

Mesoscopic phase separation in anisotropic superconductorsV. I. Yukalov^{1,2} and E. P. Yukalova^{2,3}¹*Bogolubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia*²*Freie Universität Berlin, Fachbereich Physik, Institut für Theoretische Physik, WE 2, Arnimallee 14, D-14195 Berlin, Germany*³*Department of Computational Physics, Laboratory of Information Technologies, Joint Institute for Nuclear Research, Dubna 141980, Russia*

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The general properties of anisotropic superconductors with mesoscopic phase separation are analyzed. The main conclusions are as follows: Mesoscopic phase separation can be thermodynamically stable only in the presence of repulsive Coulomb interactions. Phase separation enables the appearance of superconductivity in a heterophase sample even if it were impossible in pure-phase matter. Phase separation is crucial for the occurrence of superconductivity in bad conductors. The critical temperature for a mixture of pairing symmetries is higher than the critical temperature related to any pure gap-wave symmetry of this mixture. In bad conductors, the critical temperature as a function of the superconductivity fraction has a bell shape. Phase separation makes the single-particle energy dispersion softer. For planar structures phase separation suppresses *d*-wave superconductivity and enhances *s*-wave superconductivity. These features are in agreement with experiments for cuprates.

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I. INTRODUCTION

It is generally accepted that the majority of high-temperature superconductors, such as cuprates, possess two principal properties distinguishing them from the conventional low-temperature superconductors. These properties are mesoscopic phase separation and anisotropic pairing symmetry.

Phase separation in superconductors implies that not the whole volume of a sample is actually superconducting but is separated into regions of superconducting and normal phases. The latter may even be insulating. There exist numerous experiments confirming the occurrence of the phase separation in high-temperature superconductors, as is summarized in Refs. 1–4.

The phase separation is termed mesoscopic since the regions of coexisting phases form a kind of fog of clusters or droplets, whose typical sizes, corresponding to the coherence length l_{coh} , are in between the mean interparticle distance a and the length of the sample L , so that

$$a \ll l_{coh} \ll L.$$

These regions are intermixed, being randomly distributed in space. In general, the phase droplets are not static but can be dynamic, randomly fluctuating in time. In any case, whether they are static or not, their main features are the mesoscopic size and chaotic space location. Because of the random spatial distribution of the mesoscopic phase nuclei, they can be called heterophase fluctuations.⁵

The mesoscopic phase separation is, actually, a very general phenomenon inherent to condensed matter.^{4–7} This phenomenon happens in many systems, being responsible for a variety of unusual effects. For instance, it plays the key role in colossal magnetoresistant materials^{8–10} and relaxor ferroelectrics.^{11,12} Heterophase fluctuations, spatial or spatiotemporal, can exist in physical systems, without any no-

ticeable external influence, thus being *self-organized*.^{5,13} The action of external forces can, of course, provoke the appearance of such mesoscopic fluctuations,^{5,14} making them more intensive. However, in general, the noticeable external perturbations are not compulsory, and heterophase fluctuations can really arise in a self-organized way. In some cases, these fluctuations can be triggered by infinitesimally small stochastic noise that always exists in all realistic systems, which are never completely isolated from their surrounding but are not more than quasi-isolated.^{5,15}

The possibility of mesoscopic phase separation in superconductors was advanced^{16,17} yet before high-temperature superconductors were discovered.¹⁸ Theoretical models confirm that this phenomenon can be thermodynamically profitable, rendering the heterophase material more stable.^{17,19–23}

Another specific feature of high-temperature cuprate superconductors is the anisotropy of the gap. A number of experiments point at the predominantly *d*-wave symmetry of the superconducting order parameter^{24,25}, though in some cases one claims that the isotropic *s*-wave symmetry can be dominant.^{26–29} The majority of experiments evidence the existence of the mixed *s+d* pairing in cuprates.^{25,30–38} Several theoretical models, blending *s*-wave and *d*-wave features, describe the *s+d* superconducting gap state and provide a reasonable explanation for various experiments.^{39–52} Thus, the occurrence of anisotropy in the gap of high-temperature cuprate superconductors seems to be well established.

In the present paper, we suggest a model of superconductors, which combines two main features: *mesoscopic phase separation* and *anisotropic pairing symmetry*. We study interplay between these characteristics. We do not narrow down the consideration by fitting parameters to a particular material, but we rather concentrate on the general properties of the model. The basic goal of the paper is to formulate the principal qualitative conclusions characterizing such a superconductor with both phase separation and an anisotropic gap.

The treatment of the suggested model of superconductor is based on the theory of heterophase materials⁵ possessing the properties typical of the matter with mesoscopic phase separation. There are some important points that are worth emphasizing in order to better understand the following consideration.

First of all, one should keep in mind that the basic spatial structure of matter is defined by ions forming a crystalline lattice. Charge carriers, such as electrons and holes, exist inside the given crystalline structure of a particular solid. Therefore the properties of all spatial characteristics—e.g. interaction potentials—are prescribed by a concrete crystalline structure of ions forming the lattice.

Superconductivity or normal conductivity are the features of the charge carriers, reflecting the level of correlations between the latter. A solid with the same crystalline structure, as is well known, can be superconducting or not depending on the values of the thermodynamic parameters. The occurrence of superconductivity of carriers does not substantially change the crystalline structure of ions. Thus, superconducting and normal phases of carriers may coexist inside the same crystalline lattice.

The coexistence of different phases, typical of mesoscopic phase separation, means that the spatial regions of the sample are occupied by different phases; these regions, generally, have diverse shapes and random spatial locations. Then, in order to describe the properties of the sample as a whole, one has to average over phase configurations. The procedure of such a heterophase averaging is rather non-trivial, being analogous to the renormalization-group technique, when one averages out one type of fluctuations with temporal or spatial scales that are distinct from another type of fluctuations. In our case, the heterophase fluctuations are mesoscopic, which distinguishes them from microscopic quantum fluctuations. All mathematical details of the heterophase averaging over configurations have been thoroughly expounded in the review in Ref. 5. In order that the reader could catch the main points of this procedure, these are sketched in the Appendix.

After averaging over phase configurations, one obtains a renormalized Hamiltonian representing the phase replicas that would occupy the whole sample with a certain probability. In this way one comes to the picture where all characteristics do not involve anymore random spatial distributions but correspond to the averaged quantities resulting from their averaging over these random phase configurations. In the following sections, we deal with such averaged characteristics appearing *after* the heterophase averaging, whose essence is surveyed in the Appendix.

II. HETEROPHASE SUPERCONDUCTOR

A superconductor with phase separation is a sample consisting of intermixed regions of different thermodynamic phases. Assume there are two phases, superconducting and normal, enumerated by the index $\nu=1,2$. Let $\nu=1$ correspond to the superconducting phase, while $\nu=2$ to the normal phase. Quantum states of the phases pertain to the related spaces \mathcal{H}_ν , which are the weighted Hilbert spaces.⁵ The

phases can be distinguished by their *order parameters*, such as the gaps $\Delta_\nu(\mathbf{k})$ in momentum space, so that

$$\Delta_1(\mathbf{k}) \neq 0, \quad \Delta_2(\mathbf{k}) \equiv 0. \quad (1)$$

Another way of distinguishing phases is by the associated order indices, which are defined for reduced density matrices⁵³ and have been generalized for arbitrary operators.⁵⁴ The *order index* for a bounded operator \hat{A} is

$$\omega(\hat{A}) \equiv \frac{\ln\|\hat{A}\|}{\ln|\text{Tr}\hat{A}|}.$$

Considering, in the place of \hat{A} , p -particle density matrices $\hat{\rho}_{p\nu}$ of the phases $\nu=1,2$, we have the following.^{53,54} For the superconducting phase, the order indices of odd-density matrices are

$$\omega(\hat{\rho}_{p1}) = \frac{p-1}{2p} \quad (p=1,3,5,\dots)$$

and those of even density matrices are

$$\omega(\hat{\rho}_{p1}) = \frac{1}{2} \quad (p=2,4,6,\dots).$$

But for the normal phase, the order indices of all reduced density matrices are zero:

