the Hamiltonian starting with the eigenstate, $|0\rangle$, of H and Q and acting on this state with P. The resulting state, $|1\rangle$ should be different from the original one because P and Q do not commute: $[P,Q]|\Psi\rangle \neq 0$ for any Ψ. In a doubly degenerate system, acting again on this state with the operator P one should get back the state $|1\rangle$, so $[P^2,Q]=0$. For a set of operators, the same argument implies that in order to get a double degeneracy (and not more) one needs that $[P_iP_j,Q]=0$ and $[P,Q_iQ_j]=0$ for any i, j. Indeed, in this case one can diagonalize simultaneously the set of operators $\{Q_i\}$, $\{Q_iQ_j\}$ and $\{P_iP_j\}$. Consider a ground state, $|0\rangle$, of the Hamiltonian which is also an eigenstate of all these operators. Acting on it with, say P_1, we get a new state, $|1\rangle$, but since $|1\rangle \propto (P_iP_1)P_1|0\rangle = P_i(P_1P_1)|0\rangle \propto P_i|0\rangle$ all other operators of the same set would not produce a new state.

In the rest of the paper we introduce models that possess the symmetries satisfying these conditions (Sec. II), discuss their possible implementations in Josephson junction networks (Sec. III) and show that they are equivalent to the Chern-Simons $\mathbb{Z}_2$ gauge theory (Secs. IV and V). Section VI summarizes our results.

II. SPIN MODEL

The conditions discussed at the end of the previous section are fully satisfied by the spin $S=1/2$ model on a square $n\times n$ array described by the Hamiltonian

$$H = -J_x\sum_{i,j}\sigma^x_{i,j}\sigma^x_{i,j+1} - J_z\sum_{i,j}\sigma^z_{i,j}\sigma^z_{i+1,j}. \tag{1}$$

Here σ are Pauli matrices, note that the first term couples spins in the same row of the array while the second couples them along the columns. It is not important for the following discussion whether the boundary conditions are periodic or free, but since the latter are much easier to implement in a hardware we shall assume them in the following. Further, the signs of the couplings are irrelevant because for a square lattice one can always change it by choosing a different spin basis on one sublattice. For the sake of argument, we assumed that the signs of the couplings are ferromagnetic; this is also a natural sign for Josephson junction implementations in Sec. III. The Hamiltonian (1) was first introduced in Ref. 13 as a model for the anisotropic exchange interaction in transition metal compounds but its properties remain largely unclear.

The Hamiltonian (1) has two sets of the integrals of motion, $\{P_i\}$ and $\{Q_i\}$ with n operators each

$$P_i = \prod_j \sigma^z_{i,j},$$

$$Q_j = \prod_i \sigma^x_{i,j},$$

i.e., each P_i is the row product of $\sigma^z_{i,j}$ while Q_j is the column product of $\sigma^x_{i,j}$. Consider P_i operator first. It obviously commutes with the second term in the Hamiltonian and because the first term contains two $\sigma^x_{i,j}$ operators in the same row, P_i either contains product of σ^z_i and σ^z_j or none of them. Because different Pauli matrices anticommute, P_i commutes with each term in the Hamiltonian (1). Similarly, $[Q_i,H]=0$. Clearly, different P_i commute with themselves, $P_i^2=1$ and similarly $[Q_i,Q_j]=0$ and $Q_i^2=1$, but they do not commute with each other

$$\{P_i,Q_j\} = 0,$$

$$[P_i,Q_j]^2 = 4 \tag{2}$$

so $[P_i,Q_j]|\Psi\rangle \neq 0$ for any $|\Psi\rangle$, thus in this model all states are at least doubly degenerate. Further, because P_iP_j contains two $\sigma^z_{i,j}$ in any column, such product commutes with all Q_k operators and similarly $[Q_kQ_l,P_i]=0$. Thus, we conclude that in this model all states are doubly degenerate, there is no symmetry reason for larger degeneracy and that this degeneracy should be affected by the noise only in the nth order of the perturbation theory.

To estimate the effect of the noise (which appears in this high order) one needs to know the energy spectrum of the

model and what are its low energy states. All states of the system can be characterized by the set, $\{\lambda_i=\pm 1\}$, of the eigenvalues of P_i operators (or alternatively by the eigenvalues of the Q_j operators). The degenerate pairs of states are formed by two sets, $\{\lambda_i\}$ and $\{-\lambda_i\}$, and each operator Q_j interchanges these pairs: $Q_i\{\lambda_i\}=\{-\lambda_i\}$. We believe that different choices of $\{\lambda_i=\pm 1\}$ exhaust *all* low energy states in this model, i.e., that there are exactly 2^n low energy states. Note that this is a somewhat unusual situation; normally one expects n^2 modes in a 2D system and thus 2^{n^2} low energy states. The number 2^n low energy states is natural for a one-dimensional system and would also appear in two-dimensional systems if these states are associated with the edge. Here, however, we cannot associate them with the edge states because they do not disappear for the periodic boundary conditions. We cannot prove our conjecture in a general case but we can see that it is true when one coupling is much larger than the other and we have verified it numerically for the couplings of the same order of magnitude. We start with the analytic treatment of the $J_z \gg J_x$ case.

When one coupling is much larger than the others it is convenient to start with the system where these others are absent and then treat them as small perturbations; in the limit $J_x=0$ all columns are independent and the ground state of each column is a Ising ferromagnet. The ground state of each column is doubly degenerate: $|1\rangle_j=\Pi_i|\uparrow\rangle_{ij}$ and $|2\rangle_j=\Pi_i|\downarrow\rangle_{ij}$ giving us 2^n degenerate states in this limit. Excitations in each column are static kinks against the background of these states; each kink has energy $2J_z$. Including now J_x coupling, we see that it creates two kinks in each of the neighboring columns thereby increasing the energy of the system by $8J_z$ so the lowest order of the perturbation theory is small in $J_x/8J_z$. The splitting between the 2^n states occurs due to the high order processes which flip all spins in two columns. In the leading approximation one can calculate the amplitude of this process ignoring other columns. Thus, for this calculation we can consider the model with only two columns that can be mapped onto a single Ising chain in the transverse field in the following manner. The ground state of two independent columns belongs to the sector of the Hilbert space characterized by all $P_iP_j=1$; it is separated from the rest of the spectrum by the gap of the order of $2J_z$. Further, the Hamiltonian does not mix this sectors with different P_i, so in order to find the low energy states, it is sufficient to diagonalize the problem in the sector $P_i=1$. In this sector only two states are allowed in each row: $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$, in the basis of these states the Hamiltonian is reduced to

$$H_{\mathrm{col}} = -2J_z\sum_i \tau_i^z\tau_{i+1}^z - J_x\sum_i \tau_i^x, \tag{3}$$

where τ are Pauli matrices acting in the space of $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ states. This leads to the splitting $2\Delta\approx(J_x/2J_z)^n(2J_z)$ (see Appendix A for the details of this calculation) between the symmetric and antisymmetric combinations of the two ferromagnetic chains in this problem. Thus, we conclude that the effective Hamiltonian of the low energy states in the full system is

$$H=\Delta\sum_j \hat{\tau}_j^x\hat{\tau}_{j+1}^x,$$

where $\hat{\tau}$ are Pauli matrices acting in the space of $|1\rangle$ and $|2\rangle$ states describing the global state of the whole column. This effective low energy model also describes a ferromagnetic chain in which the excitations (static kinks) are separated from the degenerate ground state by the gap 2Δ. In the basis of these 2^n low energy states the operators $Q_j=\hat{\tau}_j^x$.

We conclude that in the limit $J_z \gg J_x 2^n$ low energy states form a narrow (of the order of Δ) band inside a much larger gap, J_z, characterized by different eigenvalues of Q_j operators and by one value $P_iP_j=1$. In the opposite limit $J_x \gg J_z$ low energy states form a narrow band inside the gap of size J_x characterized by different eigenvalues of P_j operators and by the same value $Q_iQ_j=1$. Consider now the effect of a weak noise in the former limit. To be more specific, we consider the effect of the additional single site fields

$$H_n=\sum_{i,j} h_{i,j}^z\sigma_{i,j}^z + h_{i,j}^x\sigma_{i,j}^x.$$

The first term shifts (up or down) the energies of each ferromagnetic column by $H_i^z=\Sigma_i h_{i,j}^z$ while the second term gives the transitions between up and down states in each column. These transitions appear only in the order n of the perturbation theory, so their amplitude is exponentially small: $H_j^x=(\Pi_i h_{ij}^x/J_z)J_z$. Thus, when projected onto the low energy subspace this noise part becomes

$$H_n=\sum_j H_j^z\hat{\tau}_j^z + H_j^x\hat{\tau}_j^x.$$

The effect of the first term on the ground state degeneracy appears in the nth order of the perturbation theory in H_j^z/Δ and so it is much bigger than that of the second term because Δ becomes exponentially small as $n\to\infty$ for $J_x \ll J_z$. Note that although the effect of the $h_{i,j}^z\sigma_{i,j}^z$ noise appears only in the large order of the pertubation theory, it is not small because of the small energy denominator in this parameter range. Similarly, we expect that in the opposite limit, $J_z \ll J_x$, the low lying states are characterized by the set of eigenvalues of P_i operators, the effect of the $h_{i,j}^x\sigma_{i,j}^x$ grows rapidly while the effect of the $h_{i,j}^z\sigma_{i,j}^z$ noise decreases with the J_x increase. We conclude that in the limits when one coupling is much larger than another ($J_z \gg J_x$ or $J_z \ll J_x$), the gap closes very quickly (exponentially) and the nonlinear effect of the appropriate noise grows rapidly with the system size. These qualitative conclusions should remain valid for all couplings except a special isotropic ($J_x=J_z$) point unless the system undergoes a phase transition near this point (at some $J_x/J_z=j_c\sim 1$).

In order to check these conclusions we have numerically diagonalized small spin systems containing up to 5 by 5 spins subjected to a small random field $h_{i,j}^z$ flatly distributed in the interval $(-\delta/2, \delta/2)$. We see that indeed the gap closes rather fast away from the special $J_x=J_z$ point (Fig. 1) but remains significant near $J_z=J_x$ point where it clearly has a much weaker size dependence. Interestingly, the gap between the lowest 2^n states and the rest of the spectrum expected in the limits $J_z \gg J_x$ or $J_z \ll J_x$ appears only at J_x/J_z

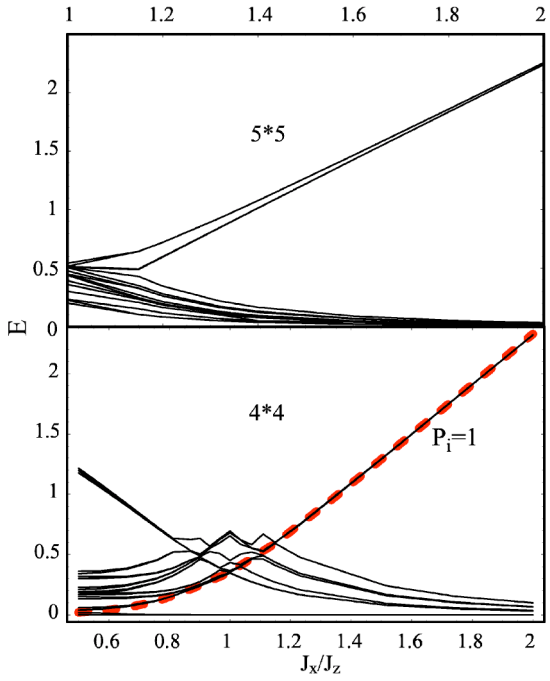

FIG. 1. Energy spectrum of the 5×5 and 4×4 systems in the units of J_z coupling as a function of J_x/J_z. We show energies of the lowest 40 states for 5×5 (upper pane) and lowest 20 states for the 4×4 system (lower pane). One clearly sees that at large anisotropy a well-defined low energy band is formed which contains 2^5 states for 5×5 system and 2^4 states for 4×4 one. In order to verify that low energy states are in one-to-one correspondence with P_i eigenvalues for large J_x/J_z we have calculated the second lowest eigenstate in $P_i=1$ sector (first one is the ground state). As shown in the lower pane by the dashed line, this state indeed has a large gap for $J_x/J_z \gtrsim 1.2$.

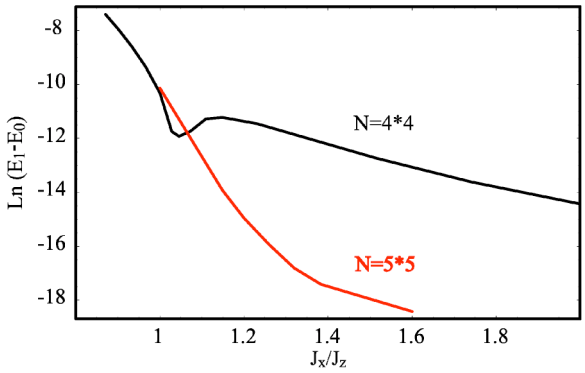

FIG. 2. Ground state splitting by random field in z direction for 5×5 and 4×4 systems. The random field acted on each spin and was randomly distributed in the interval $(-0.05,0.05)$. Note that the effect of the random field in z direction becomes larger for $J_x \ll J_z$ as expected. Because near $J_x=J_z$ isotropic point the gap for 5×5 system is significantly smaller than the gap for the 4×4 system, this relatively large disorder has almost the same effect on these systems at $J_x \sim J_z$. We have verified numerically that decrease of the disorder by a factor of 2 leads to dramatically smaller effects for 5×5 system confirming the scaling $E_1-E_0 \propto \delta^n$. For $J_x/J_z > 1.2$ the difference E_1-E_0 is difficult to resolve numerically.

