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Phases of the generalized two-leg spin ladder: A view from the SU(4) symmetry

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The zero-temperature phases of a generalized two-leg spin ladder with four-spin exchanges are discussed by means of a low-energy field theory approach starting from an SU(4) quantum critical point. The latter fixed point is shown to be a rich multicritical point that unifies different competing dimerized orders and a scalar chirality phase that breaks spontaneously the time-reversal symmetry. The quantum phase transition between these phases is dominated by spin-singlet fluctuations and belongs to the Luttinger universality class due to the existence of a U(1) self-duality symmetry.

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Multispin exchange interactions have attracted much interest over the years, both theoretical and experimental.[1] Recently, ring exchange interactions have been invoked for the description of magnetic properties of the spin ladder compound $La_xCa_{14-x}Cu_{24}O_{41}$ (Ref. 2) and for their ability to induce exotic phases in quantum magnetism.[3] In this respect, a scalar chirality phase,[4] which breaks spontaneously the time-reversal symmetry, has been found in the two-leg spin-1/2 ladder for a sufficiently strong four-spin cyclic exchange.[5] Such an exotic ground state is, in fact, not specific to this spin ladder, and exact ground states with scalar chirality long-range order have been obtained for a wider class of two-leg spin ladders with four-spin interactions.[6] A central question is the determination of all possible ground states stabilized by four-spin exchanges and the elucidation of the nature of the quantum phase transition between these phases. In this Communication, we will study a general two-leg spin-1/2 ladder with four-spin exchanges defined by[6]

$$\begin{aligned}\mathcal{H}_{\rm gen} = {} & J_l \sum_n \sum_{p=1}^{2} \mathbf{s}_{p,n}\cdot\mathbf{s}_{p,n+1} + J_\perp \sum_n \mathbf{s}_{1,n}\cdot\mathbf{s}_{2,n} \\ & + J_d \sum_n (\mathbf{s}_{1,n}\cdot\mathbf{s}_{2,n+1} + \mathbf{s}_{1,n+1}\cdot\mathbf{s}_{2,n}) + J_{rr}\sum_n (\mathbf{s}_{1,n}\cdot\mathbf{s}_{2,n}) \\ & \times(\mathbf{s}_{1,n+1}\cdot\mathbf{s}_{2,n+1}) + J_{ll}\sum_n (\mathbf{s}_{1,n}\cdot\mathbf{s}_{1,n+1})(\mathbf{s}_{2,n}\cdot\mathbf{s}_{2,n+1}) \\ & + J_{dd}\sum_n (\mathbf{s}_{1,n}\cdot\mathbf{s}_{2,n+1})(\mathbf{s}_{1,n+1}\cdot\mathbf{s}_{2,n}), \end{aligned} \qquad (1)$$

where $\mathbf{s}_{p,n}$ are spin-1/2 operators at the site n on the pth chain ($p=1,2$). The Hamiltonian (1) is the most general translation-invariant two-leg spin ladder, which (i) consists of SU(2)-symmetric interactions involving two neighboring rungs and (ii) has $\mathbb{Z}_2$ invariance under the permutation of the two chains $\mathcal{P}_{12}$. In the following, we study the phase structure of (1) around a point with the maximal symmetry in a problem consisting of two spin-1/2 operators, i.e., an SU(4) symmetry. The resulting SU(4) model displays a quantum critical behavior which enables us to develop a low-energy approach to investigate the different $T=0$ gapped phases induced by the $SU(2)\times\mathbb{Z}_2$-invariant symmetry breaking terms of Eq. (1). As will be seen, the SU(4) symmetric point is a rich multicritical point that unifies several emerging quantum orders. The nature of the quantum phase transitions between these ordered phases can then be determined within our approach and are shown to belong to the Tomonaga-Luttinger (TL) universality class[7] as a result of a nontrivial U(1) symmetry at the transition.[6]