$$\omega(\hat{\rho}_{p2}) = 0 \quad (p=1,2,3,\dots).$$

Coexisting phases occupy different spatial regions of the sample. These regions are composed of mesoscopic subregions that are randomly intermixed in space, forming complicated configurations. For each given configuration, we can define a locally equilibrium Gibbs ensemble. Then, since the spatial phase distribution is random, it is necessary to average over these phase configurations. This procedure makes the basis of the theory of statistical systems with mesoscopic phase separation.⁵ After averaging over random phase configurations, we come (see the Appendix) to the renormalized Hamiltonian

$$\tilde{H} = H_1 \oplus H_2 \quad (2)$$

defined on the fiber space

$$\mathcal{Y} = \mathcal{H}_1 \otimes \mathcal{H}_2, \quad (3)$$

being the tensor product of the weighted Hilbert spaces. The phase-replica Hamiltonians H_ν can be written in the form

$$H_\nu = w_\nu H_\nu^{kin} + w_\nu^2 H_\nu^{int}, \quad (4)$$

where H_ν^{kin} is an operator of kinetic energy, H_ν^{int} is an operator describing pair interactions, and w_ν are phase probabilities satisfying the conditions

$$w_1 + w_2 = 1, \quad 0 \leq w_\nu \leq 1. \quad (5)$$

The phase probabilities are defined as the minimizers of the thermodynamic potential

$$\Omega = -T \ln \text{Tr}_Y e^{-\beta \tilde{H}} \quad (\beta T \equiv 1). \quad (6)$$

Here and in what follows, T is temperature and $k_B=1$. Setting the notation

$$w_1 \equiv w, \quad w_2 = 1 - w, \quad (7)$$

the minimization condition reads

$$\frac{\partial \Omega}{\partial w} = 0, \quad \frac{\partial^2 \Omega}{\partial w^2} > 0. \quad (8)$$

This condition shows when mesoscopic phase separation is profitable, as compared to a pure system.

The average of an operator \hat{A} is given as

$$\langle \hat{A} \rangle \equiv \text{Tr}_Y \hat{\rho} \hat{A}, \quad \hat{\rho} \equiv \frac{e^{-\beta \tilde{H}}}{\text{Tr}_Y e^{-\beta \tilde{H}}}, \quad (9)$$

with $\hat{\rho}$ being the statistical operator. Then the first equation from condition (8) takes the form

$$\left\langle \frac{\partial \tilde{H}}{\partial w} \right\rangle = 0, \quad (10)$$

and the second equation yields the inequality, being the *condition of heterophase stability*:

$$\left[\left\langle \frac{\partial^2 \tilde{H}}{\partial w^2} \right\rangle - \beta \left\langle \left(\frac{\partial \tilde{H}}{\partial w} \right)^2 \right\rangle \right] > 0. \quad (11)$$

To illustrate the meaning of Eqs. (10) and (11), let us take the Hamiltonian (2) with the terms (4) and suppose that the kinetic part H_v^{kin} and the interaction part H_v^{int} do not depend on w_v . Let us define the mean kinetic energy K_v and the mean interaction potential Φ_v by the notation

$$K_v \equiv \langle H_v^{kin} \rangle, \quad \Phi_v \equiv 2 \langle H_v^{int} \rangle. \quad (12)$$

Then Eq. (10) gives the probability of the superconducting phase:

$$w = \frac{\Phi_2 + K_2 - K_1}{\Phi_1 + \Phi_2}. \quad (13)$$

From here, since $0 \leq w \leq 1$, we get

$$-\Phi_1 \leq K_1 - K_2 \leq \Phi_2. \quad (14)$$

The stability condition (11) yields

$$\Phi_1 + \Phi_2 > \frac{\beta}{N} \left\langle \left(\frac{\partial \tilde{H}}{\partial w} \right)^2 \right\rangle,$$

from where it follows that the necessary condition for the stability of a phase-separated sample is

$$\Phi_1 + \Phi_2 > 0. \quad (15)$$

In that case, phase separation becomes thermodynamically profitable, for which, as is seen from Eq. (15), the existence of repulsive interactions is compulsory.

III. STRUCTURE OF THE HAMILTONIAN

Employing the field representation, we deal with the field operators $\psi_{s\nu}(\mathbf{r})$, in which $s = \uparrow, \downarrow$ denotes spin and \mathbf{r} is a spatial vector. Fermi commutation relations are assumed. The kinetic part has the standard form

$$H_v^{kin} = \sum_s \int \psi_{s\nu}^\dagger(\mathbf{r}) [\hat{K}_\nu(\mathbf{r}) - \mu] \psi_{s\nu}(\mathbf{r}) d\mathbf{r}, \quad (16)$$

where $\hat{K}_\nu(\mathbf{r})$ is a kinetic transport operator and μ a chemical potential. The interaction part H_v^{int} , in general, consists of direct interactions and of effective interactions due to a kind of boson exchange. This, for instance, can be the phonon exchange if one considers the usual picture based on the Frölich Hamiltonian.^{55–58} In principle, one may consider the exchange by other types of bosons—say, excitons—but for concreteness, we shall keep in mind the conventional phonon picture. For simplicity and at the same time for generality, we take for the interaction Hamiltonian the expression

$$H_v^{int} = \frac{1}{2} \sum_{ss'} \int \psi_{s\nu}^\dagger(\mathbf{r}) \psi_{s'\nu}^\dagger(\mathbf{r}') \hat{V}_\nu(\mathbf{r}, \mathbf{r}') \psi_{s'\nu}(\mathbf{r}') \psi_{s\nu}(\mathbf{r}) d\mathbf{r} d\mathbf{r}', \quad (17)$$

where the vertex operator \hat{V}_ν models all effective interactions, direct as well as those caused by boson exchange. The vertex operator is supposed to be symmetric:

$$\hat{V}_\nu(\mathbf{r}, \mathbf{r}') = \hat{V}_\nu(\mathbf{r}', \mathbf{r}). \quad (18)$$

In agreement with condition (1), the anomalous averages for the superconducting phase are not trivial,

$$\langle \psi_{s1}(\mathbf{r}) \psi_{s'1}(\mathbf{r}') \rangle \neq 0, \quad (19)$$

at least for some spins, while such averages for the normal phase are identically zero:

$$\langle \psi_{s2}(\mathbf{r}) \psi_{s'2}(\mathbf{r}') \rangle \equiv 0. \quad (20)$$

For crystalline matter with a periodic structure, the field operator can be expanded over Bloch functions, which, for the single-zone case, is written

$$\psi_{s\nu}(\mathbf{r}) = \sum_k c_{s\nu}(\mathbf{k}) \varphi_k(\mathbf{r}), \quad (21)$$

with \mathbf{k} being a wave vector. Let us introduce the matrix elements over the Bloch functions $\varphi_k(\mathbf{r})$, resulting in the transport matrix

$$t_\nu(\mathbf{k}, \mathbf{p}) \equiv (\varphi_k, \hat{K}_\nu \varphi_p) \quad (22)$$

and the vertex

$$V_\nu(\mathbf{k}, \mathbf{k}', \mathbf{p}', \mathbf{p}) \equiv (\varphi_k \varphi_{k'}, \hat{V}_\nu \varphi_{p'} \varphi_p). \quad (23)$$

The latter, due to Eq. (18), has the symmetry property

$$V_\nu(\mathbf{k}, \mathbf{k}', \mathbf{p}', \mathbf{p}) = V_\nu(\mathbf{k}', \mathbf{k}, \mathbf{p}, \mathbf{p}'). \quad (24)$$