$> j_c$ with a practically size independent $j_c \approx 1.2$. We also see that the condition $P_i=1$ eliminates all low lying states in the $J_z \ll J_x$ limit where the lowest excited state in $P_i=1$ sector is separated from the ground state by a large gap and in fact provides a lower bound for all high energy states. The special nature of this state appears only at $J_x/J_z > j_c$. Clearly, the system behaves quite differently near the isotropic point and away from it but the size limitations do not allow us to conclude whether these different regimes correspond to two different phases (with the "isotropic" phase restricted to a small range of parameters $j_c^{-1} < J_x/J_z < j_c$) or it is a signature of the critical region which becomes narrower as the size increases. Although we do not see any appreciable change in j_c with the system size, our numerical data do not allow us to exclude the possibility that j_c tends to unity in the thermodynamic limit. We conclude that numerical data *favor* intermediate phase scenario. In contrast to this, the analytical results for two and three chains indicate that the transition occurs only at $J_z=J_x$ point. Namely, both two and three chain models with periodic boundary conditions in the transverse direction can be mapped onto solvable models with transition at $J_z=J_x$: in the case of two chains the problem is mapped onto the exactly solvable Ising model in transverse field as described above while the three chain model is mapped (see Appendix B) onto the four states Potts model in a similar way. The latter is not exactly solvable in the whole parameter range but it obeys the exact duality that allows one to determine its critical point,[14,15] further, its exponents can be determined from the conformal field theory.[16] The mapping of the three chain problem onto the Potts model is possible because the number of states of three spin rung for a given value of the conserved P operator is four while the number of different terms in the Hamiltonian that couples the adjacent rungs is three. For a larger number of chains the number of states in each rung grows exponentially while the number of terms in the Hamiltonian grows only linearly making such mappings impossible. In this sense two and three chain models are exceptional and it is fairly possible that the intermediate phase appears in models with larger number of chains.

Finally, we checked the effect of the $h^z_{i,j}\sigma^z_{i,j}$ on the ground state degeneracy splitting, and our results are shown in Fig. 2. We see that, as expected, this disorder becomes relevant for $J_x \ll J_z$ while in the opposite limit its effect quickly becomes unobservable. We conclude that at (and perhaps near) isotropic point, the gap closes slowly with the system size and the effect of even significant disorder ($\delta=0.1$) becomes extremely small for the medium sized systems.

Although it is not clear how fast the gap closes in thermodynamic limit (if it closes exponentially fast the system never becomes truly protected from the noise because the effect of the high order terms might get very large), our numerical results clearly indicate that medium size (4 by 4 or 5 by 5) systems provide an extremely good protection from the noise suppressing its effect by many orders of magnitude. This should be enough for all practical purposes. Further, if it is possible to construct the systems where $P_iP_j=1$ (in other words with an additional term in the Hamiltonian $H_P=-\Delta\Sigma_{ij}P_iP_j$ with significant Δ), this would eliminate the dangerous low energy states, leading to a good protection for

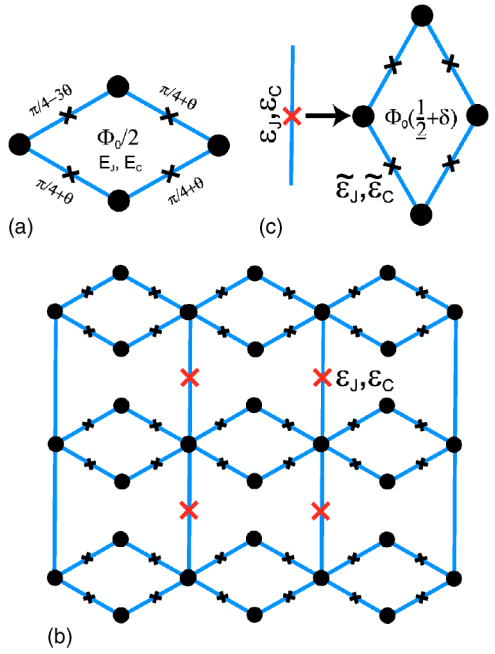

FIG. 3. Schematics of the array equivalent to the spin model with the interaction (5) in the vertical direction. (a) The main element of the array, the superconducting rhombus frustrated by magnetic flux $\Phi_0/2$. Josephson energy of each rhombus is minimal for $\theta=0$ and $\theta=-\pi/2$. Significant charging energy induces the transitions $\theta=0\leftrightarrow\theta=-\pi/2$ between these energy minima. (b) The array geometry. The superconducting boundary conditions chosen here ensure that $P_iP_j=1$ thereby eliminating all low lying states in the appropriate regime. (c) The requirement that continuous phase does not fluctuate much while the discrete variables have large fluctuations is easier to satisfy in very big arrays ($L>20$) if one replaces the vertical links by the rhombi with junctions with $\tilde{\epsilon}_J$, $\tilde{\epsilon}_C$ frustrated by the flux $\Phi_0(\delta+1/2)$.

all couplings strengths $J_x \geqslant J_z$. Indeed, in this case, we can repeat the previous analysis and conclude that the effects of the noise appear only in the nth order and that now the perturbation theory in the "dangerous" noise, H_j^z, implies the expansion in H_j^z/Δ where Δ is no longer exponentially small but is the coefficient in the Hamiltonian H_P. Thus, in this case these higher order terms become small.

III. JOSEPHSON JUNCTION IMPLEMENTATIONS

The basic ingredient of any spin 1/2 implementation in Josephson junction array is the elementary block that has two (nearly degenerate) states. One of the simplest implementation is provided by a four Josephson junction loop [shown as rhombus in Fig. 3(a)] penetrated by magnetic flux $\Phi_0/2$.[17,18] Classically, this loop is frustrated and its ground state is degenerate: it corresponds to the phase differences $\pm\pi/4$ across each junction constituting the loop. Two states (spin "up" and "down") then correspond to the states with phase difference $\pm\pi/2$ across the rhombus. For an isolated rhombus a nonzero (but small) charging energy, $E_C=e^2/2C$, would result in the transitions between these two states with the amplitude

$$r \approx E_J^{3/4} E_C^{1/4} e^{-s\sqrt{E_J/E_C}} \tag{4}$$

thereby lifting this degeneracy. Here s is the numerical coefficient of the order of unity that was found in Ref. 10, $s \approx 1.61$ and E_J is the Josephson energy of each junction.

A. Simplest Josephson junction array

We begin with the Josephson junction array that has two sets of the integrals of motion, $\{P_i\}$ and $\{Q_i\}$ discussed above, which is shown in Fig. 3. This array contains rhombi with junctions characterized by Josephson and charging energies $E_J \gtrsim E_C$ and weaker vertical junctions characterized by the energies ϵ_J, ϵ_C. As we explain below, although this array preserves the integrals of motion, $\{P_i\}$ and $\{Q_i\}$, it maps onto a spin model that differs from Eq. (1); we consider more complicated arrays that are completely equivalent to spin model (1) in the next subsection. The state of the system is fully characterized by the state of each rhombi (described by the effective spin 1/2) and by the small deviations of the continuous superconducting phase across each junction from its equilibrium (classical) values. Ignoring for the moment the continuous phase, we see that the potential energy of the array shown in Fig. 3 is given by

$$H_z = -\epsilon_J \sum_{i,j} \tau_{i,j}^z \tau_{i+1,j}^z, \quad \tau_{i,j}^z \equiv \prod_{k<j} \sigma_{i,k}^z. \tag{5}$$

Physically, the variable $\tau_{i,j}^z$ describes the phase of the right most corner of each rhombi with respect to the left (grounded) superconducting wire modulo π. The right superconducting wire (that connects the right most corners of the rhombi in the last column) ensures that the phase differences along all rows are equal. In the limit of a large phase stiffness this implies that the number of the rhombi with the phase difference $\pi/2$ should be equal for all rows modulus 2. This constraint does not allow an individual rhombus flip, instead a flip of one rhombus should be always followed by a flip of another in the same row. If, however, the phase stiffness is low, the flip of one rhombus can be also compensated by the continous phase deformations in the other rhombi constituting this row; we derive the conditions at which we can exclude these processes below. The simplest allowed process is the simultaneous flip of two rhombi in one row

$$H_x = -\sum_{i,j,k} \tilde{J}_x(j-k)\sigma_{i,j}^x \sigma_{i,k}^x, \tag{6}$$

where $\tilde{J}_x(k)$ is the amplitude to flip two rhombi a distance k apart. Both potential (5) and kinetic (6) energies commute with the integrals of motion, $\{P_i\}$ and $\{Q_i\}$, so that we expect that the main feature of this model, namely, the existence of the protected doublets, will be preserved by this array.

As explained in the previous section, in order to achieve a really good protection one needs to eliminate all low energy states (except for the degenerate ground state) characterized by different values of the $\{P_i\}$ and $\{Q_i\}$ operators. The array shown in Fig. 3 has a boundary condition implying $P_iP_j=1$ for any i, j because in this array the sum along each row of the phases across individual rhombus should be equal for all rows. Thus, for a sufficiently large tunneling amplitude $\tilde{J}_x(k)$

this array should have two degenerate ground states separated from the rest of the spectrum by a large gap. Physically, these two states correspond to two different values of the phase difference along each row. The quantitative condition ensuring that tunneling amplitude $\tilde{J}_x(k)$ is large enough depends on the range of $\tilde{J}_x(k)$. The simplest situation is realized if only the nearest neighboring rhombi flip with the significant amplitude, J_x. Because flip of the two nearest rhombi is equivalent to the flip of the phase on the island between them, in this case the spin model (5) and (6) is equivalent to the collection of independent vertical Ising chains with Hamiltonian

$$H = -\sum_{i,j} \epsilon_J \tau^z_{i,j} \tau^z_{i+1,j} - J_x \tau^x_{i,j}.$$

For $J_x \gg 2\epsilon_J$ each chain described by this Hamiltonian has a unique ground state separated by the $\Delta = 2J_x$ from the rest of the spectrum. As the ratio ϵ_J/J_x grows, the gap decreases.

In the opposite limiting case of a very long range $\tilde{J}_x(k) = J_x$, one can treat the interaction (6) in the mean field approximation

$$H_x = -J_x L_x \langle \sigma^x_{i,k} \rangle \sum_{i,j} \sigma^x_{i,j}. \tag{7}$$

At large J_x the ground state of this system is also a doublet (characterized by $\langle \sigma^x_{i,k} \rangle = \pm 1$) with all other excitations separated by the gap $\Delta = 2L_x J_x$ from the rest of the spectrum. As we increase the vertical coupling, ϵ_J, the gap for the excitations gets smaller. At very large ϵ_J the Hamiltonian is dominated by the ferromagnetic coupling in the vertical direction, so in this regime there are many low energy states corresponding to two possible magnetizations of each column. The magnitude of ϵ_J for which the gap decreases significantly can be estimated from the first order correction in ϵ_J. The dominant contribution comes from the transitions involving rhombi of the outmost rows. They occur with amplitude ϵ_J and lead to the states with energy Δ, so we expect that as long as $\epsilon_J \lesssim \Delta$, the system has a doubly degenerate ground state separated from the other states by gap of the order of Δ.

The amplitude, $\tilde{J}_x(k)$, for the simultaneous flip of two rhombi can be found from the same calculation that was used in Ref. 10 to calculate a single rhombus flip and the simultaneous flip of three rhombi. If $\epsilon_C \gg E_C$, the contribution of the vertical links to the total kinetic energy of the superconducting phase is small and can be treated as a small perturbation, in this case

$$\tilde{J}_x(k) \approx E_J^{3/4} E_C^{1/4} e^{-2s\sqrt{E_J/E_C}(1+ckE_C/\epsilon_C)}, \tag{8}$$

where $c \sim 1$. Here the factor 2 in the exponential appears because in this process one changes simultaneously the phases across two neighboring rhombi. Note that although the relative change in the action due to vertical links is always small, their contribution might suppress the flips of all rhombi except the nearest neighbors if $E_J E_C/\epsilon_C^2 \gg 1$. Note that even a relatively large ϵ_C (so that $E_C/\epsilon_C \ll 1$) can be sufficient to suppress the processes involving non-nearest neighbors. We conclude that the low energy states become absent as long as

$$\epsilon_J < L_{eff} J_x, \tag{9}$$

$$\text{where} \quad L_{eff} = 1/2 \quad \text{if} \quad E_J E_C/\epsilon_C^2 \gg 1 \quad \text{and} \quad L_{eff} \approx \min(\epsilon_C/\sqrt{E_J E_C}, L) \quad \text{if} \quad E_J E_C/\epsilon_C^2 \ll 1.$$

These estimates assume that the main contribution to the capacitance comes from the junctions and ignores the contribution from the self-capacitance. If the self-capacitance is significant, the processes involving more than one island become quickly suppressed.