The SU(4) quantum critical point. The starting point of our approach is the existence of an SU(4) symmetric point in Eq. (1), which is obtained for $J_{ll}=4J_l$ and $J_\perp=J_d=J_{rr}=J_{dd}=0$.[8] The resulting model can be regarded as the SU(4) Heisenberg spin chain when the four states on a rung are identified with those of the fundamental representation **4** of SU(4). The latter model is Bethe-ansatz solvable[9] and is known to possess an extended quantum criticality, characterized by an $SU(4)_1$ conformal field theory (CFT) with central charge $c=3$.[10] A simple description of this fixed point is provided by the conformal embedding $SU(4)_1 \sim SU(2)_2 \times SU(2)_2$ with two SU(2)s corresponding to independent rotations for the two chains. Since a single $SU(2)_2$ CFT is described by a triplet of real (Majorana) fermions, we may describe the critical properties of $SU(4)_1$ fixed point by two triplets of right- and left-moving Majorana fermions $\xi^a_{R,L}$ and $\chi^a_{R,L}$ ($a=1,2,3$). This Majorana fermion description is extremely useful to understand the symmetry properties of model (1) in the close vicinity of the SU(4) point as it has been exploited for the $SU(2)\times SU(2)$ spin-orbital chain.[11,12] Moreover, the lattice discrete symmetries of model (1), i.e., one-step translation symmetry ($\mathcal{T}_{a_0}$), time-reversal symmetry ($\mathcal{T}$), site-parity ($\mathcal{P}_S$), and the permutation $\mathcal{P}_{12}$ of the two chains, are linearly represented in terms of the Majorana fermions. For instance, the translation symmetry is described by $\xi^a_R \to -\xi^a_R$, $\chi^a_R \to -\chi^a_R$, whereas ξ^a_L and χ^a_L are left unchanged under $\mathcal{T}_{a_0}$. These results lead us to write the most general low-energy effective-field theory for the generalized two-leg spin ladder (1), which is invariant under the SU(2) spin rotational symmetry, and the discrete symmetries $\mathcal{T}_{a_0}\times\mathcal{T}\times\mathcal{P}_S\times\mathcal{P}_{12}$,

$$\mathcal{H} = \mathcal{H}_0 + (g_1 + g_2)[(\vec{\xi}_R \cdot \vec{\xi}_L)^2 + (\vec{\chi}_R \cdot \vec{\chi}_L)^2] + (g_1 - g_2) \times[(\vec{\xi}_R \cdot \vec{\chi}_L)^2 + (\vec{\chi}_R \cdot \vec{\xi}_L)^2] + 2(g_3 + g_4)(\vec{\xi}_R \cdot \vec{\xi}_L)(\vec{\chi}_R \cdot \vec{\chi}_L) + 2(-g_3 + g_4)(\vec{\xi}_R \cdot \vec{\chi}_L)(\vec{\chi}_R \cdot \vec{\xi}_L) - \frac{g_5}{2}(\vec{\xi}_R \cdot \vec{\chi}_R)(\vec{\xi}_L \cdot \vec{\chi}_L) - i\frac{g_6}{2}(\vec{\xi}_R \cdot \vec{\chi}_R + \vec{\xi}_L \cdot \vec{\chi}_L), \quad (2)$$

where $\mathcal{H}_0$ is the free Hamiltonian for the Majorana fermions: $\mathcal{H}_0 = -iv(\xi_R^a \cdot \partial_x \xi_R^a - \xi_L^a \cdot \partial_x \xi_L^a + \xi_{R,L}^a \to \chi_{R,L}^a)/2$. Strongly relevant mass terms like $\vec{\xi}_R \cdot \vec{\xi}_L$ and $\vec{\chi}_R \cdot \vec{\chi}_L$ are prohibited by the translational symmetry $\mathcal{T}_{a_0}$. The effective Hamiltonian (2) describes the low-energy properties of model (1) in the vicinity of the SU(4) point mentioned above. In particular, using the continuum expressions of the spin operators $\mathbf{s}_{1,2,n}$ at the $SU(4)_1$ fixed point found in Ref. 12, we have obtained the following identifications: $g_{1,2} \simeq (J_l \pm J_d)/2$, $g_{3,4} \simeq (J_{ll} \pm J_{dd})/8$, $g_5 \simeq J_{rr}$, and $g_6 \simeq J_\perp$.