Invoking expansion (21), the kinetic part (16) transforms to

$$H_v^{kin} = \sum_s \sum_{kp} [t_v(\mathbf{k}, \mathbf{p}) - \mu \delta_{kp}] c_{sv}^\dagger(\mathbf{k}) c_{sv}(\mathbf{p}) \quad (25)$$

and the interaction term (17) becomes

$$H_v^{int} = \frac{1}{2} \sum_{ss'} \sum_{kk'} \sum_{pp'} V_v(\mathbf{k}, \mathbf{k}', \mathbf{p}', \mathbf{p}) c_{sv}^\dagger(\mathbf{k}) c_{s'v}^\dagger(\mathbf{k}') c_{s'v}(\mathbf{p}') c_{sv}(\mathbf{p}). \quad (26)$$

To make the problem treatable, let us resort to the Hartree-Fock-Bogolubov approximation, according to which the four-operator products are expressed as

$$\begin{aligned} c_1 c_2 c_3 c_4 &= c_1 c_2 \langle c_3 c_4 \rangle + \langle c_1 c_2 \rangle c_3 c_4 - \langle c_1 c_2 \rangle \langle c_3 c_4 \rangle + c_1 c_4 \langle c_2 c_3 \rangle \\ &\quad + \langle c_1 c_4 \rangle c_2 c_3 - \langle c_1 c_4 \rangle \langle c_2 c_3 \rangle - c_1 c_3 \langle c_2 c_4 \rangle \\ &\quad - \langle c_1 c_3 \rangle c_2 c_4 + \langle c_1 c_3 \rangle \langle c_2 c_4 \rangle, \end{aligned} \quad (27)$$

where c_i represents any of the operators $c_{sv}(\mathbf{k})$ and $c_{sv}^\dagger(\mathbf{k})$ and the Fermi commutation relations are assumed. Also, we shall consider the restricted spaces of quantum states, for which the Bardeen-Cooper-Schrieffer restriction is valid:

$$\begin{aligned} c_{sv}^\dagger(\mathbf{k}) c_{s'v}(\mathbf{k}') &= \delta_{ss'} \delta_{kk'} c_{sv}^\dagger(\mathbf{k}) c_{sv}(\mathbf{k}), \\ c_{sv}^\dagger(\mathbf{k}) c_{s'v}^\dagger(\mathbf{k}') &= \delta_{-ss'} \delta_{-kk'} c_{sv}^\dagger(\mathbf{k}) c_{-s'v}^\dagger(-\mathbf{k}'). \end{aligned} \quad (28)$$

This means that the restricted spaces consist of the wave functions for which spin and momentum are conserved.

The normal average

$$n_v(\mathbf{k}) \equiv \sum_s \langle c_{sv}^\dagger(\mathbf{k}) c_{sv}(\mathbf{k}) \rangle \quad (29)$$

is the momentum distribution of particles. Introducing the anomalous average

$$\sigma_v(\mathbf{k}) \equiv \langle c_{-sv}(-\mathbf{k}) c_{sv}(\mathbf{k}) \rangle, \quad (30)$$

conditions (19) and (20) can be rewritten as

$$\sigma_1(\mathbf{k}) \neq 0, \quad \sigma_2(\mathbf{k}) \equiv 0. \quad (31)$$

With the approximations (27) and (28), the Hamiltonian (4) can be diagonalized by means of the Bogolubov canonical transformation

$$c_{sv}(\mathbf{k}) = u_v(\mathbf{k}) a_{sv}(\mathbf{k}) + v_v(\mathbf{k}) a_{-sv}^\dagger(\mathbf{k}), \quad (32)$$

in which

$$|u_v(\mathbf{k})|^2 = \frac{1}{2} \left[1 + \frac{\omega_v(\mathbf{k})}{E_v(\mathbf{k})} \right], \quad |v_v(\mathbf{k})|^2 = \frac{1}{2} \left[1 - \frac{\omega_v(\mathbf{k})}{E_v(\mathbf{k})} \right].$$

Here, the single-particle dispersion is

$$\omega_v(\mathbf{k}) = t_v(\mathbf{k}, \mathbf{k}) + w_v M_v(\mathbf{k}) - \mu, \quad (33)$$

with the mass operator

$$M_v(\mathbf{k}) \equiv \sum_p \left[V_v(\mathbf{k}, \mathbf{p}, \mathbf{p}, \mathbf{k}) - \frac{1}{2} V_v(\mathbf{k}, \mathbf{p}, \mathbf{k}, \mathbf{p}) \right] n_p(\mathbf{p}), \quad (34)$$

and the excitation spectrum

$$E_v(\mathbf{k}) = \sqrt{\Delta_v^2(\mathbf{k}) + \omega_v^2(\mathbf{k})} \quad (35)$$

contains the gap

$$\Delta_v(\mathbf{k}) = w_v \sum_p J_v(\mathbf{k}, \mathbf{p}) \sigma_v(\mathbf{p}), \quad (36)$$

where the effective interaction

$$J_v(\mathbf{k}, \mathbf{p}) \equiv -V_v(\mathbf{k}, -\mathbf{k}, -\mathbf{p}, \mathbf{p}). \quad (37)$$

Then the Hamiltonian (4) reduces to

$$H_v = w_v \sum_s \sum_k E_v(\mathbf{k}) a_{sv}^\dagger(\mathbf{k}) a_{sv}(\mathbf{k}) + w_v C_v, \quad (38)$$

with the nonoperator term

$$C_v = \sum_k \left[\omega_v(\mathbf{k}) - E_v(\mathbf{k}) + \Delta_v(\mathbf{k}) \sigma_v(\mathbf{k}) - \frac{1}{2} w_v M_v(\mathbf{k}) n_v(\mathbf{k}) \right].$$

For the averages (29) and (30), one gets

$$\begin{aligned} n_v(\mathbf{k}) &= 1 - \frac{\omega_v(\mathbf{k})}{E_v(\mathbf{k})} \tanh \frac{w_v E_v(\mathbf{k})}{2T}, \\ \sigma_v(\mathbf{k}) &= \frac{\Delta_v(\mathbf{k})}{2E_v(\mathbf{k})} \tanh \frac{w_v E_v(\mathbf{k})}{2T}. \end{aligned} \quad (39)$$

These expressions have sense for both the superconducting and normal phases; however, for the normal phase, according to condition (31), one has

$$\sigma_2(\mathbf{k}) = \Delta_2(\mathbf{k}) = 0, \quad (40)$$

from which

$$n_2(k) = \frac{2}{\exp\{\beta w_2 \omega_2(\mathbf{k})\} + 1}.$$

Note that the phase probabilities w_v enter all equations in a rather nontrivial way, which will essentially influence the properties of superconductors with mesoscopic phase separation.

IV. PHASE SEPARATION

In what follows, we shall use the notation (7), writing $w = w_1$, and, similarly, we shall omit, for the sake of simplicity, the index $\nu=1$ at all related quantities. For instance, we shall write $\Delta(\mathbf{k})$, $\sigma(\mathbf{k})$, $E(\mathbf{k})$, and so on instead of $\Delta_1(\mathbf{k})$, $\sigma_1(\mathbf{k})$, and $E_1(\mathbf{k})$.

The gap equation (36) can be presented as

$$\Delta(\mathbf{k}) = \frac{w}{2} \sum_p J(\mathbf{k}, \mathbf{p}) \frac{\Delta(\mathbf{p})}{E(\mathbf{p})} \tanh \frac{w E(\mathbf{p})}{2T}. \quad (41)$$

Looking for a positive solution for $\Delta(\mathbf{k})$, we see that this is possible when the right-hand side of Eq. (41) is also positive, which requires that the effective interaction

$$J(\mathbf{k}, \mathbf{p}) > 0 \quad (42)$$

be positive in the region of momenta making the main contribution in the summation of Eq. (41).