We now consider the effect of the continuous fluctuations of the superconducting phase. Generally, a finite phase rigidity allows single rhombus flip, described by the $\Sigma_{ij} \tilde{t} \sigma^x_{ij}$ term in the effective spin Hamiltonian. This term does not commute with the intergrals of motion P_i and thereby destroys the protected doublets. However, for a significant phase rigidity the energy of a state formed by a single rhombus flip, U_{sf}, is large. If, further, the amplitude $\tilde{t}$ of these processes is small: $\tilde{t} \ll U_{sf}$, the states corresponding to single flips can be eliminated from the effective low energy theory and the protection is restored. If $\tilde{t} > U_{sf}$, the protection is lost.

We thus begin our analysis of the effects of the finite phase rigidity with the consideration of the dangerous single rhombus flips. Generally, the continuous phase can be represented as the sum of two parts: the one that it is due to the vortices and the spin-wave part which does not change the phase winding numbers. As usual in *XY* systems, it is the vortex part that is the most relevant for the physical properties. In particular, in these arrays it is the vortex part that controls the dynamics of the discrete subsystem. Notice that, unlike the conventional arrays, the arrays containing rhombi allow two types of vortices: half vortices and full vortices because of the double periodicity of each rhombi. The flip of the individual rhombi is equivalent to the creation of the pair of half vortices. If the ground state of the system contains a liquid of half vortices, these processes become real and the main feature of the Hamiltonian, namely the existence of two sets of anticommuting variables, is lost. We now estimate the potential energy of the half vortex and of the pair associated with single flip, U_{sf}, and amplitude to create such pair, $\tilde{t}$. We begin with the potential energy which is different in different

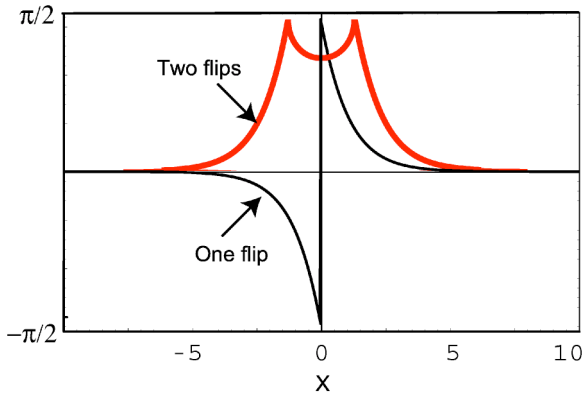

FIG. 4. Phase variation along the horizontal axis after a flip of a single rhombus (solid curve) and after consecutive flips of two rhombi (thick curve) located at a distance twice the core size of each rhombi, $\sqrt{E_J/\epsilon_J}$. The horizontal axis shows the distance, $X=\sqrt{\epsilon_J/E_J}x$, measured in the units of the vortex core size.

limits. Let us consider the simpler limiting case when rhombi flips do not affect the rigidity in the vertical direction; it remains ϵ_J. Further, we have to distinguish the case of a very large size in horizontal direction and a moderate size because the contribution from the individual chains can be dominant in a moderate system if $E_J \gg \epsilon_J$. In a very large system of linear size L with rigidity ϵ_J in the vertical direction the potential energy of one vortex is

$$E_v = \pi\sqrt{E_J\epsilon_J}\ln(L) \tag{10}$$

while the energy of the vortex-antivortex pair at a large distance R from each other is

$$U_v(R) = \pi\sqrt{E_J\epsilon_J}\ln(R). \tag{11}$$

These formulas can be derived by noting that at large scales the superconducting phase changes slowly which allows one to use the continuous approximation for the energy density: $E=\frac{1}{2}E_J(\partial_x\phi)^2+\frac{1}{2}\epsilon_J(\partial_z\phi)^2$. Rescaling then the x coordinate by $x\to\tilde{x}\sqrt{E_J/\epsilon_J}$ we get an isotropic energy density $E=\frac{1}{2}\sqrt{E_J\epsilon_J}(\nabla\phi)^2$. The continuous approximation is valid if both rescaled coordinates $\tilde{x}$, $z \gtrsim 1$. Thus, in a system with $E_J \sim \epsilon_J$ the formulas (10) and (11) remain approximately correct even at small distances $R \sim 1$, so a flip of a single rhombus creates a half vortex–anti-half vortex pair with energy $E_p \approx E_J$ but the formulas become parametrically different in a strongly anisotropic system. Consider first the limit $\epsilon_J=0$. Here the chains of rhombi are completely decoupled and the energy of two half vortices separated by one rhombus in the vertical direction (the configuration created by a single flip) is due to the phase gradients in only one chain, $U_{sf}^{(0)}=\pi^2E_J/(2L)$, which appear because the ends of the chain have the phase fixed by the boundary superconductor. A very small coupling between the chains adds $U_{sf}^{(1)}=(2/\pi)\epsilon_J L$ to this energy, so the total potential energy of the single flip inside the array is

$$U_{sf} = \pi^2E_J/(2L) + (4/\pi)\epsilon_J L \quad L \ll \sqrt{E_J/\epsilon_J}. \tag{12}$$

This formula is correct as long as the second term is much smaller than the first one; they become comparable at $L=\sqrt{E_J/\epsilon_J}$ and at larger L the potential energy associated with the single flip saturates at

$$U_{sf} = \gamma\sqrt{E_J\epsilon_J} \quad L \gg \sqrt{E_J/\epsilon_J}, \tag{13}$$

where $\gamma \approx 3.3$ Qualitatively, a single flip leads to the continuous phase configuration where phase gradients are significant in a narrow strip in the x direction of the length $\sqrt{E_J/\epsilon_J}$ and width ~ 1. The phase configuration resulting from such a process is shown in Fig. 4. These formulas assume that the rigidity of the superconducting phase remains ϵ_J which is, strictly speaking, only true if the discrete variables are perfectly ordered in the vertical direction. Indeed, the coupling in the vertical direction contains $\epsilon_J\cos(\phi)\tau^z_{i,j}\tau^z_{i+1,j}$ which renormalizes to $\epsilon_J\cos(\phi)\langle\tau^z_{i,j}\tau^z_{i+1,j}\rangle$ in a fluctuating system. In the opposite limit of strongly fluctuating rhombi, the average value of $\langle\tau^z_{i,j}\tau^z_{i+1,j}\rangle$ becomes small, we can estimate it from the perturbation theory expansion in ϵ_J which sets the lowest energy scale of the problem: $\langle\tau^z_{i,j}\tau^z_{i+1,j}\rangle \approx \epsilon_J/(L_{eff}J_x)$ which renormalizes the value of ϵ_J

$$\epsilon_J \to \tilde{\epsilon}_J = \frac{\epsilon_J^2}{L_{eff}J_x}.$$

This renormalized value of ϵ_J should be used in the estimates of the vortex energy (12) and (13). This does not affect much the estimates unless the system is deep in the fluctuating regime.

Unlike potential energy, the single flip processes occur with the amplitude

$$\tilde{t} = E_J^{3/4}E_C^{1/4}e^{-s\sqrt{E_J/E_C}} \tag{14}$$

in all regimes. This formula neglects the contribution of the continuous phase to the action of the tunneling process. The reason is that both the potential energy (13) of the half vortex and the kinetic energy required to change the continuous phase are much smaller than the corresponding energies of the individual rhombus, E_J, E_C. In order to estimate the kinetic energy, consider the contribution of the vertical links (horizontal links give equal contribution). There are roughly $\sqrt{E_J/\epsilon_J}$ such links, so their effective charging energy is about $e_C\sqrt{\epsilon_J/E_J}$. If all junctions in this array are made with the same technology their Josephson energies and capacitances are proportional to their areas, so $\epsilon_J/E_J=E_C/\epsilon_C\equiv\eta$; in the following we shall refer to such junctions as *similar*. In this case the array is characterized by two-dimensionless parameters, $\eta \ll 1$ and $E_J/E_C \gg 1$, and the additional contribution to the charging energy, $\eta^{1/2}E_C^{-1}$, coming from vertical links is smaller than the one of the individual rhombi, E_C^{-1}, and thus do not change the dynamics.

We conclude that the dangerous real single flip processes become forbidden if $\tilde{t} \ll U_{sf}$ where $\tilde{t}$ is given by Eq. (14) and U_{sf} by Eqs. (12) and (13). This condition is not difficult to satisfy in a real array because amplitude $\tilde{t}$ is typically much smaller than E_J. Further, for moderately sized arrays (with $L=5-10$ which already provide a very good protection) the

energy of a single rhombus flip is only slightly smaller than E_J, so the condition $\tilde{t} \ll U_{sf}$ is not really restrictive. Note, however, that in order to eliminate low energy states of the discrete subsystem we also need to satisfy the condition (9) which implies that the tunneling processes should occur with a significant amplitude. While this might be difficult in the infinite array made from the similar junctions (with the same product of charging and Josephson energies), this is not really a restrictive condition for moderately sized arrays. One can choose, for instance, for a system of $L \times L$ rhombi with $L=5-10$ Josepshon contacts with $E_J=10E_C$. This choice would give $\tilde{t} \approx 0.35E_C$ and $J_x \approx 0.2E_C$ for a system of disconnected horizontal chains. The condition $\tilde{t} \ll U_{sf}$ is well satisfied. Choosing now the vertical links with $\epsilon_J=0.5E_C$ and corresponding $\epsilon_C=20E_C$ we get $L_{eff} \approx 5$, so that the condition (9) is satisfied as well and there are no low energy states. It is more difficult, however, to eliminate the low energy states in the infinite array of coupled chains shown in Fig. 3 and to satisfy the condition $\tilde{t} \ll U_{sf}$ at the same time, especially if all junctions are to be similar in the sense defined above. This can be achieved, however, by replacing the vertical links by rhombi frustrated by the flux $\Phi=(1/2+\delta)\Phi_0$ with $\delta \ll 1$ with each junction characterized by $\tilde{\epsilon}_J \lesssim E_J$ and $\epsilon_C \gtrsim E_C$. These rhombi would provide a significant rigidity to the continuous phase fluctuations (with effective rigidity $\tilde{\epsilon}_J$) but only weak coupling ($\epsilon_J=\delta\tilde{\epsilon}_J$) between discrete degrees of freedom.

Finally, we discuss the effect of the finite phase rigidity on the amplitude of the two rhombi processes, $J_x(k)$. The condition that real single flip processes do not occur does not exclude the virtual processes that flip consequitively two rhombi in the same chain. This would lead to an additional contribution to $\tilde{J}_x(k)$ (8). To estimate this contribution we note that immediately after two flips the continuous phase has a configuration shown in Fig. 4, which is associated with the energy $\sim U_{sf}$. The amplitude for two such consecutive flips is $\tilde{t}^2/U_{sf}$; it can become of the order of $\tilde{J}_x(k)$ in a large system (where U_{sf} is small). However, the amplitude of the full process involves additional action which further suppresses this amplitude. This happens because the two consecutive flips lead to the high energy virtual state sketched in Fig. 4 and in order to get back to the low energy state the resulting continuous phase has to evolve dynamically. To estimate the action corresponding to this evolution, we note that its dynamics is controlled by $\sqrt{E_J/\epsilon_J}$ junctions with charging energy ϵ_C. For the estimate we can replace these junctions by a single junction with capacitive energy $\epsilon_C\sqrt{\epsilon_J/E_J}$. Thus, the final stage of this process leads to the additional term in the action $\delta S \sim E_J/\epsilon_C=\eta(E_J/E_C)$. Depending on the parameter, $\sqrt{E_JE_C}/\epsilon_C=\eta\sqrt{E_J/E_C}$, this additional contribution to the action is smaller or bigger than the total action, but even if it is smaller, it is still large compared with unity if $E_J \gg \epsilon_C$. In this case, the processes that do not change the continuous phase dominate. We emphasize again that in any case the transitions involving two flips in the same row commute with both integrals of motion P, Q and thus do not affect the qualitative conclusions of the previous section. For practically important similar junctions, it means the following. If $\eta \gg \sqrt{E_C/E_J}$ only nearest rhombi flip with the amplitude J_x given by Eq. (8). If $\sqrt{E_C/E_J} \gg \eta \gg E_C/E_J$ the flips occur for the rhombi in the same row if they are closer than $L_{eff} < \eta^{-1}\sqrt{E_C/E_J}$. Finally, for $\eta \ll E_C/E_J$ the distance between flipped rhombi exceeds the size of the half vortex and the two rhombi flips in a large ($L \gg \sqrt{E_J/\epsilon_J}$) array happen via virtual half vortices in the continuous phase.