Order parameters and duality symmetries. Before investigating the infrared (IR) phases of the low-energy effective-field theory (2), let us first discuss its symmetries and possible orders. The $SU(4)_1$ fixed point Hamiltonian, i.e., $\mathcal{H}_0$, is invariant under chiral SO(2) rotations $\mathcal{R}_r(\theta)$, $r=R,L$ on the Majorana fermions

$$\xi_r^a \to \xi_r^a \cos \theta/2 - \chi_r^a \sin \theta/2, \qquad \chi_r^a \to \xi_r^a \sin \theta/2 + \chi_r^a \cos \theta/2. \quad (3)$$

This rotation defines a first U(1) symmetry $\mathcal{U}_\theta = \mathcal{R}_L(\theta) \times \mathcal{R}_R(\theta)$ which acts on the fields of the $SU(4)_1$ CFT. Now we introduce a first set of order parameters—the staggered dimerization operator $\mathcal{O}_{\rm SD} = (-1)^n(\mathbf{s}_{1,n} \cdot \mathbf{s}_{1,n+1} - \mathbf{s}_{2,n} \cdot \mathbf{s}_{2,n+1})$ and the scalar chirality order parameter.[5,6] $\mathcal{O}_{\rm SC} = (-1)^n[(\mathbf{s}_{1,n} + \mathbf{s}_{2,n})(\mathbf{s}_{1,n+1} \wedge \mathbf{s}_{2,n+1}) + (n \leftrightarrow n+1)]$. They have a simple continuum description in terms of the Majorana fermions: $\mathcal{O}_{\rm SD} \sim i(\vec{\xi}_R \cdot \vec{\xi}_L - \vec{\chi}_R \cdot \vec{\chi}_L)$ and $\mathcal{O}_{\rm SC} \sim i(\vec{\xi}_R \cdot \vec{\chi}_L + \vec{\chi}_R \cdot \vec{\xi}_L)$. From Eq. (3) we deduce that these order parameters transform as a doublet under $\mathcal{U}_\theta$,

$$\begin{pmatrix} \mathcal{O}_{\rm SD} \\ \mathcal{O}_{\rm SC} \end{pmatrix} \to \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \mathcal{O}_{\rm SD} \\ \mathcal{O}_{\rm SC} \end{pmatrix}. \quad (4)$$

In particular, for $\theta = \pi/2$, the two phases characterized by $\mathcal{O}_{\rm SD}$ and $\mathcal{O}_{\rm SC}$ are interchanged under $\mathcal{D} = \mathcal{U}_{\pi/2}$, which can thus be viewed as a $\mathbb{Z}_2$ duality for a pair of order parameters. Remarkably, the duality symmetry $\mathcal{D}$ and the U(1) transformation $\mathcal{U}_\theta$ have lattice expressions $\mathcal{U}_\theta^{\rm lat} = \exp[-i\theta \Sigma_n(\mathbf{s}_{1,n} \cdot \mathbf{s}_{2,n} - 1/4)]$ and $\mathcal{D} = \mathcal{U}_{\theta=\pi/2}^{\rm lat}$, respectively, which have been discovered previously[6,13] and called the spin-chirality duality. On top of them, we introduce two additional order parameters, expressed again as bilinears of Majorana fermions: $\mathcal{O}_D \sim i(\vec{\xi}_R \cdot \vec{\xi}_L + \vec{\chi}_R \cdot \vec{\chi}_L)$ and $\mathcal{O}_{\rm RD} \sim i(\vec{\xi}_R \cdot \vec{\chi}_L - \vec{\chi}_R \cdot \vec{\xi}_L)$, which are left invariant under the spin-chirality rotation $\mathcal{U}_\theta$ and are thus self-dual with respect to $\mathcal{D}$. In fact, these order parameters are the continuum representation of the columnar dimerization operator $\mathcal{O}_D = (-1)^n(\mathbf{s}_{1,n} \cdot \mathbf{s}_{1,n+1} + \mathbf{s}_{2,n} \cdot \mathbf{s}_{2,n+1})$ and the rung dimerization operator $\mathcal{O}_{\rm RD} = (-1)^n \mathbf{s}_{1,n} \cdot \mathbf{s}_{2,n}$. The latter order parameter characterizes a phase with alternation of rung singlets and rung triplets, which has been found in some integrable two-leg spin ladder.[14] The second set of order parameters is closely related, in the continuum limit, to the existence of a second U(1) symmetry: $\widetilde{U}_\theta = \mathcal{R}_L(\theta) \times \mathcal{R}_R(-\theta)$. It leaves invariant the $\mathcal{O}_{\rm SD}$ and $\mathcal{O}_{\rm SC}$ order parameters, whereas $\mathcal{O}_D$ and $\mathcal{O}_{\rm RD}$ transform now as a doublet under $\widetilde{\mathcal{U}}_\theta$ as in Eq. (4). A second $\mathbb{Z}_2$ duality, $\widetilde{\mathcal{D}} = \widetilde{\mathcal{U}}_{\pi/2}$, can thus be considered as a transformation which interchanges $\mathcal{O}_D$ and $\mathcal{O}_{\rm RD}$ while keeping $\mathcal{O}_{\rm SD}$ and $\mathcal{O}_{\rm SC}$ intact. Finally, the Kramers-Wannier duality ($\vec{\chi}_R \to -\vec{\chi}_R$) on the underlying Ising models, corresponding to the Majorana fermions $\chi_{R,L}^a$, interchanges $\mathcal{O}_D$ (respectively, $\mathcal{O}_{\rm RD}$) with $\mathcal{O}_{\rm SD}$ (respectively, $\mathcal{O}_{\rm SC}$). The $SU(4)_1$ fixed point is therefore a rich multicritical point that unifies four different competing orders (dimerized $\mathcal{O}_D, \mathcal{O}_{\rm SD}, \mathcal{O}_{\rm RD}$, and $\mathcal{T}$-breaking $\mathcal{O}_{\rm SC}$; see Fig. 1). The SU(4) $\to SU(2) \times \mathbb{Z}_2$ symmetry-breaking perturbations will select one of these quantum orders as we are going to see now.