Another necessary condition is the condition (15) for the profitability of phase separation. For the mean interaction potential, defined in Eq. (12), we find

$$\Phi_\nu = \sum_k M_\nu(\mathbf{k})n_\nu(\mathbf{k}) - 2 \sum_{kp} J_\nu(\mathbf{k}, \mathbf{p})\sigma_\nu(\mathbf{k})\sigma_\nu(\mathbf{p}). \quad (43)$$

Then the stability condition (15) yields

$$\sum_\nu \sum_k M_\nu(\mathbf{k})n_\nu(\mathbf{k}) > 2 \sum_{kp} J(\mathbf{k}, \mathbf{p})\sigma(\mathbf{k})\sigma(\mathbf{p}). \quad (44)$$

From here, we get the necessary condition

$$\sum_\nu \sum_k M_\nu(\mathbf{k})n_\nu(\mathbf{k}) > 0, \quad (45)$$

which tells us that some sufficiently strong repulsive interactions are to be present in the system. Such natural interactions are, of course, Coulomb interactions. Thus, we come to the first conclusion:

Mesoscopic phase separation in superconductors can be thermodynamically stable only in the presence of repulsive Coulomb interactions.

Now let us recall that, as is discussed in the Introduction, the gap of the hole-doped high-temperature cuprate superconductors displays strong anisotropic dependence on momentum. To describe the anisotropy, one may introduce a basis $\{\chi_i(\mathbf{k})\}$ of functions $\chi_i(\mathbf{k})$ characterizing the lattice symmetry, with the index $i=1, 2, \dots$ enumerating irreducible representations of the symmetry group. Let such a basis be defined, being orthonormal and complete:

$$\sum_k \chi_i^*(\mathbf{k})\chi_j(\mathbf{k}) = \delta_{ij}, \quad \sum_i \chi_i^*(\mathbf{k})\chi_i(\mathbf{p}) = \delta_{kp}.$$

Then one may expand (see Refs. 25 and 59) over this basis the effective interaction

$$J(\mathbf{k}, \mathbf{p}) = \sum_{ij} J_{ij}\chi_i(\mathbf{k})\chi_j^*(\mathbf{p}) \quad (46)$$

and the gap

$$\Delta(\mathbf{k}) = \sum_i \Delta_i \chi_i(\mathbf{k}). \quad (47)$$

Using this, the gap equation (41) reduces to

$$\Delta_i = \sum_j A_{ij}\Delta_j, \quad (48)$$

where

$$A_{ij} \equiv \sum_p \frac{wJ_{ij}}{2E(\mathbf{p})} \tanh\left[\frac{wE(\mathbf{p})}{2T}\right] \chi_i^*(\mathbf{p})\chi_j(\mathbf{p}). \quad (49)$$

The system of uniform algebraic equations (48) possesses nontrivial solutions when

$$\det(\hat{1} - \hat{A}) = 0, \quad (50)$$

where $\hat{1} = [\delta_{ij}]$ is the unity matrix and the matrix $\hat{A} = [A_{ij}]$ is composed of elements (49).

The effective interaction (46) consists of an attractive part, caused by phonon exchange, and a repulsive part, due

to direct Coulomb interactions,⁶⁰ because of which J_{ij} has the structure

$$J_{ij} = \left(\frac{|\alpha|^2}{\tilde{\omega}_0^2} - M_0 \right) b_{ij}, \quad (51)$$

in which α is the charge-lattice coupling, $\tilde{\omega}_0$ is the characteristic lattice frequency in the presence of heterostructural fluctuations,^{61,62} connected by the relation

$$\tilde{\omega}_0 = \sqrt{w}\omega_0, \quad (52)$$

with the characteristic lattice frequency ω_0 of a pure sample, and M_0 is an effective intensity of direct Coulomb interactions. The latter approximately equals

$$M_0 \approx \frac{\pi e^2}{k_F^2} \ln \left| 1 + 4 \left(\frac{k_F}{\kappa} \right)^2 \right|,$$

where k_F is a Fermi momentum of charge carriers, κ^{-1} is a screening radius, for which $\kappa^2 \approx 4m_0e^2(3\rho/\pi)^{1/3}$, and m_0 , e , and ρ are mass, charge, and density of carriers.

Keeping in mind inequality (42), we set $J_{ij} > 0$ and $b_{ij} > 0$. Then the condition for the existence of superconductivity reads

$$\frac{|\alpha|^2}{\omega_0^2} - wM_0 > 0. \quad (53)$$

This differs from the Bardeen-Cooper-Schrieffer criterion for superconductivity⁶³ by the presence of the superconducting phase probability w , which essentially changes the meaning of Eq. (53). It may happen that Coulomb interactions are so strong, with $M_0 > |\alpha/\omega_0|^2$, that superconductivity in a pure sample is impossible. However, since $w < 1$, criterion (53) may be valid, which implies the occurrence of superconductivity. In this way, we get the second conclusion:

Phase separation enables the appearance of superconductivity in a heterophase sample even if it were impossible in pure-phase matter.

Defining the dimensionless quantity

$$\mu^* \equiv M_0 \frac{\omega_0^2}{|\alpha|^2}, \quad (54)$$

condition (53) can be presented as

$$1 - w\mu^* > 0. \quad (55)$$

The parameter (54) is of the order⁶⁰ of $\mu^* \sim \omega_0/\omega_p$, where ω_p is the ion plasma frequency. For good conductors, $\omega_0 \ll \omega_p$; hence, $\mu^* \ll 1$, and inequality (55) is easy to satisfy even for a pure sample, with $w=1$. For bad conductors, $\omega_0 \geq \omega_p$, so that $\mu^* \geq 1$. In particular, if $\mu^* > 1$, superconductivity cannot arise in a pure sample, though may appear in phase-separated matter, with $w < 1$. This yields the third conclusion:

Phase separation is crucial for the occurrence of superconductivity in bad conductors.

V. CRITICAL TEMPERATURE

At the critical temperature T_c , the gap tends to zero, $\Delta(\mathbf{k}) \rightarrow 0$; hence, $E(\mathbf{k}) \rightarrow \omega(\mathbf{k})$. Then Eq. (49) becomes

$$A_{ij}(T_c) = \sum_p \frac{wJ_{ij}}{2\omega(\mathbf{p})} \tanh\left[\frac{w\omega(\mathbf{p})}{2T_c}\right] \chi_i^*(\mathbf{p})\chi_j(\mathbf{p}). \quad (56)$$

Using the density of states

$$N_{ij}(\omega) \equiv \sum_p \delta(\omega - \omega(\mathbf{p})) \chi_i^*(\mathbf{p})\chi_j(\mathbf{p}) \quad (57)$$

and satisfying the normalization

$$\int_{-\infty}^{+\infty} N_{ij}(\omega) d\omega = \delta_{ij},$$

Eq. (56) can be written as

$$A_{ij}(T_c) = wJ_{ij} \int_{-\infty}^{+\infty} \frac{N_{ij}(\omega)}{2\omega} \tanh\left(\frac{w\omega}{2T_c}\right) d\omega. \quad (58)$$

The density of states (57), with the standard replacement of summation by integration,

$$\sum_{p \in \mathcal{B}} \rightarrow \frac{1}{\rho} \int_{\mathcal{B}} \frac{d\mathbf{p}}{(2\pi)^3},$$

where \mathcal{B} implies the Brillouin zone, transforms to

$$N_{ij}(\omega) = \frac{1}{\rho} \int_{\mathcal{B}} \delta(\omega - \omega(\mathbf{p})) \chi_i^*(\mathbf{p})\chi_j(\mathbf{p}) \frac{d\mathbf{p}}{(2\pi)^3}. \quad (59)$$

Assuming, as usual, that the density of states $N_{ij}(\omega)$ is the largest on the Fermi surface and quickly decreases after $\omega > \tilde{\omega}_0$, Eq. (58) can be reduced to

$$A_{ij}(T_c) = wJ_{ij}N_{ij}(0) \int_0^{\tilde{\omega}_0} \frac{1}{\omega} \tanh\left(\frac{w\omega}{2T_c}\right) d\omega. \quad (60)$$

Introducing the coupling matrix $\hat{\lambda}$, with the elements

$$\lambda_{ij} \equiv N_{ij}(0) \frac{|\alpha|^2}{\omega_0^2} b_{ij}, \quad (61)$$

effective coupling matrix $\hat{\Lambda}$, with

$$\Lambda_{ij} \equiv wJ_{ij}N_{ij}(0) = (1 - w\mu^*)\lambda_{ij}, \quad (62)$$

and the characteristic integral

$$I_c \equiv \int_0^1 \frac{1}{x} \tanh\left(\frac{w^{3/2}\omega_0}{2T_c}x\right) dx, \quad (63)$$

we can present Eq. (60) as

$$A_{ij}(T_c) = I_c \Lambda_{ij}. \quad (64)$$

Substituting this into condition (50) gives the equation

$$\det(\hat{1} - I_c \hat{\Lambda}) = 0 \quad (65)$$

for the critical temperature T_c .