In the discussion above we have implicitly assumed superconducting boundary conditions such as shown in Fig. 3. These boundary conditions imply that in the absence of significant continuous phase fluctuations $P_iP_j=1$. Physically, it means that if the array as a whole is a superconductor, it still has two states characterized by the phase difference, $\Delta\phi=0$ or π between the left and the right boundaries even in the regime where individual phases in the middle fluctuate strongly between values 0 and π. In this regime of strong discrete phase fluctuations, the external fields are not coupled to the global degree of freedom $\Delta\phi$ describing the array as a whole. In principle, it is also possible to have a similar array with open boundary conditions but in this case it is more difficult to eliminate low lying states because there is no reason for the constraint $P_iP_j=1$ in this case.

B. Array equivalent to a spin model with local interactions

In order to construct the array equivalent to the spin model (1) we need to couple the rhombi in such a way that the transitions involving only one rhombus in a row are not allowed but the superconducting phase varies significantly between one rhombus and the next in the row. This is achieved if rhombi are connected in a chain by a weak Josephson link, characterized by Josephson energy e_J and Coulomb energy e_C, so that $E_J \gtrsim e_J$ and $e_C \ll E_C$, as shown in Fig. 5(b). In this case the simultaneous tunneling of two rhombi which does not change the phase difference across the weak junction is not affected, its amplitude, J_x, is still given by Eq. (8).

In this array the flip of a single rhombus is due to two alternative mechanisms. The first one is that it involves the creation of the half vortex (as discussed in the previous subsection) and does not involve the change in the phase accross the weak junction. Alternatively, it can be due to the phase flip by π across the weak junction. Because $e_C \ll E_C$ this process is slow and its amplitude is low

$$t = E_J^{3/4} e_C^{1/4} e^{-s'\sqrt{E_J/e_C}},$$

where $s' \approx 1$. Further, this process increases the energy of the system by e_J, so if its amplitude is small: $t \ll e_J$ it can be completely neglected. Normally junctions with smaller Josephson energy, $e_J \lesssim E_J$, have a bigger charging energy, not a smaller one as required here. To avoid this problem, we note that these weak junctions can be in practice implemented as two junction superconducting quantum interference device (SQUID) loops containing the flux $\Phi=(1/2+\delta)\Phi_0$ as shown in Fig. 5(c). The effective Josephson coupling provided by such loop is $e_J=\sin(2\pi\delta)E_J^{(0)}$ (where $E_J^{(0)}$ is the energy of the individual junction) while its effective charging energy is $e_C=E_J^{(0)}/2$. This allows one to use bigger junctions for these weak junctions and has another advantage in that it provides an additional controlling parameter on the system. This construction seems somewhat similar to the partially frustrated

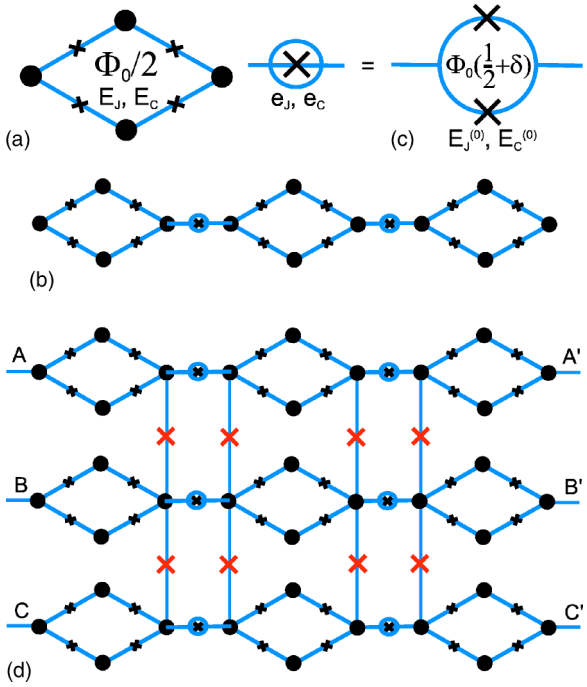

FIG. 5. Schematics of the array. (a) The elementary Josephson circuit emulating spin 1/2 consists of a four junction loop penetrated by magnetic flux $\Phi_0/2$. (b) Implementation of the spin chain by Josephson junction loops; here elementary rhombi are connected by weak links, the appropriate parameters for these links can be obtained if each link in fact consists of two elementary Josephson junctions as shown in (c). (d). Full array implementing the spin model (1). The natural boundary conditions for this array are periodic, i.e., the point A should be connected to A', B to B', etc.

rhombi that one needs to introduce as vertical links in large arrays discussed in the previous subsection [Fig. 3(c)] but it serves a completely opposite purpose: it increases the density of the full vortices while keeping the discrete variables coupled. The partially frustrated rhombi, on the other hand, suppress the fluctuations of the continous phase while allowing independent fluctuations of the discrete variables in different rows. Such a dramatic difference is made possible by a combination of two reasons. First, the frustration has a different effect on these elements: in the case of the SQUID loops half flux eliminates the Josephson coupling completely while in a case of rhombi it leads to the exact degeneracy between two discrete states and to a rigidity for the continuous phase. Second, the values of the charging energies are rather different: in the case of SQUID loops the charging energy of its junctions completely dominates all processes in which phase changes across this loop thereby prohibiting the single rhombus flips. In the case of partially frustrated rhombi their charging energy is of the order of the charging energy of the contacts in horizontal rhombi and thus it suppresses the double flips of distant rhombi in one chain but has relatively minor effect on the nearest ones.

The J_z interaction between the spins is provided by the pairs of the weak Josephson contacts as shown in Fig. 5(d) with the Josephson energy $\epsilon_J \lesssim E_J$ and charging energy $\epsilon_C \gtrsim E_C$. These junctions do not affect the tunneling process of each rhombi but provide the weak interaction between them of the form (1) with the strength

$$J_z = \epsilon_J. \tag{15}$$

Similar to the array discussed in the previous subsection, we have to choose the parameters so that the energy of the half vortex and states resulting from a single flip is sufficiently high compared to the amplitude of the single flips. The discussion of the previous subsection carries over to this array. The only change is that the energy of the vortex in the infinite system contains the weakest link in the horizontal direction, i.e., e_J instead of E_J

$$\tilde{t} \ll U_{sf},$$

$$U_{sf} = \pi^2 e_J/(2L) + (4/\pi)\epsilon_J L \quad L \ll \sqrt{e_J/\epsilon_J},$$

$$U_{sf} = \gamma\sqrt{e_J \epsilon_J} \quad L \gg \sqrt{e_J/\epsilon_J}.$$

Finally, we have to ensure that the phases of the consecutive rhombi are decoupled and the interaction between discrete degrees of freedom is purely local. This is satisfied if the continuous phase across weak junctions fluctuates strongly, i.e., that the energy of a usual (not half) vortex is smaller than the kinetic energy: $\sqrt{e_J \epsilon_J} \ll e_C$. Physically, it means that the array as a whole is an insulator, similar to the topological insulator considered in Ref. 11 due to the full vortices that move in a vertical direction thereby decoupling different columns of rhombi. This condition does not contradict the condition $\tilde{t} \ll U_{sf}$ because the latter involves the exponentially small amplitude of flipping a single rhombus. If both conditions are satisfied the absolute value of the phase on each island constituting a rhombus fluctuates but the difference across the rhombus remains a slow varible taking two discrete values and it flips only simultaneously with the other one in the same row. Note that the interaction between discrete variables belonging to one column is due to the loop formed by these rhombi and two vertical junctions; it is therefore always local by construction, its value is given by Eq. (15). Repeating the arguments of the previous subsection, we see that in order to suppress the simultaneous flips of distant rhombi in the same row one needs also to satisfy the condition $E_J E_C/\epsilon_C^2 \gg 1$, but in contrast to the case of the regime discussed there, here the conditions on the vertical junctions are not difficult to satisfy because one does not need to keep the long range order in a continuous phase. Under these conditions the dynamics of the array is described by the Hamiltonian (1).

Although in this regime the system as a whole is an insulator, it does not allow a half vortex to move across. So, physically, the two states of the global system can be observed in the array with the periodic boundary conditions shown in Fig. 5. Here two different states correspond to the half vortex trapped or not trapped inside the big loop formed by the array as a whole due to the periodic boundary conditions.

IV. LATTICE CHERN-SIMONS GAUGE THEORIES WITH A FINITE ABELIAN GROUP

In this section we discuss the general properties of the Chern-Simons theories with discrete gauge group $\mathbb{Z}_n$. *Con-*

tinuous Chern-Simons theories received a lot of attention in the past because they provide a natural mechanism for the flux attraction to the charged particles which thereby change their statistics.[19] In spite of some technical difficulties, it is possible to construct lattice versions of Chern-Simons theories,[20,21] at least for *continuous* groups. The case of a *discrete* gauge group is slightly more delicate because one cannot define canonically conjugate pairs such as the "magnetic" (A_{ij}) and the "electric" ($\partial/i\partial A_{ij}$) variables. Before we discuss the peculiarities associated with discrete groups, we briefly recapitulate the main properties of the continuous Chern-Simons theories with the noncompact and compact $U(1)$ groups which we describe in the Hamiltonian formalism that we need to generalize these models to discrete groups. The $U(1)$ Chern-Simons model is usually described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\lambda(\dot{A}_x^2 + \dot{A}_y^2) - \frac{1}{2}\mu B^2 + \nu(\dot{A}_x A_y - \dot{A}_y A_x), \qquad (16)$$

where $B=\partial_x A_y - \partial_y A_x$ and dots stand for time derivatives. We have used the gauge in which the time component A_0 of the vector potential is zero. Because of this, we shall use only invariance under time-independent gauge transformations in this discussion. The canonical variables conjugate to A_x and A_y are, respectively, $\Pi_x=\lambda\dot{A}_x+\nu A_y$ and $\Pi_y=\lambda\dot{A}_y-\nu A_x$. The gauge transformations of the classical fields A_ρ are the usual ones $A_\rho \to A_\rho + \partial_\rho \phi$, but because of the Chern-Simons term this also induces a transformation of the conjugate fields $\Pi_\rho \to \Pi_\rho + \nu\epsilon_{\rho\sigma}\partial_\sigma\phi$, where $\epsilon_{xy}=-\epsilon_{yx}=1$.

In quantum theory, Π_ρ and A_ρ become operators with the commutation relation $[\Pi_\rho, A_\mu]=-i\delta_{\rho,\mu}(r-r')$ and the gauge transformation is generated by the operator R_ϕ defined by $R_\phi=\int d^2r R(r)\phi(r)$ with $R(r)=\partial_\rho\Pi_\rho(r)+\nu B(r)$, since $[R(r), A_\rho(r')]=-i\partial_\rho\delta(r-r')$ and $[R(r), \Pi_\rho(r')]=-i\nu\epsilon_{\rho\sigma}\partial_\sigma\delta(r-r')$. In more physical terms, introducing an electric field (which is equal to $\lambda\dot{A}$ in the classical theory) by $E_x=\Pi_x-\nu A_y$ and $E_y=\Pi_y+\nu A_x$, the generator of the gauge transformations can be expressed by $R(r)=\partial_\rho E_\rho + 2\nu B$. It is simple to check that the gauge transformations commute among themselves and also commute with the Hamiltonian density $\mathcal{H}=\frac{1}{2\lambda}E^2-\frac{1}{2}\mu B^2$. In the absence of matter, the equations of motion for the fields read

$$\dot{E}_\rho + \epsilon_{\rho\sigma}\left(\frac{2\nu}{\lambda}E_\sigma + \mu\partial_\sigma B\right) = 0, \qquad (17)$$

$$\lambda\dot{B} = \epsilon_{\rho\sigma}\partial_\rho E_\sigma. \qquad (18)$$

This yields a massive branch of excitations, which are no longer purely transverse, but also develop a longitudinal component proportional to ν. Their dispersion relation is $\lambda^2\omega_k^2=4\nu^2+\lambda\mu k^2$. These propagating modes are then pushed to very high energies in the limit where $\lambda \ll \nu$. In the presence of static external charges e_i at positions r_i, the Gauss constraint becomes: $R(r)=\Sigma_i e_i\delta(r-r_i)$. In the ground state, these charges induce a static field configuration according to: $(2\nu-\lambda\mu/2\nu\nabla^2)B(r)=\Sigma_i e_i\delta(r-r_i)$. So each particle is bound to a flux tube carrying a flux equal to $e_i/2\nu$ and smeared over a typical length $\xi=(\lambda\mu)^{1/2}/\nu$. The pure Chern-Simons limit is then recovered as ξ goes to zero. In this limit, when one particle of charge e_1 goes around another charge e_2, the total wave function of the system is multiplied by an Aharonov-Bohm phase factor equal to $\exp[i(e_1e_2/2\nu)]$, so we a get a factor $\exp[i(e_1e_2/2\nu)]$ upon exchanging these two particles. We note that in the limit $\mu\to 0$, H commutes with operators $\tilde{\Pi}_\mu=\Pi_\mu+\nu\epsilon_{\mu\eta}A_\eta$. These operators provide a generalization of the usual shift operators (Π_μ) for the Chern-Simons theory. Physically, the additional term in these operators appears because in the presence of Chern-Simons one cannot only shift fields, one has also to change the phase of the wave function accordingly.