Renormalization group (RG) analysis. The next step of the approach is to perform a one-loop RG calculation to determine the nature of the IR phases of the low-energy effective Hamiltonian (2). First of all, the SU(4) model in Eq. (1), perturbed by a standard rung interaction $J_\perp \neq 0$, is Bethe-ansatz integrable;[15] for a small value of $J_\perp$, the gapless behavior of the SU(4) model extends up to a critical point $J_{\perp c} = 4J_l$ above which the standard gapped rung-singlet phase of the two-leg spin ladder appears. In the close vicinity of the $SU(4)_1$ quantum critical point, when $|g_i| \ll 1$, we can thus forget the perturbation with coupling constant g_6 in Eq. (2) and the resulting one-loop RG equations read as follows:

$$\begin{aligned} \dot{g}_1 &= g_1^2 + g_2^2 + 5g_3^2 + g_4^2, \\ \dot{g}_2 &= 2g_1g_2 + 6g_3g_4 + g_4g_5, \\ \dot{g}_3 &= 6g_1g_3 + 2g_2g_4, \\ \dot{g}_4 &= 2g_1g_4 + 6g_2g_3 + g_2g_5, \\ \dot{g}_5 &= -16(g_1g_3 - g_2g_4). \end{aligned} \quad (5)$$

We are now going to investigate the different gapped phases that emerge in the IR limit of the RG Eqs. (5). First of all, it is important to note that the interaction part of the Hamiltonian (2) with $g_{5,6} = 0$ can be recast as

$$\mathcal{H}_{\rm int} = -\lambda_{\rm SD}\mathcal{O}_{\rm SD}^2 - \lambda_{\rm SC}\mathcal{O}_{\rm SC}^2 - \lambda_{\rm D}\mathcal{O}_{\rm D}^2 - \lambda_{\rm RD}\mathcal{O}_{\rm RD}^2, \quad (6)$$

where the couplings $\lambda_{\rm SD}$, etc., are functions of $g_1, \ldots, g_4$ and Eq. (6) describes the competition between the four quantum orders introduced previously. Then we apply an ansatz, proposed by Lin *et al.*[16] in the context of the half-filled two-leg Hubbard ladder, that the IR asymptotics of Eq. (5) is described by $g_i(t) = r_i/(t_0 - t)$, where t is the RG time and t_0 marks the crossover point where the weak-coupling perturbation breaks down. The sets of coefficients r_i, which are obtained as solutions of nonlinear equations, indicate the symmetric rays that attract the RG flow in the IR limit and

