

Normal-to-planar superfluid transition in ^3He

Martino De Prato,¹ Andrea Pelissetto,² and Ettore Vicari^{3,*}

¹*Dipartimento di Fisica, Università di Roma Tre, and INFN, I-00146 Roma, Italy*

²*Dipartimento di Fisica, Università di Roma La Sapienza, and INFN, I-00185 Roma, Italy*

³*Dipartimento di Fisica, Università di Pisa, and INFN, I-56127 Pisa, Italy*

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We study the nature of the ^3He superfluid transition from the normal to the planar phase, which is expected to be stabilized by the dipolar interactions. We determine the renormalization group flow of the corresponding Landau–Ginzburg–Wilson theory by exploiting two fixed-dimension perturbative schemes: the massive zero-momentum scheme and the minimal-subtraction scheme without ϵ expansion. The analysis of the corresponding six-loop and five-loop series shows the presence of a stable fixed point in the relevant coupling region. Therefore, we predict the transition to be continuous. We also compute critical exponents. The specific-heat exponent α is estimated as $\alpha=0.20(15)$, while the magnetic susceptibility and magnetization exponents γ_H and β_H for ^3He are $\gamma_H=-0.34(5)$, $\beta_H=1.07(9)$.

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I. INTRODUCTION AND SUMMARY

The superfluid phases in ^3He have been extensively studied.^{1–4} In the absence of magnetic field two superfluid phases, named A and B, respectively, have been identified. However, if one takes into account the weak dipolar interaction,^{1,5} a new phase, called planar phase, should separate the normal state from the A and B superfluid states.^{6,7} At present, there is no experimental evidence for a normal-to-planar transition. This is due to the fact that dipolar interactions are expected to become relevant only within the interval $|t| \leq 2 \times 10^{-6}$, where $t \equiv 1 - T/T_c$. For pressures between 10 and 30 bar, T_c varies between 2 and 2.6 mK, so that the interesting temperature interval is approximately $|T - T_c| \approx (4 - 5) \times 10^{-9}$ K. Outside this interval normal Fermi liquid or A/B superfluid behavior should be observed. Despite the narrowness of such temperature interval, Ref. 6 argued that it should be possible to observe the planar phase if the temperature T can be controlled to high precision, as it has been done recently for ^4He .⁸ It is therefore interesting to investigate its nature in detail.

The normal-to-superfluid transition in ^3He is supposed to be driven by p -wave spin-triplet pairing states. Near the transition the critical behavior is controlled by an order parameter that is a complex 3×3 matrix, see, e.g., Refs. 1 and 9–12. The weak dipolar interaction breaks the original $\text{SO}(3) \times \text{SO}(3) \times \text{U}(1)$ symmetry down to $\text{SO}(3) \times \text{U}(1)$ and the corresponding order parameter becomes a complex three-component field ψ . The Landau–Ginzburg–Wilson Hamiltonian describing the critical modes at the normal-to-planar transition can be written as^{6,7,13}

$$\mathcal{H} = \int d^d x \left[\partial_\mu \psi^* \cdot \partial_\mu \psi + r \psi^* \cdot \psi + \frac{1}{4} f_0 (\psi^* \cdot \psi)^2 + \frac{1}{4} g_0 |\psi \cdot \psi|^2 \right] \quad (1)$$

with $f_0 > 0$ and $g_0 < 0$. We have neglected a further anisotropic gradient term that may arise in the case of anisotropic

superfluids,^{1,14} and that is expected to be irrelevant at the transition,⁶ although it may lead to slowly decaying scaling corrections, see also Ref. 15. Approximate estimates^{3,6,16–18} give $g_0/f_0 \approx -1/3$. Note that stability requires $f_0 \geq 0$ and $f_0 + g_0 \geq 0$. When $g_0 < 0$ the Hamiltonian is minimized by fields $\psi_a = e^{i\theta} n_a$, where n_a is a constant real vector, and the corresponding ground-state manifold is $(S^2 \times S^1)/\mathbb{Z}_2$, where S^n is the n -dimensional sphere. The case $g_0 > 0$ with a two-component and a three-component field ψ is also physically interesting because it describes the critical properties of frustrated spin models with noncollinear order.^{19,20} The same Hamiltonian has also been considered to discuss the critical properties of Mott insulators.²¹ It should also be relevant for two-dimensional quantum phase transitions in cuprate superconductors²² and for transitions in heavy-fermion compounds such as UPt_3 .²³