We now turn to the lattice versions of the Chern-Simons theory. For the sake of simplicity, we shall work here on a square lattice, although these constructions could be generalized to other periodic systems. In this case, the vector potential describing the gauge field is defined on the links of the lattice, and will be denoted by A_{ij} for the oriented link connecting sites i and j. If we reverse the orientation of the link, we obtain $A_{ji}=-A_{ij}$. We shall adopt the Hamiltonian description from now on. In the absence of a Chern-Simons term, the local electric fields are simply the canonical conjugate operators Π_{ij} of the A_{ij}'s. The corresponding commutation relations become

$$[A_{ij}, A_{kl}] = 0,$$

$$[\Pi_{ij}, \Pi_{kl}] = 0,$$

$$[A_{ij}, \Pi_{kl}] = i\delta_{(ij),(kl)}.$$

The natural lattice Hamiltonian whose continuum limit is the same as before reads

$$H = \frac{1}{2\lambda}\sum_{<ij>}\Pi_{ij}^2 + \frac{\mu}{2}\sum_{ijkl}(A_{ij}+A_{jk}+A_{kl}+A_{li})^2. \qquad (19)$$

In the presence of a Chern-Simons term, the electric field operators are modified. The electric field E_{ij} will contain, besides Π_{ij}, terms associated with the vector potential in the direction perpendicular to the link (ij). On a square lattice, there are four links immediately perpendicular to this link, and containing either site i or j. Let us denote by $\mathcal{N}(ij)$ this set of four links. To reflect the signs which appear in the continuous version discussed above, the links in $\mathcal{N}(ij)$ have to be oriented in such a way as when (ij) runs from left to right, (kl) runs from bottom to top, as illustrated in Fig. 6(a) below.

With these notations, the lattice Hamiltonian in the presence of a Chern-Simons term may be written as

$$H_{CS} = \frac{1}{2\lambda}\sum_{<ij>}\left(\Pi_{ij} - \frac{\nu}{4}\sum_{(kl)\in\mathcal{N}(ij)}A_{kl}\right)^2 + \frac{\mu}{2}\sum_{ijkl}(A_{ij}+A_{jk}+A_{k\ell}+A_{li})^2. \qquad (20)$$

The important fact is that the generalized electrical field operators $E_{ij}=\Pi_{ij}-\nu/4\Sigma_{(kl)\in\mathcal{N}(ij)}A_{kl}$ no longer commute. The

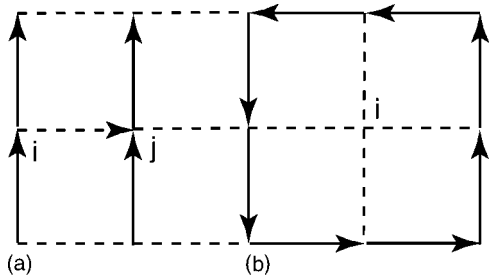

FIG. 6. (a) An oriented link (ij) (dashed line) and the four oriented links adjacent to it (thick lines) which enter in the set $\mathcal{N}(ij)$. (b) For a given site i, the loop $\mathcal{L}(i)$ defined in text (thick lines).

relation $[E_x(r),E_y(r')]=-2i\nu\delta(r-r')$ from the continuous case becomes now: $[E_{ij},E_{kl}(r')]=-_{i/2}\nu$, whenever (kl) is one of the four links in $\mathcal{N}(ij)$. Here (ij) is oriented along the positive x axis, and (kl) along the positive y axis. The apparent difference in the normalization of the delta function on the right-hand sides of these expressions is compensated by the fact that $\mathcal{N}(ij)$ contains four elements.

The energy spectrum for the eigenmodes associated with this system now reads

$$\lambda^2\omega_k^2 = 4\nu^2\cos^2\left(\frac{k_x}{2}\right)\cos^2\left(\frac{k_y}{2}\right) + 4\lambda\mu\left(\sin^2\left(\frac{k_x}{2}\right) + \sin^2\left(\frac{k_y}{2}\right)\right).$$

This spectrum is specially interesting in the limit where μ goes to zero, since then it exhibits lines of zero modes at the boundaries of the first Brillouin zone, namely for $k_x=\pm\pi$ or $k_y=\pm\pi$. These modes are directly related to two remarkable nonlocal conservation laws which appear in the limit of vanishing μ. More precisely, for each row and each column, we may define two conserved quantities $\tilde{P}_{\text{row}}$ and $\tilde{Q}_{\text{column}}$ in the following way:

$$\tilde{P}_{\text{row}} = \sum_{j\in\text{row}} (-1)^{x_j} E_{j,j+\hat{\mathbf{x}}}, \tag{21}$$

$$\tilde{Q}_{\text{column}} = \sum_{j\in\text{column}} (-1)^{y_j} E_{j,j+\hat{\mathbf{y}}}. \tag{22}$$

Unlike the row and column operators discussed in Sec. II these operators mutually commute. Note that in the $\mu=0$ limit, the system exhibits a large set of *local* conserved quantities $\tilde{\Pi}_{ij}$, i.e., one for each link, defined by

$$\tilde{\Pi}_{ij} = \Pi_{ij} + \tfrac{\nu}{4}\textstyle\sum_{(kl)\in\mathcal{N}(ij)} A_{kl}. \tag{23}$$

Similar to the local electric fields, these variables do not mutually commute, so they cannot be simultaneously diagonalized. Furthermore, they are not invariant under gauge transformations, since the electric field operators are gauge invariant and since $E_{ij}-\tilde{\Pi}_{ij}=-\nu/2\Sigma_{(kl)\in\mathcal{N}(ij)}A_{kl}$ which is clearly gauge dependent. This situation is very similar to what occurs in the process of quantizing the motion of a particle on a plane in the presence of a uniform perpendicular magnetic field. The local electrical field operators are analogous to the gauge invariant velocity operators, $P-eA$, in the particle problem. Because of the magnetic field, the two components of this vector no longer commute. Since the vector potential A is not translation invariant, the usual translation operators have to be combined with gauge transformations in order to commute with the kinetic energy $(P-eA)^2$. These deformed generators of translations are analogous to the two components of the vector $\tilde{\Pi}=P+eA$. Apparently, the only way to construct gauge-invariant symmetry operators in the lattice gauge model is to use the nonlocal combinations $\tilde{P}_{\text{row}}$ and $\tilde{Q}_{\text{column}}$ defined in Eqs. (21) and (22). Note that a model with similar conservation laws, for interacting Bosons on a square lattice, has been analyzed in Ref. 22.

A next step is to construct a lattice Chern-Simons gauge theory for the continuous but *compact* $U(1)$ group. This is simply achieved in the absence of a Chern-Simons term, by assuming that the vector potential variables A_{ij} are *periodic*, with a period chosen, for instance, equal to 2π. This implies that the spectrum of the conjugate operators Π_{ij} are *discrete*, containing only integer values. Because of this, the form of the Hamiltonian has to be modified from Eq. (19) above, and a natural choice respecting the requirements of gauge invariance and periodicity in the gauge potentials reads

$$H = \frac{1}{2\lambda}\sum_{<ij>}\Pi_{ij}^2 - \mu\sum_{ijkl}\cos(A_{ij}+A_{jk}+A_{kl}+A_{li})^2. \tag{24}$$

Using this Hamiltonian as a starting point we add a Chern-Simons term by the standard deformation of the electrical operators

$$H_{CS} = \frac{1}{2\lambda}\sum_{<ij>}\left(\Pi_{ij} - \frac{\nu}{4}\sum_{(kl)\in\mathcal{N}(ij)}A_{kl}\right)^2 - \mu\sum_{ijkl}\cos(A_{ij}+A_{jk}+A_{kl}+A_{li})^2. \tag{25}$$

We now discuss what can be the Hilbert space associated with this Hamiltonian. This is a nontrivial problem because we can no longer impose the periodicity in the local gauge potentials in the usual way, assuming that the wave function of the system considered as a function of the A_{ij}'s is 2π periodic with respect to any A_{ij}. This is not possible because the naive shift operator $S_{ij}^{naive}=\exp(-i2\pi\Pi_{ij})$ that changes A_{ij} into $A_{ij}+2\pi$ no longer commutes with the kinetic part of the Hamiltonian. The appropriate definition of these shift operators becomes $S_{ij}=\exp(-i2\pi\tilde{\Pi}_{ij})$, with the $\tilde{\Pi}_{ij}$ defined in Eq. (23). In order to construct invariant states under this full set of shift operators, we need them to be mutually commuting. This is realized only for a discrete set of values of $\nu=m/\pi$, where m is any integer. Thus, the compact gauge theory is compatible with the Chern-Simons term only for special, "quantized," values of ν (see also Refs. 23 and 24). This statement is very similar to the well-known fact that in order to quantize the problem of a particle on a torus in a perpendicular uniform magnetic field, the total magnetic flux through this torus should be an integer multiple of the flux quantum h/e. In one particle problem this requirement simply expresses the need for mutual commutation between two magnetic translations that are used to construct the torus from an infinite plane.

Note that, even for these special values of ν, the Hilbert space of the theory is peculiar because the shift operators S_{ij} are not gauge invariant. More precisely, let us denote the generator of the gauge transformation sending A_{ij} into $A_{ij}+\alpha$, for any site j connected to site i by $U_i(\alpha)=\exp(-i\alpha\Sigma_j^{(i)}\tilde{\Pi}_{ij})$. In this expression, the sum is taken over the nearest neighbors j of site i. Clearly, these unitary operators commute with the Hamiltonian (25). For a site i, and a link (jk) belonging to $\mathcal{L}_i$, where $\mathcal{L}_i$ is the oriented loop [see Fig. 6(b)] defined by the edges of the square built from four elementary plaquettes whose center is located at site i, we have the following relations:

$$U_i(\alpha)S_{jk}=e^{\pm i\pi\nu\alpha}S_{jk}U_i(\alpha). \tag{26}$$

The sign in the phase factor depends on the orientation of the link (jk), and is negative if it is oriented along $\mathcal{L}_i$. This equation implies that *one cannot enforce at the same time a condition of gauge invariance on the physical states, and invariance of the wave function under the* 2π *shift operators*. If we choose to work in a basis of eigenvectors for the gauge transformations, namely states $|\Psi\rangle$ satisfying $U_i(\alpha)=\exp(i\alpha Q_i)|\Psi\rangle$, where Q_i is the electrical charge at site i, applying S_{jk} to $|\Psi\rangle$ for (jk) in $\mathcal{L}_i$ modifies the charge Q_i by $\pm m$ if $\nu=m/\pi$. Conversely, a subspace where $S_{jk}|\Psi\rangle=e^{i\theta_{jk}}|\Psi\rangle$ is *not* gauge invariant, since after applying $U_i(\alpha)$ on such states, we get the new periodic conditions with $\theta'_{jk}=\theta_{jk}+\pi\nu\alpha$, for any link (jk) belonging to $\mathcal{L}_i$ with the corresponding orientation. However, it appears that $\Sigma_k^{(j)}\theta'_{jk}=\Sigma_k^{(j)}\theta_{jk}$ because $\Pi_k^{(j)}S_{jk}=U_j(2\pi)$ commutes with any gauge transformation. This relation shows that the gauge invariant quantity $\Sigma_k^{(j)}\theta_{jk}$ is nothing but $2\pi Q_j$ modulo 2π.