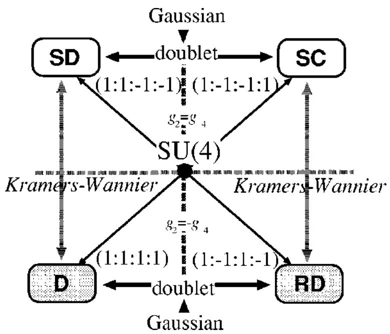

FIG. 1. Relationship between four orders. On manifolds $g_2=g_4$ ($g_2=-g_4$) the system acquires a U(1) symmetry under $\mathcal{U}_\theta$ ($\tilde{\mathcal{U}}_\theta$) and order parameters form a doublet. The value $r_1:r_2:r_3:r_4$ along each ray is also shown.

define the different strong-coupling phases of the problem. Along the symmetric ray $(r_1,r_2,r_3,r_4,r_5)=(1/8,-1/8,-1/8,1/8,0)$, the low-energy physics is described by the effective-field theory

$$\mathcal{H}_{\text{eff}}^{(\text{SC})} = \mathcal{H}_0 + \lambda_{\text{SC}}^{*}(\vec{\xi}_R \cdot \vec{\chi}_L + \vec{\chi}_R \cdot \vec{\xi}_L)^2,$$

$$=\mathcal{H}_0 - \lambda_{\text{SC}}^{*}\mathcal{O}_{\text{SC}}^2, \tag{7}$$

with $\lambda_{\text{SC}}^{*}>0$. After a transformation $\vec{\chi}_R \leftrightarrow \vec{\xi}_R$, the Hamiltonian (7) takes the form of the SO(6) Gross-Neveu (GN) model, which is an integrable field theory with a spectral gap.[17] As is suggested from the form of Eq. (7), the strong-coupling phase is characterized by the nonvanishing scalar chirality $\langle\mathcal{O}_{\text{SC}}\rangle=\pm\Delta_0\neq 0$. A staggered scalar chirality phase is thus stabilized and breaks both the time-reversal- and the translation symmetry spontaneously. Applying the duality $\mathcal{D}$ $(g_2\leftrightarrow g_4)$, we move to the second ray $(r_1,r_2,r_3,r_4,r_5)=(1/8,1/8,-1/8,-1/8,0)$ and the effective action then reads

$$\mathcal{H}_{\text{eff}}^{(\text{SD})} = H_0 - \lambda_{\text{SD}}^{*}\mathcal{O}_{\text{SD}}^2. \tag{8}$$

A similar analysis concludes $\langle\mathcal{O}_{\text{SD}}\rangle=\pm\Delta_0\neq 0$, i.e., the system is in the staggered dimer phase as in the SU(2) $\times$ SU(2) spin-orbital chain.[11,12] Note that the above results are directly obtained by applying the transformation $\mathcal{D}=\mathcal{U}_{\theta=\pi/2}$ onto the Hamiltonian (7). All these are consistent with the observations in Ref. 6. Actually, our approach predicts that there are two more SO(6)-symmetric rays $(r_1,r_2,r_3,r_4,r_5)=(1/8,\pm 1/8,1/8,\pm 1/8,0)$. Similar arguments tell us that these rays correspond, respectively, to the dimerization ($\mathcal{O}_{\text{D}}$) and the rung dimerization ($\mathcal{O}_{\text{RD}}$) which are interchanged now under the second duality $\tilde{\mathcal{D}}=\tilde{\mathcal{U}}_{\theta=\pi/2}$. In summary, the four phases ($\mathcal{O}_{\text{SD}}$, $\mathcal{O}_{\text{SC}}$, $\mathcal{O}_{\text{D}}$, and $\mathcal{O}_{\text{RD}}$), related two by two through the duality symmetries $\mathcal{D}$, $\tilde{\mathcal{D}}$, and the Kramers-Wannier transformation (see Fig. 1), are the four different gapped phases which are characterized by the physics of the SO(6) GN model in the IR limit.