If Λ_{ij} is diagonal, then T_c is defined by the largest Λ_{ii} . However, in general, for anisotropic superconductors, Λ_{ij} is not diagonal and $\Lambda_{ij} \neq 0$ for $i \neq j$. The latter means that, generally, the gap (47) is presented by a mixture of waves of different symmetry. There is a very nontrivial relation be-

tween the gap being such a mixture and the magnitude of the critical temperature, which we describe below.

Let us define an *effective coupling* Λ_{eff} by the identity

$$\Lambda_{eff} I_c \equiv 1 - \det(\hat{1} - I_c \hat{\Lambda}). \quad (66)$$

Then Eq. (65) for the critical temperature takes the form

$$\Lambda_{eff} I_c = 1. \quad (67)$$

Note that, since, according to definition (63), the integral $I_c > 0$ is positive, then $\Lambda_{eff} > 0$. From Eq. (67) it follows

$$\frac{\partial T_c}{\partial \Lambda_{eff}} = \frac{2T_c^2 I_c}{\omega_0 w^{3/2} \Lambda_{eff} I_c'} > 0, \quad (68)$$

where

$$I_c' \equiv \int_0^1 \text{sech}^2\left(\frac{w^{3/2}\omega_0}{2T_c}x\right) dx.$$

Inequality (68) tells us that T_c is higher for larger Λ_{eff} . This is valid for all $\Lambda_{eff} > 0$ and can be explicitly illustrated for the particular cases:

$$T_c \simeq 1.14 w^{3/2} \omega_0 \exp\left(-\frac{1}{\Lambda_{eff}}\right) (\Lambda_{eff} \ll 1),$$

$$T_c \simeq \frac{1}{2} w^{3/2} \omega_0 \Lambda_{eff} (\Lambda_{eff} \gg 1). \quad (69)$$

Another important inequality is

$$\Lambda_{eff} > \max_i \Lambda_{ii}, \quad (70)$$

provided that $\Lambda_{ij} \neq 0$ for some $i \neq j$. And if $\Lambda_{ij} = \delta_{ij} \Lambda_{ii}$, then $\Lambda_{eff} = \max_i \Lambda_{ii}$. This property is easy to explicitly demonstrate for the case when there are two prevailing wave symmetries. Thus, if $i = 1, 2$, then Eq. (66) yields

$$\Lambda_{eff} = \frac{1}{2} [\Lambda_{11} + \Lambda_{22} + \sqrt{(\Lambda_{11} - \Lambda_{22})^2 + 4\Lambda_{12}^2}]. \quad (71)$$

Without loss of generality, we may use an enumeration such that $\Lambda_{11} > \Lambda_{22}$. As is seen from Eq. (71),

$$\Lambda_{eff} = \Lambda_{11} (\Lambda_{12} = 0).$$

In addition,

$$\frac{\partial \Lambda_{eff}}{\partial \Lambda_{12}^2} = \frac{1}{\sqrt{(\Lambda_{11} - \Lambda_{22})^2 + 4\Lambda_{12}^2}} > 0; \quad (72)$$

hence, Λ_{eff} increases with increasing $|\Lambda_{12}|$. Thus, for a mixture of waves, when $|\Lambda_{ij}| > 0$, where $i \neq j$, the effective coupling Λ_{eff} becomes larger than the maximal Λ_{ii} . But in agreement with Eq. (68), the larger Λ_{eff} , the higher is the transition temperature T_c . Therefore, we come to the following conclusion:

The critical temperature for a mixture of gap waves is higher than the critical temperature related to any pure gap wave from this mixture.

In superconductors with phase separation, the critical temperature can be a nonmonotonic function of the supercon-

ducting fraction w . To show this, let us consider the case when one of the gap symmetries is prevailing, so that one of Λ_{ii} is essentially larger than other Λ_{ii} . Let us denote this maximal Λ_{ii} as

$$\max_i \Lambda_{ii} \equiv (1 - w\mu^*)\lambda, \quad (73)$$

where the relation (62) is taken into account. Equation (67) for the critical temperature can be written as

$$(1 - w\mu^*)\lambda I_c = 1.$$

The characteristic integral (63) possesses the following asymptotic properties:

$$I_c \approx \frac{w^{3/2}\omega_0}{2T_c} (w \rightarrow 0),$$

$$I_c \approx \ln\left(1.14 \frac{w^{3/2}\omega_0}{T_c}\right) (T_c \rightarrow 0).$$

Using these properties, we see that the critical temperature tends to zero in two limits, if the superconducting fraction tends to zero, when

$$\frac{T_c}{\omega_0} \approx \frac{1}{2}(1 - w\mu^*)\lambda w^{3/2} (w \rightarrow 0), \quad (74)$$

and also if this fraction tends to a finite value $1/\mu^*$, when

$$\frac{T_c}{\omega_0} \approx 1.14w^{3/2} \exp\left\{-\frac{1}{(1 - w\mu^*)\lambda}\right\} \left(w \rightarrow \frac{1}{\mu^*}\right). \quad (75)$$

For good conductors, when $\mu^* \ll 1$, the limit (75) is unachievable. But for bad conductors, for which $\mu^* \geq 1$, this limit can be achieved. This can be formulated as another conclusion:

In bad conductors, the critical temperature as a function of the superconducting fraction w has the bell shape, tending to zero at $w \rightarrow 0$ and at $w \rightarrow 1/\mu^$.*

To estimate the point where the critical temperature is maximal, we may keep in mind that experiments with high-temperature superconductors show that only part of a given sample is in a superconducting phase, this part often being just a few percent.^{64,65} This means that $w \ll 1$ and, hence, the value w_{max} , where $T_c = T_{max}$ is maximal, is also small, $w_{max} \ll 1$. Taking this into account, from the above equations we obtain

$$T_{max} \approx \frac{1}{5}\lambda\omega_0 w_{max}^{3/2}, \quad w_{max} \approx \frac{3}{5\mu^*}.$$

Such a bell shape of the critical temperature as a function of doping is typical of experimental curves for high-temperature cuprate superconductors,^{1,66,67} where the maximal critical temperature occurs at the optimal doping 0.15. Assuming that the superconducting fraction is proportional to the doping, so that $w_{max} \approx 0.15$, we have $\mu^* \approx 4$. Then the function $T_c(w)$ has a striking similarity with the behavior of T_c as a function of doping, studied in experiments with cuprates. Figure 1 illustrates $T_c(w)$ found by solving numeri-

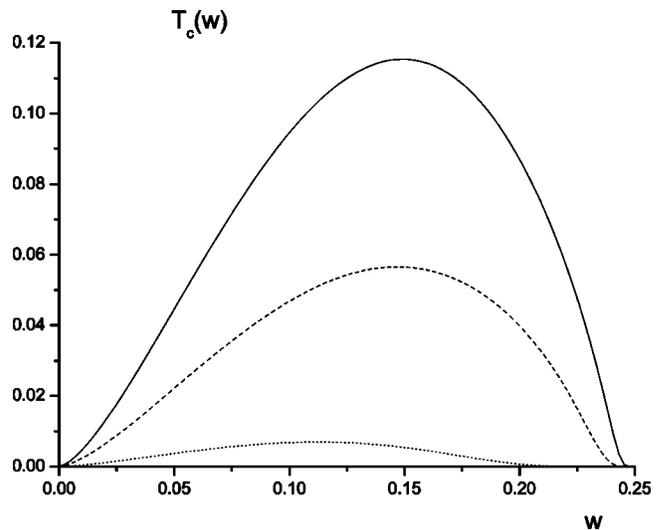


FIG. 1. Critical temperature T_c , in units of ω_0 , as a function of w for $\mu^* = 4$ and different couplings: $\lambda = 1$ (dotted line), $\lambda = 5$ (dashed line), and $\lambda = 10$ (solid line).

cally Eq. (67).