The properties of this model have been investigated by several groups. In the absence of a Chern-Simons term, the periodicity of the $U(1)$ gauge field allows quantum tunneling processes where the local flux on a plaquette changes by $\pm 2\pi$. In a $2+1$ dimensional path integral description, these instantons (called here monopoles) interact via a Coulomb-like $1/r$ potential, leading to Debye screening in this monopole plasma. The proliferation of such tunneling events drives the system into a phase where the magnetic variables A_{ij} are strongly disordered, and in which external static electric charges interact by a confining potential increasing linearly with their separation.[25,26] In the presence of a Chern-Simons term, the properties of the monopole plasma are deeply altered, and several groups have reached the conclusion that a linear interaction now binds pairs of monopoles of opposite charges, thus destroying the confinement of electrical charges.[21,27,28]

To extend this construction to a discrete $\mathbb{Z}_n$ group, we replace the continuous vector potential on the link joining sites i and j by $A_{ij}=2/\pi n p_{ij}$, where p_{ij} is an integer. In the absence of a Chern-Simons term, the generator of the gauge transformation based at site i sending p_{jk} into $p_{jk}+\delta_{ji}-\delta_{ki}$ is $U_i=\Pi_j^{(i)}\pi_{ij}^+$, in analogy with the continuous case discussed above. The unitary operator π_{ij}^+ is analogous to the operator $\exp[-i(2/\pi n)\Pi_{ij}]$ of the continuous model, namely it transforms A_{ij} into $A_{ij}+2\pi/n$ or equivalently p_{ij} into $p_{ij}+1$. In order to attach flux to particles, the generator U_i has to be modified by a phase factor $\exp(-i(v/4)[2\pi/n]^2\Sigma_{(jk)\in\mathcal{L}_i}p_{jk})$, where the (jk)'s belong to the loop $\mathcal{L}_i$ already defined above [see Fig. 6(b)]. For each value of ν, we define the generators by

$$U_i=\prod_j^{(i)}\pi_{ij}^+\exp\left(-i\frac{\nu}{4}\left(\frac{2\pi}{n}\right)^2\sum_{(jk)\in\mathcal{L}(i)}p_{jk}\right). \tag{27}$$

This definition preserves the fact that these generators mutually commute. Note that since the generators U_i commute simultaneously with the local fluxes, a convenient gauge-invariant basis of the Hilbert space is obtained by simultaneous diagonalization of the local fluxes and the U_i's. With a discrete basis to describe each link, the kinetic part in the Hamiltonian (25) has to be modified. The most natural way to do this is to replace the local electrical field E_{ij} by a gauge-invariant operator $\mathcal{E}_{ij}^+$ which shifts A_{ij} by the minimal possible amount $2\pi/n$. This operator is defined as follows:

$$\mathcal{E}_{ij}^+=\pi_{ij}^+\exp\left(i\frac{\nu}{4}\left(\frac{2\pi}{n}\right)^2\sum_{(kl)\in\mathcal{N}(ij)}p_{kl}\right).$$

The Hamiltonian may now be written as

$$H_{CS}=-\frac{1}{\lambda_n}\sum_{<ij>}(\mathcal{E}_{ij}^+ +\mathcal{E}_{ij}^-) -\mu\sum_{ijkl}\cos\left(\frac{2\pi}{n}(p_{ij}+p_{jk}+p_{kl}+p_{li})\right)^2, \tag{28}$$

where $\mathcal{E}_{ij}^-$ is the Hermitian conjugate of $\mathcal{E}_{ij}^+$. To recover the Hamiltonian (25) in the limit where n becomes very large, we notice that $\mathcal{E}_{ij}^+$ acts in the same way as $\exp[-i(2\pi/n)E_{ij}]$, therefore we have to choose λ_n so that $2(2\pi/n)^2\lambda_n^{-1}=\lambda^{-1}$.

With these notations, the operators shifting the link variables p_{ij} by one unit, and which commute with all the gauge-invariant kinetic terms $\mathcal{E}_{ij}^{\pm}$ read

$$\tilde{\pi}_{ij}^+=\pi_{ij}^+\exp\left(-i\frac{\nu}{4}\left(\frac{2\pi}{n}\right)^2\sum_{(kl)\in\mathcal{N}(ij)}p_{kl}\right)$$

since they are analogous to $\exp[-i(2\pi/n)\tilde{\Pi}_{ij}]$ in the continuous model. The 2π shift operators S_{ij} previously introduced are then equal to $(\tilde{\pi}_{ij}^+)^n$. Note that the parameter ν is quantized, in the same way as before (namely $\nu=m/\pi$, with integer m), since we impose the model to be periodic when p_{ij} is changed into $p_{ij}+n$. More precisely, as for the compact $U(1)$ group, this notion of periodicity requires the mutual commutation between all the 2π shift operators S_{ij}. Models obtained from two values of ν which differ by an integer multiple of $2n^2/\pi$ are clearly identical. We also note that changing m into $2n^2-m$ amounts to replacing all the phase factors entering in the definition of various operators such as U_i by their complex conjugates, and the corresponding models exhibit similar properties. In the special case $m=n^2$, the operators $\mathcal{E}_{ij}^+$ mutually commute, so this case is equivalent to $m=0$. Therefore, it is sufficient to choose m in the set of integers between 0 and n^2-1. Among those n^2 possible values of ν, there is an interesting subset of n values for which m is an integer multiple of n. If this condition holds, the generators of the el-

ementary gauge transformations U_i defined in Eq. (27) commute with all the 2π shift operators S_{jk}. It is then possible to apply *simultaneously* the condition of gauge invariance and the periodic boundary conditions $S_{jk}|\Psi\rangle=|\Psi\rangle$ for the allowed physical states.

If $\nu=0$, the resulting discrete gauge theory (without Chern-Simons term) has two regimes according to the relative size of the two terms entering in H_{CS}. When $\mu \gg \lambda_n^{-1}$, $H(\nu=0)$ describes a phase where localized flux excitations, called fluxons, have the energy gap $\mu(1-\cos(2\pi/n))$. In this phase, quantum fluctuations of the magnetic variables p_{ij} are small, and the effective interaction between two static external electric charges varies logarithmically with their separation. Notice that this phase owes its existence to the *discrete* nature of the symmetry group, as illustrated by the vanishing of the corresponding energy gap as n is taken to infinity. A small λ_n^{-1} term simply gives some amount of dispersion to these excitations. When λ is decreased further below a critical value λ_n^c, the fluxon gap closes, and the system enters the charge confining disordered phase similar to the one obtained in the *compact* $U(1)$ case, for all values of λ.

Let us first consider the effect of a switching on a Chern-Simons term in the former regime, where the potential energy (proportional to μ) dominates. When λ_n is very large, we do not expect the flux binding mechanism to operate. Indeed, a unit charge at site j corresponds to imposing the Gauss law constraint $U_k|\Psi\rangle=\exp[i(2\pi/n)\delta_{jk}]|\Psi\rangle$. When λ_n^{-1} is small, it is energetically more favorable to keep a vanishing flux everywhere, because of the low value of the kinetic term compared to the fluxon gap. So we expect the flux attachment mechanism to work only if λ_n is smaller than a critical value λ_n^*. When λ_n is further reduced, below λ_n^c, the fluxon gap eventually closes, and the qualitative properties of the system change drastically. In Sec. V we present a simple perturbative estimate of the critical value λ_n^c and argue that it is in fact equal to λ_n^*, i.e., both transitions happen simultaneously. As already discussed for the case of the compact $U(1)$ group, the presence of the Chern-Simons term modifies deeply the picture obtained for vanishing ν in the strongly fluctuating regime of small λ_n. Note that, by contrast to the $\nu=0$ case, analysis of the $\lambda \to 0$ limit is difficult since the E_{ij} operators no longer commute if they involve two perpendicular links sharing a common site. In the $n \to \infty$ limit, we expect to recover the continuous, but *compact* $U(1)$, theory for which we still do not know how to write down explicitly the ground state wave function, even in the $\lambda \to 0$ limit.

V. MAPPING CHERN-SIMONS THEORIES ONTO SPIN MODELS

Here we shall study in more detail the interesting limit of the vanishing magnetic energy and show explicitely how, in the $n=2$ case, it maps precisely on the models studied in the beginning of this paper. As we have discussed, it is possible to propose a design of Josephson junction arrays which directly implements this limit.

As a first step, it is convenient to introduce a basis in the Hilbert space of gauge invariant states (i.e., states $|\Psi\rangle$ such that $U_i|\Psi\rangle=0$ for any site i), which keeps track of the flux variables. For any square plaquette $(ijkl)$ centered at $\mathbf{r}$, the corresponding flux $\sigma_{\mathbf{r}}$ is the integer $p_{ij}+p_{jk}+p_{kl}+p_{li}$. For any flux configuration $\{\sigma_{\mathbf{r}}\}$, we may then define a gauge-invariant quantum state $|\Psi(\{\sigma_{\mathbf{r}}\})\rangle$. Our main task is to represent the algebra of gauge invariant operators $\mathcal{E}_{ij}^+$ in such a basis. These operators obey two families of constraints. First we have

$$\mathcal{E}_{ij}^+ \mathcal{E}_{jk}^+ = \exp\left(i2\pi \frac{m}{n^2} \right) \mathcal{E}_{jk}^+ \mathcal{E}_{ij}^+. \tag{29}$$

Here the sign corresponds to the geometry when link (jk) is perpendicular to (ij) and located on its left. Notions of left and right are defined for an observer moving along the link in agreement with its orientation. We have used the condition $\nu=m/\pi$. Second, these operators are related to the generators of local gauge transformations by

$$\prod_j^{(i)} \mathcal{E}_{ij}^+ = \exp\left(i2\pi \frac{m}{n^2} \sum_{\mathbf{r}}^{(i)} \sigma_{\mathbf{r}} \right) U_i. \tag{30}$$

As usual, the product in the left-hand side runs over the nearest neighbors of site i, whereas the sum in the right-hand side involves the four plaquettes adjacent to i. In this expression, attention should be paid to the ordering of the various operators. We assume that the two operators involving bonds along a given direction directly follow each other. Once this is enforced, any of the eight possible residual permutations compatible with this criterion does not change the result. For any oriented link (ij), let us call $\mathbf{r}$ (resp. $\mathbf{r}'$) the adjacent plaquette located at the left (resp. right) of (ij). The operator $\mathcal{E}_{ij}^+$ decreases the local flux $\sigma_{\mathbf{r}'}$ and increases $\sigma_{\mathbf{r}}$ by one unit. We see that $\mathcal{E}_{ij}^+$ should be proportional to $\sigma_{\mathbf{r}}^+ \sigma_{\mathbf{r}'}^-$ up to a phase factor which depends on the configuration of local fluxes on the whole lattice. This phase factor is required in order to satisfy the constraints (29) and (30) above. In a general case this phase factor might become a very nonlocal function of the flux configuration but it remains simple in the case of $\mathbb{Z}_2 m=2$ model. We discuss now its construction in different cases, starting with the simplest ones.

We start with the simplest case $n=2$, and $m=2$. Then, the electrical operators $\mathcal{E}_{ij}^+$ on two adjacent and perpendicular links *anticommute.* Furthermore, according to Eq. (26), the generators of the local gauge transformations commute with the shift operators $S_{jk}=(\tilde{\pi}_{jk}^+)^2$. It is then possible to impose *simultaneously* the gauge invariance constraint and the $\mathbb{Z}_2$ periodicity on the links. As a result, the two operators $\sigma_{\mathbf{r}}^+$ and $\sigma_{\mathbf{r}}^-$ are equal. We may then represent the above algebra by the following Pauli operators, associated with a quantum Ising model attached to the plaquettes of the lattice

$$\mathcal{E}_{ij}^+ = \sigma_{\mathbf{r}}^x \sigma_{\mathbf{r}'}^x, \quad \text{for vertical (ij)} \tag{31}$$

or

$$\mathcal{E}_{\mathrm{ij}}^+ = \sigma_{\mathbf{r}}^z \sigma_{\mathbf{r}'}^z, \quad \text{for horizontal } (ij) \tag{32}$$

$$\exp(i\pi\sigma_{\mathbf{r}}) = \sigma_{\mathbf{r}}^y. \tag{33}$$

The symmetry operators defined in Eqs. (21) and (22) are easily generalized to the case of a discrete group. We now have

$$\tilde{P}_{\mathrm{r}} = \prod_{j \in \mathrm{row(r)}} \mathcal{E}_{j,j+\hat{\mathbf{x}}}^{\mathrm{sign}(j)}, \tag{34}$$

$$\tilde{Q}_{\mathrm{s}} = \prod_{j \in \mathrm{column(s)}} \mathcal{E}_{j,j+\hat{\mathbf{y}}}^{\mathrm{sign}(j)}. \tag{35}$$

In these equations, the symbol sign (j) stands for $+$ or $-$, according to the parity of x_j (resp. y_j) in the first (resp. second) equation, row(r) denotes row number r and column(s) denotes column number s. For a generic (not boundary) row and column these operators commute.