Quantum phase transition. In addition to these SO(6)-symmetric rays, there are special manifolds where the RG Eqs. (5) also display a larger symmetric behavior. On the two manifolds $g_2=\pm g_4$, the SU(2) $\times\mathbb{Z}_2$ symmetric model (2) acquires a larger continuous symmetry SU(2) $\times$ U(1), being invariant under arbitrary rotations $\mathcal{U}_\theta$ and $\tilde{\mathcal{U}}_\theta$, respectively (see Fig. 1). On these self-dual manifolds, the RG flow is attracted in the IR limit toward two new asymptotes: $(r_1,r_2,r_3,r_4,r_5)=(1/6,0,\mp 1/6,0,\pm 4/9)$. Along the first line ($r_3=-r_1$), the interacting part of the low-energy self-dual theory takes the form

$$\mathcal{H}_{\text{eff}}^{\text{int}} = -\lambda^{*}(\mathcal{O}_{\text{SD}}^2 + \mathcal{O}_{\text{SC}}^2) - \frac{4\lambda^{*}}{3}(\vec{\xi}_R \cdot \vec{\chi}_R)(\vec{\xi}_L \cdot \vec{\chi}_L), \tag{9}$$

which describes the competition between the staggered dimerization and scalar chirality orders, i.e., it governs the nature of the quantum phase transition between these two phases. Similarly, the second asymptote accounts for the competition between the columnar dimerization and the rung dimerization. The emerging effective-field theory (9) displays a larger symmetry than the U(1) $\times$ SU(2) symmetry of the initial manifold $g_2=g_4$. First, model (9) turns out to be invariant not only under $\mathcal{U}_\theta$ but under a larger $\mathcal{U}_\theta\times\tilde{\mathcal{U}}_\theta$ symmetry (note that at the lattice level, $\tilde{\mathcal{U}}_\theta$ may be broken by umklapp interactions). Second, it is also invariant under a hidden SU(3) symmetry. A way to reveal the second symmetry is to combine the six Majorana fermions into three Dirac fermions: $\Psi_{aR,L}=(\xi_{R,L}^a+i\chi_{R,L}^a)/\sqrt{2}$. Then the two rotations, $\mathcal{U}_\theta$ and $\tilde{\mathcal{U}}_\theta$, are translated, respectively, to the chiral U(1) (charge) symmetries of the Dirac fermions: $\Psi_{aR,L}\rightarrow e^{i\theta/2}\Psi_{aR,L}$ and $\Psi_{aR,L}\rightarrow e^{\pm i\theta/2}\Psi_{aR,L}$. In addition, we can see that the order parameters $\mathcal{O}_{\text{SD}}$ and $\mathcal{O}_{\text{SC}}$ also have a simple interpretation as Cooper pairs by taking a combination $\mathcal{O}_{\text{SD}}+i\mathcal{O}_{\text{SC}}\sim\vec{\Psi}_R\cdot\vec{\Psi}_L$. The role of these pseudocharge degrees of freedom, introduced by the spin-chirality U(1) symmetry, becomes manifest with the help of a bosonization of the Dirac fermions. To this end, we define three right-left moving bosonic fields $\varphi_{aR,L}$ such as $\Psi_{aR,L}\sim\exp[\pm i\sqrt{4\pi}\varphi_{aR,L}]$, and switch to a basis where the pseudocharge degrees of freedom may be singled out as follows: $\varphi_{cR,L}=(\varphi_{1R,L}+\varphi_{2R,L}+\varphi_{3R,L})/\sqrt{3}$. The low-energy Hamiltonian (9) then exhibits a "spin"[SU(3)]-"charge"[U(1)] separation $\mathcal{H}_{\text{eff}}=\mathcal{H}_c+\mathcal{H}_s$, with $[\mathcal{H}_c,\mathcal{H}_s]=0$. The charge degrees of freedom are described by the TL Hamiltonian

$$\mathcal{H}_c = \frac{v}{2}[(\partial_x\varphi_c)^2 + (\partial_x\vartheta_c)^2], \tag{10}$$

where $\varphi_c=\varphi_{cR}+\varphi_{cL}$ is the total charge bosonic field and $\vartheta_c=\varphi_{cL}-\varphi_{cR}$ is its dual field. The Hamiltonian $\mathcal{H}_s$ for the remaining degrees of freedom can be recast in a fully SU(3) symmetric form in terms of chiral SU(3)$_1$ currents $\mathcal{J}_{R,L}^A$,

$$\mathcal{H}_s = \frac{\pi v}{2}\sum_{A=1}^{8}(\mathcal{J}_R^A\mathcal{J}_R^A + \mathcal{J}_L^A\mathcal{J}_L^A) + 8\lambda^{*}\sum_{A=1}^{8}\mathcal{J}_R^A\mathcal{J}_L^A. \tag{11}$$