VI. DENSITY OF STATES

To concretize the consideration, let us take into account that the crystalline structure of cuprates is such that the carriers move mainly in planes, only rarely jumping between the latter, which are separated by a distance essentially exceeding the mean distance a between lattice sites on the plane. Neglecting the interplane jumps, one comes to a two-dimensional motion of carriers on the plane. In the case of such a planar motion, the single-particle dispersion $\omega(\mathbf{p}) = \omega_1(\mathbf{p})$, given by Eq. (33), depends only on two momentum components—say, p_1 and p_2 . Then in the following formulas it is easy to integrate out the third component p_3 .

In many cuprates, the Cu and O atoms arrange themselves in a square lattice with the point group symmetry C_{4v} (Ref. 25). This concrete case will be employed in what follows.

For a square lattice $a \times a$, the Brillouin zone is defined by the wave vectors $p_\alpha \in [-\pi/a, \pi/a]$, with $\alpha = 1, 2$. It is convenient to introduce the dimensionless wave vectors $\mathbf{k} = \{k_\alpha\}$, in which $k_\alpha \equiv p_\alpha a$, so that $k_\alpha \in [-\pi, \pi]$. The point group C_{4v} of a square lattice is characterized by three one-dimensional irreducible representations labeled as A_1 , B_1 , and B_2 . The representation A_1 is of rank 3, having three types of symmetries denoted as s , s^* , and s_{xy} . The representation B_1 is of rank 1, with the symmetry $d_{x^2-y^2}$. And the irreducible representation B_2 is of rank 1, with the type of symmetry denoted by d_{xy} . The corresponding basis functions are

$$\chi_1(\mathbf{k}) = 1 (s),$$

$$\chi_2(\mathbf{k}) = \frac{1}{2}(\cos k_1 + \cos k_2) (s^*),$$

$$\chi_3(\mathbf{k}) = \cos k_1 \cos k_2 (s_{xy}),$$

$$\chi_4(\mathbf{k}) = \sin k_1 \sin k_2 (d_{xy}),$$

$$\chi_5(\mathbf{k}) = \frac{1}{2}(\cos k_1 - \cos k_2) (d_{x^2-y^2}).$$

For the density of states (57), we have

$$N_{ij}(\omega) = \int_{-\pi}^{\pi} \delta(\omega - \omega(\mathbf{k})) \chi_i^*(\mathbf{k}) \chi_j(\mathbf{k}) \frac{dk_1 dk_2}{(2\pi)^2}. \quad (76)$$

The basis functions $\chi_i^*(\mathbf{k}) = \chi_i(\mathbf{k})$ are real and symmetric with respect to the inversion of \mathbf{k} , so that $\chi_i(-\mathbf{k}) = \chi_i(\mathbf{k})$. The single-particle dispersion is also symmetric, $\omega(-\mathbf{k}) = \omega(\mathbf{k})$. It is convenient to introduce the dimensionless dispersion

$$\bar{\omega}(\mathbf{k}) \equiv \frac{\omega(\mathbf{k})}{\mu}. \quad (77)$$

Then the density of states (76) on the Fermi surface becomes

$$N_{ij}(0) = \frac{1}{\pi^2 \mu} \int_0^{\pi} \delta(\bar{\omega}(\mathbf{k})) \chi_i(\mathbf{k}) \chi_j(\mathbf{k}) dk_1 dk_2. \quad (78)$$

To write down an explicit expression for the single-particle dispersion $\omega(\mathbf{k})$, one usually resorts to the tight-binding approximation,^{25,39,40,45} in which, for a square lattice, one has

$$\omega(\mathbf{k}) = -t_{eff}(\cos k_1 + \cos k_2) - \mu, \quad (79)$$

where t_{eff} is an effective transport parameter. In our case, taking into account Eq. (33), we see that t_{eff} consists of two parts,

$$t_{eff} = t_0 + wM_0, \quad (80)$$

the nearest-neighbor hopping integral t_0 and the intensity of the repulsive Coulomb interaction M_0 . These terms correspond to the transport matrix and mass operator, respectively. As is well established, both experimentally and theoretically, a strong Coulomb repulsion is present in all cuprates.²⁵ With the parameter

$$t \equiv \frac{t_{eff}}{\mu} = \frac{1}{\mu}(t_0 + wM_0), \quad (81)$$

the dimensionless dispersion (77) takes the form

$$\bar{\omega}(\mathbf{k}) = -t(\cos k_1 + \cos k_2) - 1. \quad (82)$$

When the mesoscopic phase separation happens in a superconductor, so that $w < 1$, then, as is seen from Eq. (80), the parameter t_{eff} decreases. As a result of this, the dispersion (79) becomes softer. Thus, we get an important conclusion:

Phase separation softens the single-particle dispersion.

To proceed further, we calculate the density of states (78). For this purpose, we use the dispersion (82) and note that

$$\delta(\bar{\omega}(\mathbf{k})) = \frac{\delta(k_2 - k_2(k_1))}{|t \sin k_2(k_1)|},$$

where the function $k_2(k)$ is defined by the equation

$$\cos k_2(k) = -\frac{1}{t} - \cos k.$$

Then the density of states (78) can be transformed to

$$N_{ij}(0) = \frac{1}{\pi^2 \mu} \int_{k_0}^{\pi} \frac{\varphi_i(k) \varphi_j(k) dk}{\sqrt{t^2 - (1 + t \cos k)^2}}, \quad (83)$$

where

$$\varphi_i(k) \equiv \chi_i(k, k_2(k)), \quad k_0 \equiv \arccos\left(1 - \frac{1}{t}\right),$$

$$0 \leq k_0 \leq \pi \quad \left(\frac{1}{2} \leq t < \infty\right).$$

Explicit expressions for the functions $\varphi_i(k)$ are

$$\varphi_1(k) = 1, \quad \varphi_2(k) = -\frac{1}{2t}, \quad \varphi_3(k) = -\left(\frac{1}{t} + \cos k\right) \cos k,$$

$$\varphi_4(k) = \sin k \sqrt{1 - \left(\frac{1}{t} + \cos k\right)^2}, \quad \varphi_5(k) = \frac{1}{2t} + \cos k.$$

Accomplishing in Eq. (83) the change of variables

$$\psi_i(x) \equiv \varphi_i(\arccos x) \quad (x \equiv \cos k),$$

we obtain the form

$$N_{ij}(0) = \frac{1}{\pi^2 \mu t} \int_{-1}^{1-1/t} \frac{\psi_i(x) \psi_j(x)}{\psi_4(x)} dx, \quad (84)$$

in which

$$\psi_1(x) = 1, \quad \psi_2(x) = -\frac{1}{2t}, \quad \psi_3(x) = -\left(\frac{1}{t} + x\right)x,$$

$$\psi_4(x) = \sqrt{(1-x^2) \left[1 - \left(\frac{1}{t} + x\right)^2\right]}, \quad \psi_5(x) = \frac{1}{2t} + x.$$

The value of the quantity $N_{ij}(0)$ plays an important role in defining the coupling parameters (61) and (62). The larger is the density of states $N_{ii}(0)$, the more profitable is the occurrence of the related gap symmetry labeled by the index i .