It is now possible to recover the symmetry operations P_{i} and Q_{j} introduced in Sec. II for Chern-Simons theories with proper boundary conditions. First, notice that for $\mathbb{Z}_2$ model operators $\tilde{P}_{\mathrm{r}}$ and $\tilde{Q}_{\mathrm{s}}$ correspond to the product of two P (or two Q) operators: $\tilde{P}_{\mathrm{r}} = P_r P_{r+1}$ when expressed in terms of the spin operators of Sec. II. If operator $P_0 = 1$ is trivial $(P_0 = 1)$, $\tilde{P}_1 = P_1$ and we can recover a single P_r operator taking the product of all $P_{r'}$ operators with $r' < r$. In terms of the gauge theory, the Chern-Simons Hamiltonian [Eq. (28)] in its $\mu = 0$ limit should not contain the dynamical variables (gauge fields) associated with the bonds along the edge of the lattice. In this case the operators $\tilde{P}_1$ and $\tilde{Q}_1$ indeed do not commute because they contain only one pair of noncommuting electric fields located in the corner of the lattice. With this assumption, we obtain two families of operators commuting with the Chern-Simons Hamiltonian at $\mu = 0$

$$P_{\mathrm{r}} = \prod_{y_j \leqslant y_{\mathrm{row(r)}}} \mathcal{E}_{j,j+\hat{\mathbf{x}}}^{\mathrm{sign}(j)},$$

$$Q_{\mathrm{s}} = \prod_{x_j \leqslant x_{\mathrm{column(s)}}} \mathcal{E}_{j,j+\hat{\mathbf{y}}}^{\mathrm{sign}(j)}.$$

Here sign (j) is defined as above. These operators satisfy the generalized commutation relations

$$P_{\mathrm{r}} Q_{\mathrm{s}} = \exp\left(i 2\pi \frac{m}{n^2}\right) Q_{\mathrm{s}} P_{\mathrm{r}}. \tag{36}$$

In the special case $m = 2$, these operators anticommute. Using Eqs. (31) and (32) above, we may write

$$P_{\mathrm{r}} = \prod_{\mathbf{r} \in \mathrm{row(r)}} \sigma_{\mathbf{r}}^{z},$$

$$Q_{\mathrm{s}} = \prod_{\mathbf{r} \in \mathrm{column(s)}} \sigma_{\mathbf{r}}^{x}.$$

We now discuss the mapping in a more general case. First, we assume that the generators of local gauge transformations still commute with the shift operators S_{jk} on the links. According to Eq. (26), this requires m to be a multiple of n. In this case, we may still view the local fluxes $\sigma_{\mathbf{r}}$ as defined modulo n. This implies in particular that $(\sigma_{\mathbf{r}}^{+})^n$ can be chosen to act as the identity operator. After enforcing the gauge invariance and the $\mathbb{Z}_n$ periodicity in the bond variables, we may represent $\mathcal{E}_{ij}^{+}$ as

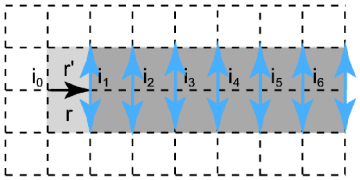

FIG. 7. The electrical field operator, $\mathcal{E}_{i_0 i_1}$ moves one flux quantum from the plaquette r to r'. In order to represent this process in the gauge where all horizontal links A_{ij} are 0, we use the gauge transformation on the string of sites $i_1 \ldots i_n$ that produce the electric field operators on vertical links (shown by light arrow) and the phase factors associated with the shaded area. In the flux representation the effect of vertical electric field is to remove the flux from plaquette r and to add one flux to plaquette r'. Note that the phase factors associated with all plaquettes except r and r' are doubled.

$$\mathcal{E}_{ij}^{+} = \exp(i\mathcal{A}_{\mathbf{r},\mathbf{r}'}(\{\sigma_{\mathbf{r}}\}))\sigma_{\mathbf{r}}^{+}\sigma_{\mathbf{r}'}^{-}. \tag{37}$$

The "statistical" gauge field $\mathcal{A}_{\mathbf{r},\mathbf{r}'}$ should not be confused with the original link variables A_{ij}. This new entity is imposed to us by the necessity to satisfy the constraints (29) and (30). Note that as usual, there is a large amount of freedom in the definition of $\mathcal{A}_{\mathbf{r},\mathbf{r}'}$, reflecting the arbitrariness in choosing a global phase for each state $|\Psi(\{\sigma_{\mathbf{r}}\})\rangle$. If we could ignore the first constraint (29), our system of fluxons would be completely equivalent to a collection of particles obeying fractional statistics, since the second constraint (30) relates the total statistical flux seen by a fluxon hopping around an elementary plaquette of the dual lattice centered at i, to the number of fluxons in the immediate neighborhood of i. The presence of the first constraint is an original feature of fluxon dynamics in Chern-Simons theories.

Let us now show how to construct explicitly one realization for this statistical gauge field $\mathcal{A}_{\mathbf{r},\mathbf{r}'}$. This amounts to making a definite choice for the global phase of the basis states $|\Psi(\{\sigma_{\mathbf{r}}\})\rangle$. If we have one fluxon on the plaquette centered at $\mathbf{r}$, it is possible to represent this by a string-like configuration of bond variables $p_{ij}(\mathbf{r})$ such that p_{ij} vanishes on any horizontal bond, and on most vertical bonds, with the exception of all the links located on the same row as $\mathbf{r}$ and on its right, for which it takes the value 1. This may be summarized by the following definition:

$$|\mathbf{r}\rangle = \mathcal{P}_{\mathrm{inv}} \prod_{(ij) > \mathbf{r}} \mathcal{E}_{ij}^{+} |0\rangle.$$

Here, the operator $\mathcal{P}_{\mathrm{inv}}$ is the projector on the subspace of the gauge invariant states. The notation $(ij) > \mathbf{r}$ stands for all the links (ij) on the right-hand side of the plaquette center $\mathbf{r}$ and on the same row, and the reference state $|0\rangle$ is simply the state where all the bond variables p_{ij} are equal to 0. For a general flux configuration $\{\sigma_{\mathbf{r}}\}$, we simply choose a reference configuration $p_{ij}(\{\sigma_{\mathbf{r}}\})$ obtained by superposing the configurations associated with each fluxon excitation in the system. Therefore, the integers $p_{ij}(\{\sigma_{\mathbf{r}}\})$ are defined as follows. They

are equal to zero for any *horizontal* bond (ij). For a *vertical* bond, we set

$$p_{ij}(\{\sigma_{\mathbf{r}}\}) = \sum_{\mathbf{r}<(ij)} \sigma_{\mathbf{r}}.$$

Using this, we define

$$|\Psi(\{\sigma_{\mathbf{r}}\})\rangle = \mathcal{P}_{\text{inv}} \prod_{(ij)} (\mathcal{E}_{ij}^{+})^{p_{ij}(\{\sigma_{\mathbf{r}}\})} |0\rangle. \tag{38}$$

We could have used operators π_{ij}^{+} instead of $\mathcal{E}_{ij}^{+}$ to generate a state with the same desired flux configuration $\{\sigma_{\mathbf{r}}\}$, but the advantage of $\mathcal{E}_{ij}^{+}$ is that they commute with the projector $\mathcal{P}_{\text{inv}}$. These states $|\Psi(\{\sigma_{\mathbf{r}}\})\rangle$ form a complete basis of the gauge invariant subspace, since any configuration $\{p_{ij}\}$ of bond variables producing the flux pattern $\{\sigma_{\mathbf{r}}\}$ may be deduced from the configuration $\{p_{ij}(\{\sigma_{\mathbf{r}}\})\}$ by a gauge transformation. With this choice of gauge, we have

$$\mathcal{E}_{ij}^{+} = \sigma_{\mathbf{r}}^{+} \sigma_{\mathbf{r}'}^{-} \tag{39}$$

for any *vertical* link (ij). This means that $\mathcal{A}_{\mathbf{r},\mathbf{r}'}$ vanishes whenever the vector joining $\mathbf{r}$ and $\mathbf{r}'$ is equal to $\pm\hat{\mathbf{x}}$. For a *horizontal* link (ij) oriented from left to right, we make a repeated use of the second constraint (30) above for all the sites k located on the same row as (ij) and on its right, which, if applied on a gauge invariant state, yields

$$\mathcal{E}_{ij}^{+} = \exp\left(i2\pi \frac{m}{n^2} (\sigma_{\mathbf{r}} + \sigma_{\mathbf{r}'} + 2 \sum_{\mathbf{r}''>\mathbf{r},\mathbf{r}'} \sigma_{\mathbf{r}''}) \right) \mathcal{M}_{ij},$$

where the "string" operator $\mathcal{M}_{ij}$ is defined by

$$\mathcal{M}_{ij} = \prod_{(kl)>\mathbf{r}} \mathcal{E}_{kl}^{+} \prod_{(kl)>\mathbf{r}'} \mathcal{E}_{kl}^{-}.$$

As before, $\mathbf{r}$ (resp. $\mathbf{r}'$) denotes the adjacent plaquette located above (resp. below) the oriented link (ij). The notation $(kl)>\mathbf{r}$ refers to all the vertical links (kl) on the right of plaquette $\mathbf{r}$ and on the same row. In our gauge the string operator becomes

$$\mathcal{M}_{ij} |\{\sigma_{\mathbf{r}}\}\rangle = \sigma_{\mathbf{r}}^{+} \sigma_{\mathbf{r}'}^{-} |\{\sigma_{\mathbf{r}}\}\rangle.$$

Finally, we get

$$\mathcal{E}_{ij}^{+} = \sigma_{\mathbf{r}}^{+} \sigma_{\mathbf{r}'}^{-} \exp\left(i2\pi \frac{m}{n^2} \left(\sigma_{\mathbf{r}} + \sigma_{\mathbf{r}'} + 2 \sum_{\mathbf{r}''>\mathbf{r},\mathbf{r}'} \sigma_{\mathbf{r}''} \right) \right), \tag{40}$$

where the formula applies only on the gauge invariant subspace. This shows that

$$\exp(i\mathcal{A}_{\mathbf{r},\mathbf{r}'}) = \exp\left(i2\pi \frac{m}{n^2} \left(\sigma_{\mathbf{r}} + \sigma_{\mathbf{r}'} + 2 \sum_{\mathbf{r}''>\mathbf{r},\mathbf{r}'} \sigma_{\mathbf{r}''} \right) \right)$$

whenever the vector joining $\mathbf{r}$ and $\mathbf{r}'$ is equal to $\pm\hat{\mathbf{y}}$. The right-hand side of this expression is most of the time highly *nonlocal*. The only exception is the case when $2m$ is an integer times n^2. For m chosen in the interval $[0, n^2-1]$, this occurs only when $2m=n^2$, which requires n to be even. If this condition is satisfied, we get

$$\exp(i\mathcal{A}_{\mathbf{r},\mathbf{r}+\hat{\mathbf{y}}}) = \exp\big(i\pi(\sigma_{\mathbf{r}} + \sigma_{\mathbf{r}'})\big).$$

In physical terms, this set of models corresponds to an effective Bose statistics for fluxons.

Let us now address the most general case. Since according to Eq. (26), exchanging U_i and S_{jk} involves a phase factor $\exp[\pm i2\pi(m/n)]$, we may always diagonalize simultaneously the local gauge generators U_i with the operators $S_{jk}^{n'}$ where n' is equal to n divided by the greatest common divisor of m and n. This implies that dimension of the Hilbert space associated with one flux is increased to $N=nn'$ and we may view the local fluxes as elements of $\mathbb{Z}_N$. All the expressions already written are then valid. In particular, we may check that $\exp(i\mathcal{A}_{\mathbf{r},\mathbf{r}'})$ is unchanged, if we change $\sigma_{\mathbf{r}}$ into $\sigma_{\mathbf{r}}+Nb_{\mathbf{r}}$, where $b_{\mathbf{r}}$ is any integer.

The representation in terms of fluxes constructed in this section becomes especially convenient when the gap of a single fluxon is the largest energy scale in the problem which occurs if $\mu >> \lambda_n^{-1}$ [see Hamiltonian (28)]. In this limit, the term proportional to λ_n^{-1} induces tunneling processes where a fluxon jumps from a plaquette to one of its neighbors. Using the constraints (29) and (30) and, we see that the operator which moves a fluxon around the elementary plaquette of the dual lattice centered at site i is simply equal to U_i, provided that it acts on a state with only one fluxon on a plaquette adjacent to i. Indeed, this operator may be written as: $\mathcal{E}_{i,i-\hat{\mathbf{y}}}^{+} \mathcal{E}_{i,i-\hat{\mathbf{x}}}^{+} \mathcal{E}_{i,i+\hat{\mathbf{y}}}^{+} \mathcal{E}_{i,i+\hat{\mathbf{x}}}^{+}$; notice that this ordering is different from Eq. (30). To recast this product in the form of the latter expression, one may permute the last two operators on the right, which according to Eq. (29) produces a phase factor $\exp[-i2\pi(m/n^2)]$. But for a state with exactly one fluxon located next to site i, this factor is exactly cancelled by the term $\exp[i2\pi(m/n^2)\Sigma_{\mathbf{r}}^{(i)}\sigma_{\mathbf{r}}]$ present in Eq. (30), leaving only the local gauge generator U_i. This shows that a single fluxon moves in this limit as a free quantum particle on the dual lattice, with an energy spectrum

$$\epsilon(k) = \mu\left(1 - \cos\left(\frac{2\pi}{n}\right)\right) - 2\lambda_n^{-1}\big(\cos(k_x) + \cos(k_y)\big).$$

This spectrum is gapped as long as λ remains larger than λ_n^c given by

$$\frac{1}{\lambda_n^c} = \frac{\mu}{4}\left(1 - \cos\left(\frac{2\pi}{n}\right)\right). \tag{41}$$

These equations neglect the renormalization of the fluxon spectrum by the fluctuations, but we expect that it remains qualitatively correct when these fluctuations are taken into account. In particular, even in the regime of the strong fluctuations, close to the transition, the picture of the fluxons moving with a spectrum $\epsilon(k)=\delta\epsilon+(1/2m)\mathbf{k}^2$ should remain valid at sufficiently long scales as long as $\lambda_n > \lambda_n^c$. For a system with an external charge Q_i located at site i, we have $U_i=\exp[i(2\pi n)Q_i]$, so a single fluxon will experience the usual Aharonov-Casher effect from this static charge, independently of the value of the Chern-Simons coefficient ν. Since this interference effect always raises the value of the fluxon energy, in comparison to the case $Q_i=0$, we see that the flux attachment mechanism predicted in the continuous

$U(1)$ Chern-Simons theory does not operate as long as the single fluxon spectrum remains gapped. In the notations of this section [cf. (IV) above], this implies that $\lambda_n^* = \lambda_n^c$.