In the mean-field approximation the model with Hamiltonian (1) has a continuous second-order transition. On the other hand, ϵ -expansion calculations for an N -component field ψ find^{6,7,19,24} a stable fixed point (FP) in the region $g_0 < 0$ only for $2 - \epsilon \leq N \leq 2.20 - 0.57\epsilon$, $\epsilon \equiv 4 - d$. Apparently no FP is found for $N=3$, suggesting a fluctuation-induced first-order transition. This scenario has been further supported by three-loop calculations within a three-dimensional (3D) massive zero-momentum (MZM) scheme²⁵ and by a nonperturbative renormalization-group (RG) study of the effective average action in the lowest-order approximation of the derivative expansion.²⁶ As we shall argue below mentioning a few specific physical examples, these results may not be conclusive. For instance, in some physically interesting cases, low-order, and in some cases also high-order, ϵ -expansion calculations fail to provide the correct physical picture. The location and the stability of the FPs may drastically change approaching $d=3$, and new FPs, not present for $d \approx 4$, may appear in three dimensions. We mention the Ginzburg–Landau model of superconductors, in which a complex scalar field couples to a gauge field. One-loop ϵ -expansion calculations²⁷ indicate that no stable FP exists unless the number N of real components of the scalar field is

larger than $N_c = 365 + O(\epsilon)$. This number is much larger than the physical value $N=2$. Consequently, a first-order transition was always expected.²⁷ Later, exploiting 3D theoretical approaches (see, e.g., Ref. 28) and Monte Carlo simulations (see, e.g., Ref. 29), it was realized that 3D systems described by the Ginzburg–Landau model can also undergo a continuous transition—this implies the presence of a stable FP in the 3D Ginzburg–Landau theory—in agreement with experiments.³⁰ A similar phenomenon probably occurs—but this is still a controversial issue, see, e.g., Refs. 20, 31, and 32—in model (1) when $g_0 > 0$. ϵ -expansion calculations find a stable FP with attraction domain in the region $g_0 > 0$ only for $N > N_c$ with^{19,24,33,34} $N_c = 21.80 + O(\epsilon)$, which seems to exclude the physically interesting cases $N=2, 3$. These conclusions were apparently confirmed by three-loop 3D perturbative calculations in the MZM scheme²⁵ and by a nonperturbative RG study of the effective average action,³⁵ using approximations based on the lowest orders of the derivative expansion. On the other hand, recent studies based on 3D high-order field-theoretical calculations found a stable FP with attraction domain in the region $g_0 > 0$ for both $N=2$ and $N=3$. Two different renormalization schemes were used: the MZM scheme (six loops)^{36,37} and the minimal-subtraction (MS) scheme without ϵ expansion (five loops).³⁸ These results support the existence of new chiral universality classes.

In this paper we investigate the nature of the normal-to-planar transition in ^3He by exploiting two different 3D perturbative field-theoretical approaches based on the Hamiltonian (1). The first one, the MZM scheme, is defined in the massive (disordered) phase.³⁹ It is based on a zero-momentum renormalization procedure and an expansion in powers of zero-momentum quartic couplings. As already mentioned, this perturbative scheme was considered in Ref. 25, where three-loop series were analyzed without finding evidence for stable FPs. The second scheme is the so-called MS scheme without ϵ expansion,⁴⁰ which is defined in the massless critical theory. This scheme is strictly related to the ϵ expansion, but, unlike it, no expansion in powers of ϵ is performed. Indeed, ϵ is set to its physical value $\epsilon=1$, and one analyzes the expansions in powers of the renormalized quartic couplings.

Here we shall present an analysis of the six-loop MZM series that have been computed in Ref. 36 and of the five-loop MS series that have been computed in Ref. 34. We find a stable FP in the region $g < 0$ whose attraction domain includes the region where $g_0/f_0 \approx -1/3$, which should be the relevant one for the transition in ^3He . The two perturbative schemes give consistent results, providing a nontrivial cross-check of the results. We thus predict a continuous normal-to-planar transition, belonging to a new universality class. These conclusions are at variance with earlier theoretical studies, see, e.g., Refs. 6, 25, and 26.

We also determine the standard critical exponents for model (1). We obtain $\nu=0.59(4)$, $\eta=0.079(7)$, $\gamma=1.14(8)$, $\beta=0.32(3)$ in the MZM scheme and $\nu=0.63(8)$, $\eta=0.086(24)$, $\gamma=1.20(15)$, $\beta=0.34(4)$ in the MS scheme. A weighted average gives

$$\nu = 0.60(5), \quad \eta = 0.08(1), \quad \gamma = 1.16(10), \quad \beta = 0.33(4). \quad (2)$$

The errors are such to include the MZM results (with their errors) that apparently are the most precise ones. By using hyperscaling we also obtain $\alpha=0.20(15)$. Note that, while ν and α are indeed the critical exponents associated with the specific heat and with the correlation length, γ , η , and β are not directly accessible in ^3He and are *not* related with the magnetic critical behavior of ^3He . Indeed, the operator that couples with the external magnetic field in ^3He is the so-called chiral operator C_a ,^{19,20,41} defined in Eq. (12). We compute the exponents γ_H and β_H that characterize the singular behavior of the magnetic susceptibility and of the magnetization, and η_H , related to the large-momentum behavior of the magnetic structure factor (in field-theoretical terms it is defined in terms of the Fourier transform of $\langle C_a(0)C_a(x) \rangle$ that behaves as $q^{-2+\eta_H}$ for large momenta q). We find $\eta_H=2.72(7)$, $\gamma_H=-0.34(5)$, $\beta_H=1.05(7)$ in the MZM scheme and $\eta_H=2.63(19)$, $\gamma_H=-0.34(11)$, $\beta_H=1.11(12)$ in the minimal-subtraction scheme. A weighted average gives