Let us analyze the behavior of Eq. (84) as a function of t changing from $t=1/2$ to larger $t > 1/2$. For convenience, we consider the dimensionless quantity

$$D_{ij} \equiv \pi \mu N_{ij}(0), \quad (85)$$

normalized by means of

$$N_{11}(0) = N_{22}(0) = \frac{1}{\pi \mu} \quad \left(t = \frac{1}{2}\right).$$

At $t=1/2$, the integral (84) can be calculated analytically, giving

$$D_{11} = D_{13} = D_{22} = D_{33} = 1, \quad D_{14} = D_{24} = D_{34} = D_{44} = D_{55} = 0,$$

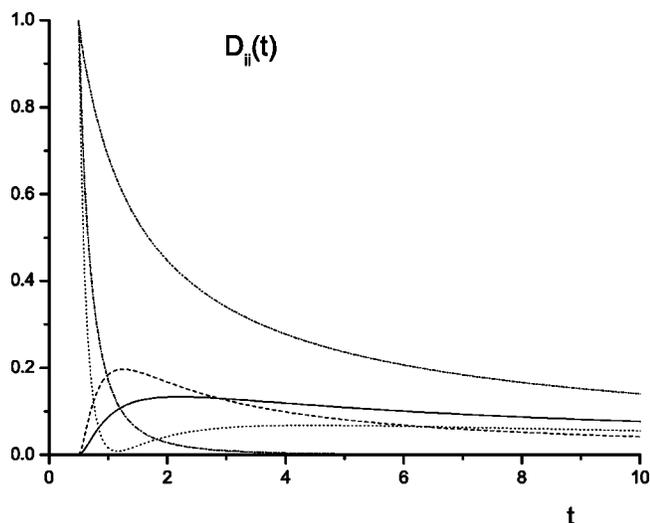


FIG. 2. Density of states D_{ii} as a function of the effective transport parameter t : D_{11} (upper dash-double-dotted line), D_{22} (dash-dotted line), D_{33} (dotted line), D_{44} (dashed line), and D_{55} (solid line).

$$D_{12} = D_{23} = -1 \quad \left(t = \frac{1}{2} \right).$$

Also, for all $t \geq 1/2$, one has

$$D_{15} = D_{25} = D_{35} = D_{45} = 0.$$

For $t > 1/2$, we calculated the integral (84) numerically. The corresponding densities of states are shown in Fig. 2. Among all D_{ii} , only D_{11} and D_{22} monotonically decrease; all other D_{ii} are nonmonotonic functions of t . Comparing the density of states D_{ii} for different $i=1,2,3,4,5$, we have the following. Since at $t=1/2$ only $D_{11}=D_{22}=D_{33}=1$ are nonzero, while $D_{44}=D_{55}=0$; from this, it follows that only the s , s^* , and s_{xy} waves can exist. But at $t=1$, we get $D_{11} \approx 0.6$, $D_{22} = D_{44} \approx 0.2$, $D_{33} \approx 0$, and $D_{55} \approx 0.1$, which tells us that now the gap symmetries s , s^* , d_{xy} , and $d_{x^2-y^2}$ may exist, while the s_{xy} wave disappears. With increasing t , the density D_{55} increases. For instance, at $t=3$, we have $D_{11} \approx 0.3$, D_{22} , and D_{33} are close to zero, $D_{44}=D_{55} \approx 0.1$. Therefore, here the probable symmetries are s , d_{xy} , and $d_{x^2-y^2}$. For $t > 3$, the highest densities are D_{11} , diminishing below 0.3, and $D_{55} \approx 0.1$, all other densities being smaller. In this way, for $t > 3$ the most probable symmetries are s and $d_{x^2-y^2}$.

To estimate the magnitude of t for high-temperature superconductors, we may take the values of parameters typical of cuprates^{25,39,40,45}—that is, $t_0 \approx (0.5-1)$ eV, $M_0 \approx 1$ eV, and $\mu \approx 0.5$ eV. Then the parameter t , defined in Eq. (81), with $w \approx 1$, equals $t \approx 3-4$. In this region of t , the largest densities are D_{11} and D_{55} ; hence, the symmetries s and $d_{x^2-y^2}$ are preferable. But when phase separation occurs, the superconducting fraction w becomes less than unity, which diminishes the value of t . The decrease of t suppresses the density of states D_{55} and enhances D_{11} , which means that the relative weights of s and $d_{x^2-y^2}$ symmetries are changing. This can be formulated as the following conclusion:

Mesoscopic phase separation suppresses the contribution

of d -wave superconductivity and enhances that of s -wave superconductivity.

The value of the superconducting fraction w depends on such parameters as temperature, pressure or external magnetic fields. Therefore the relative contribution of different wave symmetries will be varying under the action of these parameters. This can explain why in different experiments one observes alternately the dominance of either s or d gap symmetries.

VII. SUMMARY

We have studied the main properties of superconductors, such as cuprates, exhibiting two principal features common for these high-temperature superconductors: mesoscopic phase separation and anisotropic gap symmetry. The interplay between these two phenomena is investigated by means of a model incorporating the presence of mesoscopic phase separation into the randomly distributed regions of superconducting and normal phases. The following general conclusions are obtained.

(i) Mesoscopic phase separation in superconductors can be thermodynamically stable only in the presence of repulsive Coulomb interactions.

(ii) Phase separation enables the appearance of superconductivity in a heterophase sample even if it were impossible in pure-phase matter.

(iii) Phase separation is crucial for the occurrence of superconductivity in bad conductors.

(iv) The critical temperature for a mixture of gap waves is higher than the critical temperature related to any pure gap wave from this mixture.

(v) In bad conductors, the critical temperature as a function of the superconducting fraction has the bell shape.

(vi) Phase separation softens the single-particle energy dispersion.

(vii) Mesoscopic phase separation suppresses the contribution of d -wave superconductivity and enhances that of s -wave superconductivity.

These conclusions are in good qualitative agreement with experiments for high-temperature superconductors. Since in colossal magnetoresistance materials there also occurs the phenomenon of mesoscopic phase separation,^{8-10,68,69} such materials may possess some of the features described in this paper. Moreover, the mesoscopic phase separation is a rather general phenomenon appearing in different kinds of condensed matter and it can be described in the frame of the general theory,^{5,70} which was used in this paper for studying the general properties of anisotropic phase-separated superconductors.

ACKNOWLEDGMENT

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APPENDIX: AVERAGING OVER PHASE CONFIGURATIONS

Let the considered sample occupy in the real space $\{\mathbf{r}\}$ a region V having the volume $V \equiv \int_V d\mathbf{r}$. Assume that meso-

copie phase separation occurs in the sample, so that it is filled by several thermodynamic phases, which are enumerated by an index ν . Say, $\nu=1$ corresponds to the superconducting phase, while $\nu=2$ to the normal phase. At each given instant of time, distinct phases are located at different spatial regions, which can also be labeled by the phase index ν . This means that the total sample volume V can be divided into subregions V_ν , filled by the related phases. A family $\{V_\nu\}$ of subregions V_ν forms an orthogonal covering of V . This covering can be characterized by a family

$$\xi \equiv \{\xi_\nu(\mathbf{r}) | \mathbf{r} \in V\}$$

of the manifold indicator functions

$$\xi_\nu(\mathbf{r}) \equiv \begin{cases} 1, & \mathbf{r} \in V_\nu, \\ 0, & \mathbf{r} \notin V_\nu. \end{cases}$$

The family ξ uniquely defines a phase configuration in the real-space volume V .

From the physical point of view, distinct thermodynamic phases possess different properties, because of which such phases can be distinguished from each other. For example, the phases can be distinguished by their order parameters. In the case of superconductor, a convenient order parameter can be chosen as the anomalous average,

$$\eta_\nu(\mathbf{r}) \equiv \langle \psi_\nu(\mathbf{r}) \psi_\nu(\mathbf{r}) \rangle \xi_\nu(\mathbf{r}),$$

where $\psi_\nu(\mathbf{r})$ is a field operator of the field of carriers, when the latter are in the phase ν , and $\xi_\nu(\mathbf{r})$ is the related indicator function. For the superconducting regions, one has $\eta_1(\mathbf{r}) \neq 0$, provided that $\mathbf{r} \in V_1$, while for the normal parts, one has $\eta_2(\mathbf{r}) \equiv 0$, with $\mathbf{r} \in V_2$. The anomalous average, as is known, is directly related to the gap in the spectrum of excitations. Thus, different phases could be distinguished by the existence or absence of the gap in the spectrum. For a while, when dealing with a nonuniform sample consisting of separate phases, the anomalous average, depending on the spatial variable \mathbf{r} , is a more convenient order parameter.