VI. CONCLUSION

We presented (Sec. II) the general symmetry analysis of physical systems with protected degeneracies, i.e., with degeneracies that are exponentially weakly affected by local perturbations. We have shown that such protected degeneracies appear in a system described by a wide class of Hamiltonians that commute with two sets of integrals of motion, $\{P_i\}$ and $\{Q_j\}$ but which do not commute themselves. These sets of noncommuting operators should allow for a finite dimensional representation. In the simplest case, if $P_i P_j$ and $Q_k Q_\ell$ commute with all other operators the algebra of these operators allows two-dimensional representation and the states of the system are exactly doubly degenerate. For the effect of local perturbations to become really small in the thermodynamic limit one also needs that the gap to the low energy excitations remains finite.

We have explicitly constructed a two-dimensional lattice spin model with local interactions and which has these integrals of motion. In this model all states are exactly doubly degenerate. The behavior of this spin model is characterized by the dimensionless parameter, J_z/J_x which physically corresponds to the anisotropy of the couplings in different directions. We were able to treat it analytically in the regime of large (or small) values of this parameter. In this regime the spectrum of the system contains 2^L low energy modes where L is the linear size of the spin array. The gap between these modes and the ground state decreases exponentially with the system size. The number of these low energy modes is the same as would be the number of edge states but, unlike the latter, they are not sensitive to the boundary conditions. In order to check the validity of these conclusions for all values of J_z/J_x we have also performed the diagonalization of small arrays (up to 25 spins) and concluded that the gap to low energy states remains a decreasing function of the system size for all values but this decrease becomes very slow for $J_z/J_x \sim 1$. It remains unclear to us, however, whether the system exhibits a new phase at these values of the parameter or this apparently slower decrease of the gap is a consequence of a critical behavior.

We have suggested and studied (Sec. III) two designs of the Josephson junction arrays and showed that their effective low energy Hamiltonians satisfy the symmetry requirements described above and thus their states are doubly degenerate and protected from the external noise. The simplest of these arrays can be mapped onto a spin model with nonlocal interactions. The nonlocality of these interactions, however, is not important for the protection from the external noise. Further, in these systems one can completely eliminate the dangerous low energy modes by appropriate boundary conditions. The mapping of the Josephson junction array onto a spin system with symmetric Hamiltonian implies that the continuous superconducting phase can be integrated out. We have examined the conditions when this can be done and when low energy degrees of freedom corresponding to the continuous phase are irrelevant. Summarizing the requirements for a physical Josephson junctions we conclude that they are relatively easy to satisfy in medium sized arrays (up to 10×10 elements) which should be quite sufficient to get a noise suppression by ten orders of magnitude.

The spin models studied in Sec. II can also be mapped onto a discrete Chern-Simons theory on a lattice. In order to establish this mapping we have constructed (Sec. IV) a Hamiltonian framework for lattice Chern-Simons theories with Abelian groups. We argued that, in contrast to the continuous theories, such theories generally have low energy modes corresponding to the excitations with large momentum, comparable to the inverse lattice spacing. Further, we showed that in a theory with a compact group (in particular, in a theory with discrete group) the Chern-Simons coefficient, ν, is quantized similar to the quantization of the magnetic flux through the torus: $\nu = \pi/m$. In the gauge invariant space of magnetic fluxes the kinetic part of the Hamiltonian of these theories can be described as a flux dynamics. Due to the presence of Chern-Simons term the motion of fluxes in x and z directions does not commute. In the simplest case of the $\mathbb{Z}_2$, $m=2$ theory, the fluxes take only two values and their motions in x and z directions anticommute allowing us to map this theory onto the spin model studied in Sec. II. In a general case the Hamiltonian in flux representation becomes very nonlocal but still this representation is convenient in the limit of large magnetic energy when the fluxes are rare. Using this limit, we show that the flux attachment to the charge only occurs if a single fluxon is gapless.

The main theoretical issue raised by Sec. IV and V of the paper is the precise connection between lattice and continuous versions of Chern-Simons theories. For the continuous case, there is a sharply isolated ground state subspace, whose degeneracy directly reflects the topology of the two-dimensional space on which the model is defined. In a recent series of papers,[30,31] various descriptions of these models (in terms of wave functions defined on equivalence classes of loops) have been advocated to construct candidate lattice models which would exhibit a ground state sector equivalent to a pure (topological) Chern-Simons theory. The approach we have followed here starts from a direct quantization of a lattice Hamiltonian inspired from the continuous Chern-Simons theory with an additional kinetic term. We found that such construction typically leads to the degenerate modes attached to the Brillouin zone boundary; it remains to be investigated whether or not the presence of these modes spoils the expected properties of continuous models (such as statistical transmutation of external charges). It would be also interesting to see whether these degeneracies remain for other lattice structures. Finally, extending this construction to non-Abelian discrete groups is clearly desirable, from the perspective of enlarging the set of unitary operations generated by adiabatic exchanges between charge and/or vortex excitations.[8,32,33]

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APPENDIX A: EXACT SOLUTION OF TWO CHAIN PROBLEM

For the diagonalization of the Hamiltonian (3) it is convenient to rotate the Pauli matrices: $\tau_i^z \to \tilde{\sigma}_i^x$, $\tau_i^x \to \tilde{\sigma}_i^z$ and use the Jordan-Wigner fermionic representation (see Ref. 29)

$$\tilde{\sigma}_i^+ = a_i^+ \exp\left\{ i\pi \sum_{i'=1}^{i-1} a_{i'}^+ a_{i'} \right\},$$

$$\tilde{\sigma}_i^- = \exp\left\{ -i\pi \sum_{i'=1}^{i-1} a_{i'}^+ a_{i'} \right\} a_i,$$

$$\tilde{\sigma}_i^z = 2a_i^+ a_i - 1 \tag{A1}$$

so that Eq. (3) takes the form

$$H = 2J_z \left\{ \sum_{l=1}^{n-1} (a_l^+ - a_l)(a_{l+1} + a_{l+1}^+) + 2\Lambda \sum_{l=1}^{n} (a_l^+ a_l - 1/2) \right\}, \tag{A2}$$

where $\Lambda = J_x/2J_z$. By means of the Bogoliubov-like transformation the Hamiltonian (A2) can be diagonalized

$$H = \sum_k E_k (2b_k^+ b_k - 1),$$

where b_k^+, b_k are the fermionic operators of the eigenmodes with eigenenergies

$$E(k) = 2J_z \sqrt{\Lambda^2 + 1 + 2\Lambda \cos k}.$$

The quasicontinuous spectrum of this Hamiltonian can be found from the quantization rule for the quasimomentum k

$$k(n+1) - \arctan\left(\frac{\sin k}{\Lambda + \cos k} \right) = \pi m \tag{A3}$$

with integer m. For $\Lambda > 1$ this equation has exactly n distinct nontrivial solutions, and the set of the corresponding eigenfunctions is complete. It is not the case for $\Lambda < 1$, however. Here the number of the continuous spectrum eigenstates is only $n-1$, so that there should be one additional mode—the bound state. To find the latter one should look for complex solutions of the dispersion Eq. (A3). Substituting $k = \pi + i\gamma$, we arrive at

$$\gamma(n+1) = \frac{1}{2} \ln\left\{ \frac{\Lambda - e^{\gamma}}{\Lambda - e^{-\gamma}} \right\} \tag{A4}$$

and, introducing $e^{-\gamma} = \Lambda + x$ with small $x \ll 1$, we get $x \approx [(1/\Lambda) - \Lambda]\Lambda^{2(n+1)}$, so that and the bound state energy are

$$2\Delta = 2E(\pi + i\gamma) \approx 4J_z[1 - (J_x/2J_z)^2](J_x/2J_z)^n. \tag{A5}$$

The formula (A5) is valid for $(J_x/2J_z)^n \ll 1$. In the fermionic representation the bound-state eigenfunction is localized near the ends of the chain within the range $\xi \sim -\ln \Lambda$. The localized character of the mode responsible for the ground state doublet splitting is, however, an artifact of the nonlocal representation (A1) and, apparently, does not have much physical meaning.

A similar result for the splitting can also be obtained for an Ising chain with periodic boundary conditions. Here the chain does not have ends, and the dispersion equation has only solutions, corresponding to the continuous spectrum. The splitting arises from the following effect: the effective boundary conditions for the Jordan-Wigner fermions a are periodic or antiperiodic, depending on the parity I of their total number (the latter is a good quantum number)

$$kn = \begin{cases} 2\pi m, & \text{for } I = 1, \\ 2\pi(m + 1/2), & \text{for } I = -1 \end{cases} \tag{A6}$$

with $m = 0, 1, \ldots, L-1$.

The energy of the first excited state is just due to this effect; it comes not from any specific single-particle state, but from the entire Fermi sea of the filled energy levels, each of which is slightly shifted when the parity I is changing

$$2\Delta = \sum_{m=0}^{n-1} \left\{ E\left(k = \frac{2\pi m}{n} \right) - E\left(k = \frac{2\pi(m+1/2)}{n} \right) \right\}. \tag{A7}$$

Using the Poisson summation formula we arrive at the result

$$2\Delta \approx 4J_z \sqrt{\frac{1 - (J_x/2J_z)^2}{\pi n}} (J_x/2J_z)^n, \tag{A8}$$

which differs from (A5) only in the preexponential factor.

APPENDIX B: THE CRITICAL POINT OF THREE CHAIN PROBLEM

We consider a three-strings ladder with periodic boundary conditions along each rung. The corresponding Hamiltonian reads

$$H = -J_z \sum_{i=1}^{n-1} \sum_{j=1}^{3} \sigma_{ij}^z \sigma_{i+1j}^z - J_x \sum_{i=1}^{n} \sum_{j=1}^{3} \sigma_{ij}^x \sigma_{ij+1}^x, \tag{B1}$$

where $j = 4$ is identical to $j = 1$. We introduce a basis of four states ψ_m (with $m = 0$, 1, 2, 3) on a particular rung i, corresponding to the sector with all $P_i = 1$

$$\psi_0 = \begin{pmatrix} \uparrow \\ \uparrow \\ \uparrow \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} \uparrow \\ \downarrow \\ \downarrow \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} \downarrow \\ \uparrow \\ \downarrow \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} \downarrow \\ \downarrow \\ \uparrow \end{pmatrix}.$$

Then (up to an irrelevant additive constant) the Hamiltonian (B1) can be rewritten as

$$H = -4J_z \sum_{i=1}^{n-1} \delta_{m_i m_{i+1}} - J_x \sum_{i=1}^{n} \Gamma_i,$$

where the matrix Γ

$$\hat{\Gamma} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix},$$

plays a role, similar to that of the σ_x-operator for the two chain problem. This is the Hamiltonian of the one-dimensional $q=4$ Potts model in a "transverse field."

Consider now an asymmetric classic two-dimensional $q=4$ Potts model with the Hamiltonian

$$\beta H_{2d} = -K_z \sum_{ik} \delta_{m_{ik} m_{i+1k}} - K_x \sum_{ik} \delta_{m_{ik} m_{ik+1}}. \tag{B2}$$

The transfer matrix for this system (in the k direction) is

$$\hat{T} = \exp\left\{ K_z \sum_{i}^{n-1} \delta_{m_i m_{i+1}} \right\} \prod_{i=1}^{n} \left(\hat{1} e^{K_x} + \hat{\Gamma}_i \right), \tag{B3}$$

where $\hat{1}$ is 4×4 unity matrix. Using the identity

$$e^{h\hat{\Gamma}} = \frac{1}{4}\left\{ (e^{3h} + 3e^{-h}) + (e^{3h} - e^{-h})\hat{\Gamma} \right\}$$

for matrix Γ, we can rewrite (B3) in the form

$$\hat{T} = C \exp\left\{ K_z \sum_{i}^{n-1} \delta_{m_i m_{i+1}} + h \sum_{i}^{n} \Gamma_i \right\}, \tag{B4}$$

where C is an irrelevant constant and h is determined by $e^{K_x} = (e^{4h}+3)/(e^{4h}-1)$.

The line of critical points for the asymmetric two-dimensional $q=4$ Potts model is governed by the relation (see Ref. 14)

$$(e^{K_x} - 1)(e^{K_z} - 1) = 4;$$

in terms of K_z, h this relation takes the simple form $K_z = 4h$. On the other hand, the matrix (B4) describes the time evolution of the quantum system with the Hamiltonian (B1) and with $4J_z = K_z/\Delta t$, $J_x = h/\Delta t$, where $\Delta t \to 0$ is an infinitesimal time interval. Thus, we conclude, that the quantum phase transition in our initial three chain system takes place at the symmetric point $J_x = J_z$.

Unfortunately, the solution of the two-dimensional $q=4$ Potts model away from the critical line is not known and, in contrast to the exactly solvable two-chain model, we cannot find the dependence of the gap Δ on the parameter J_x/J_z in the full range of this parameter.

*Also at: Landau Institute for Theoretical Physics, Kosygina 2, Moscow, 117940 Russia.

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