The latter model is the SU(3) GN model, which is a massive $(\lambda^{*}>0)$ integrable field theory.[18] A spectral gap is thus formed by the interactions in the "spin" sector and the low-lying excitations are known, from the exact solution,[18] to be gapped SU(3) spinons and antispinons. The low-energy physics of (9) is dominated by the gapless *spin-singlet* fluctuations (10) of the "charge" degrees of freedom, which stem from the remarkable U(1) symmetry of Eq. (9). Therefore, we may conclude that the quantum phase transition between the staggered dimerized- and scalar chirality phases belongs to the $c=1$ TL universality class. The physical properties at the transition can also be determined within our approach. At the transition, all order parameters have zero expectation values. The first doublet $(\mathcal{O}_{\mathrm{SD}},\mathcal{O}_{\mathrm{SC}})$ has a fixed modulus and correlation functions decaying as $x^{-2/3}$, i.e., has long-range coherence, whereas the second one ($\mathcal{O}_{\mathrm{D}}$ and $\mathcal{O}_{\mathrm{RD}}$) is exponentially decaying due to strong quantum fluctuations. Now it is straightforward to discuss the effect of a small deviation from the self-dual manifold (9) by switching on the perturbation $\mathcal{V}=\varepsilon(\mathcal{O}_{\mathrm{SD}}^2-\mathcal{O}_{\mathrm{SC}}^2)$, $|\varepsilon|\ll 1$ which breaks in particular the $\mathcal{U}_\theta$ symmetry of model (9). This small symmetry-breaking perturbation does not close the spin gap but introduces to the charge Hamiltonian (10) a "pinning" term $\mathcal{V}_c \simeq -\varepsilon\cos(\sqrt{16\pi/3}\varphi_c)$ that locks the direction of the doublet ($\mathcal{O}_{\mathrm{SD}}$, $\mathcal{O}_{\mathrm{SC}}$). The interaction has scaling dimension $\Delta=4/3<2$ so that the perturbation opens a charge gap; for $\varepsilon<0$ (respectively, $\varepsilon>0$), the staggered dimerization (respectively, scalar chirality) order is stabilized by the small symmetry-breaking term. The same argument applies to the second ray ($r_1=r_3$) as well after the replacement $\varphi_c\leftrightarrow\vartheta_c$, and describes now the competition between $\mathcal{O}_{\mathrm{D}}$ and $\mathcal{O}_{\mathrm{RD}}$. A simple understanding of the quantum phase transition can be obtained by noting that $\mathcal{S}_i^z=\mathbf{s}_{1,i}\cdot\mathbf{s}_{2,i}+1/4$ plays the role of S_i^z in the spin-1/2 *XYZ* Heisenberg model. The spin-chirality rotation $\mathcal{U}_\theta$ is then the analog of the U(1) rotation generated by $\Sigma_i S_i^z$. The *XY*-anisotropy term stabilizes two Néel $\mathbb{Z}_2$ gapped phases ($\langle S_i^x\rangle\neq 0$ and $\langle S_i^y\rangle\neq 0$, respectively) which are related through a duality transformation, i.e., a $\pi/2$ rotation $(\mathcal{D}=\mathcal{U}_{\theta=\pi/2})$ along the *z*-spin axis. The U(1) quantum phase transition between the staggered and scalar chirality phases is thus similar to the Gaussian criticality that occurs in the spin-1/2 *XYZ* Heisenberg chain.

In summary, we have shown, in the continuum approach, that four different gapped phases around the SU(4) multicritical point are unified by the hidden $\mathbb{Z}_2$ symmetries $\mathcal{D}$ and $\tilde{\mathcal{D}}$. The spin-chirality U(1) symmetry $\mathcal{U}_\theta$ plays an essential role in the self-dual manifold, and as a consequence, a second-order phase transition that separates the staggered dimerized- and scalar chirality phases is characterized by the $c=1$ TL universality class. On the basis of this fact, we explained how an exotic phase with a broken time-reversal symmetry is stabilized. Finally, we have revealed another *hidden* relationship between a dimerized phase and a rung-dimer phase, together with a corresponding U(1) symmetry $\tilde{\mathcal{U}}_\theta$. The resulting emerging U(1) quantum critical behavior of the transition separating these two orders may be viewed as a one-dimensional analog of the so-called deconfined quantum criticality introduced by Senthil *et al.*[19]

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