Mesoscopic phase separation occurs in the real space in a random way, which means that the locations and shapes of the phase subregions V_ν are random. Then observable quantities should be determined with averaging over these random phase configurations. Since each phase configuration is uniquely defined by the set ξ of the manifold indicator functions, it is necessary to describe an ensemble $\{\xi\}$ of all possible sets ξ , corresponding to all possible phase configurations. For this purpose, we introduce an orthogonal subcovering $\{V_{\nu i}\}$ of each region V_ν , such that

$$V_\nu = \cup_{i=1}^{n_\nu} V_{\nu i}, \quad V_{\mu i} \cap V_{\nu j} = \delta_{\mu\nu} \delta_{ij} V_{\nu i}.$$

To each subregion $V_{\nu i}$, we ascribe a vector $\mathbf{a}_{\nu i} \in V_{\nu i}$, called the center, playing the role of a local center of coordinates, so that moving $\mathbf{a}_{\nu i}$ implies a congruent motion of $V_{\nu i}$. The introduced subcovering is uniquely characterized by a family of the indicator functions

$$\xi_{\nu i}(\mathbf{r} - \mathbf{a}_{\nu i}) \equiv \begin{cases} 1, & \mathbf{r} \in V_{\nu i}, \\ 0, & \mathbf{r} \notin V_{\nu i}, \end{cases}$$

with the property

$$\sum_{i=1}^{n_\nu} \xi_{\nu i}(\mathbf{r} - \mathbf{a}_{\nu i}) = \xi_\nu(\mathbf{r}).$$

By moving the centers $\mathbf{a}_{\nu i}$ and changing the measure of V_ν , it is possible to construct various phase configurations. To explicitly realize the averaging over these configurations, we define the differential functional measure

$$D\xi \equiv \lim_{\{n_\nu \rightarrow \infty\}} \prod_\nu \prod_{i=1}^{n_\nu} \frac{d\mathbf{a}_{\nu i}}{V} \delta\left(\sum_\nu x_\nu - 1\right) \prod_\nu dx_\nu,$$

in which

$$x_\nu \equiv \frac{1}{V} \int_V \xi_\nu(\mathbf{r}) d\mathbf{r}.$$

This measure, with varying $\mathbf{a}_{\nu i} \in V$ and $x_\nu \in [0, 1]$, induces a topology on the manifold $\{\xi\}$, which results in the topological configuration space $\mathcal{X} \equiv \{\xi | \mathcal{D}\xi\}$, composed of all admissible phase configurations.

For each fixed phase configuration, the observable quantities, represented by Hermitian operators, depend on the given configuration and have the structure

$$A(\xi) = \oplus_\nu A_\nu(\xi_\nu),$$

being defined on the fiber space $\mathcal{Y} = \otimes_\nu \mathcal{H}_\nu$, whose fibering yields the weighted Hilbert spaces \mathcal{H}_ν . The dependence of the operators of observable quantities on the indicator functions, marking the space filled by the corresponding phases, naturally comes from the identity

$$\int_{V_\nu} d\mathbf{r} = \int_V \xi_\nu(\mathbf{r}) d\mathbf{r}.$$

This identity is employed when representing the operators through the integrals of their operator densities:

$$\hat{A}(\xi) = \int \hat{A}(\xi, \mathbf{r}) d\mathbf{r} = \oplus_\nu \hat{A}_\nu(\xi_\nu),$$

$$\hat{A}(\xi, \mathbf{r}) = \oplus_\nu \hat{A}_\nu(\xi_\nu, \mathbf{r}), \quad \hat{A}_\nu(\xi_\nu) = \int \hat{A}_\nu(\xi_\nu, \mathbf{r}) d\mathbf{r}.$$

Here and everywhere in the paper, we denote, for simplicity, the integration over the whole sample as

$$\int d\mathbf{r} \equiv \int_V d\mathbf{r}.$$

For example, let us write down the Hamiltonian density

$$\begin{aligned} \hat{H}_\nu(\xi_\nu, \mathbf{r}) &= \xi_\nu(\mathbf{r}) \psi_\nu^\dagger(\mathbf{r}) \hat{K}_\nu(\mathbf{r}) \psi_\nu(\mathbf{r}) \\ &+ \frac{1}{2} \int \xi_\nu(\mathbf{r}) \xi_\nu(\mathbf{r}') \psi_\nu^\dagger(\mathbf{r}) \psi_\nu^\dagger(\mathbf{r}') \hat{V}_\nu(\mathbf{r}, \mathbf{r}') \psi_\nu(\mathbf{r}') \\ &\times \psi_\nu(\mathbf{r}) d\mathbf{r} d\mathbf{r}', \end{aligned}$$

in which $\hat{K}_\nu(\mathbf{r})$ is an operator of kinetic energy and $\hat{V}_\nu(\mathbf{r}, \mathbf{r}')$ is an effective interaction. Analogously, the number-of-particle operator density is

$$\hat{N}_\nu(\xi, \mathbf{r}) = \xi_\nu(\mathbf{r}) \psi_\nu^\dagger(\mathbf{r}) \psi_\nu(\mathbf{r}).$$

The field operators here are assumed to be columns with respect to spin indices.

A nonuniform system, composed of several thermodynamic phases, must be described by the quasiequilibrium (or locally equilibrium) Gibbs ensemble, with a statistical operator proportional to $e^{-\hat{X}(\xi)}$, where

$$\hat{X}(\xi) \equiv \int \beta(\xi, \mathbf{r}) [\hat{H}(\xi, \mathbf{r}) - \mu(\xi, \mathbf{r}) \hat{N}(\xi, \mathbf{r})] d\mathbf{r}.$$

Here the local inverse temperature $\beta(\xi, \mathbf{r})$ and the local chemical potential $\mu(\xi, \mathbf{r})$ model the system nonuniformity corresponding to a given phase configuration characterized by a set ξ . The natural thermodynamic potential for a quasiequilibrium system, with random ξ , is

$$Q = -\ln \text{Tr}_\gamma \int e^{-\hat{X}(\xi)} D\xi.$$

Assuming, as usual, the existence of the thermodynamic limit and accomplishing the averaging over phase configurations, which is characterized by the differential measure $D\xi$, it is possible to prove⁵ that the thermodynamic potential Q reduces to the form

$$Q = -\ln \text{Tr}_\gamma e^{-\beta \tilde{H}},$$

with an effective renormalized Hamiltonian $\tilde{H} = \oplus_\nu H_\nu$, in which

$$H_\nu = \int [\hat{H}_\nu(w_\nu, \mathbf{r}) - \mu \hat{N}_\nu(w_\nu, \mathbf{r})] d\mathbf{r},$$

and where the average inverse temperature and average chemical potentials, respectively, are

$$\beta = \int \beta(\xi, \mathbf{r}) D\xi, \quad \mu = \int \mu(\xi, \mathbf{r}) D\xi.$$

Then the potential

$$Q = \beta \Omega$$

is simply related to the Gibbs grand potential Ω . The phase probabilities w_ν are defined as the minimizers of either Q or Ω .

In this way, after averaging over heterophase configurations, we come to the description of the system by means of a renormalized Hamiltonian, containing only averaged quantities and not involving anymore random phase distributions that have been averaged out. All expressions throughout the paper correspond to averaged quantities resulting from the described procedure of heterophase averaging. A complete and detailed mathematical foundation for this averaging procedure is given in the reviews in Refs. 5 and 70.

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