$$\eta_H = 2.70(9), \quad \gamma_H = -0.34(5), \quad \beta_H = 1.07(9), \quad (3)$$

where the errors have been computed as before. Note that γ_H is negative so that in the high-temperature phase the analytic background gives the dominant contribution to the magnetic susceptibility χ . Close to T_c , we have therefore $\chi \approx \text{const} + (T_c - T)^{0.34}$. In the low-temperature phase instead χ is infinite due to the presence of Goldstone modes.

The paper is organized as follows. In Sec. II we present the analysis of the six-loop series in the MZM scheme. It provides a rather robust evidence for the presence of a stable FP with attraction domain in the region $g_0 < 0$, which is the relevant one for the ^3He superfluid transition. This result is fully confirmed by the analysis of the five-loop series in the massless MS scheme that is presented in Sec. III. We report the perturbative expansions analyzed here in the MZM and MS schemes in Appendices A and B, respectively.

II. MASSIVE ZERO-MOMENTUM SCHEME

In this section we consider the fixed-dimension massive zero-momentum (MZM) scheme that describes the disordered massive phase. The relevant RG functions were computed to six loops in Refs. 36 and 41. In Appendix A we report the series for the relevant case $N=3$.

The perturbative expansions are asymptotic. Therefore, perturbative series must be resummed in order to study the RG flow. Consider a generic quantity $S(f, g) = \sum_{ij} c_{ij} f^i g^j$. In order to determine $S(f, g)$, one may resum the series

$$S(xf, xg) = \sum_k s_k(f, g) x^k \quad (4)$$

in powers of x and then evaluate it at $x=1$. This can be done by using the conformal-mapping method^{42,43}, which exploits the knowledge of the large-order behavior of the expansion, or the so-called Padé–Borel method. The large-order behavior of the coefficients is generally given by

$$s_k(f, g) \sim k![-A(f, g)]^k k^b [1 + O(k^{-1})]. \quad (5)$$

Using semiclassical arguments, one can argue that³⁶ the expansion is Borel summable for

$$f \geq 0, \quad f + g \geq 0. \quad (6)$$

In this domain we have

$$A(f, g) = a \text{Max}[f, f + g] > 0, \quad (7)$$

where $a \approx 0.147\,774\,22$. Under the additional assumption that all the singularities of the Borel transform lie on the negative axis, the conformal-mapping method turns the original expansion into a convergent one⁴² when conditions (6) are satisfied. Outside this region the expansion is not Borel summable. However, if the condition

$$f + \frac{1}{2}g > 0 \quad (8)$$

holds, then the singularity of the Borel transform that is closest to the origin is still in the negative axis. Therefore, the leading large-order behavior is still given by Eq. (5) with $A(f, g)$ given by Eq. (7). In this case, the conformal-mapping method is still able to take into account the leading large-order behavior, although it does not provide a convergent series. Therefore, one may hope to get an asymptotic expansion with a milder behavior, which still provides reliable results.

The RG flow of the theory is determined by the fixed points (FPs). Two FPs are easily identified: the Gaussian FP ($f^* = g^* = 0$) and the O(6) FP at⁴⁴ $f^* \approx 0.86$, $g^* = 0$. The results of Ref. 45 on the stability of the three-dimensional O(M)-symmetric FPs under generic perturbations can be used to prove that the O(6) FP is unstable. Indeed, the perturbation term $|\psi|^2$ in Hamiltonian (1) is a particular combination of quartic operators transforming as the spin-0 and spin-4 representations of the O(6) group, and any spin-4 quartic perturbation is relevant at the O(M) FP for $M \geq 3$ (Ref. 45), since its RG dimension $y_{4,4}$ is positive for $M \geq 3$. In particular, $y_{4,4} \approx 0.27$ at the O(6) FP.⁴⁵

The analyses of the six-loop series reported in Refs. 36 and 37 provided a rather robust evidence of the presence of a stable FP with $g^* > 0$ and attraction domain in the region $g_0 > 0$, at $f^* = -0.4(2)$ and $g^* = 1.5(1)$. As already mentioned in the introduction, this FP should describe continuous transitions in frustrated three-component spin models with non-collinear order. However, it is of no relevance for the critical behavior of ^3He . Indeed the $g=0$ axis plays the role of a separatrix, and thus the RG flow corresponding to $g_0 < 0$ cannot cross the $g=0$ axis. Therefore, the RG flow relevant for the normal-to-planar transition is determined by FPs in the region $g < 0$. We shall now show that the RG flow obtained by resumming the six-loop β functions provides a rather robust evidence for another stable FP in the region $g < 0$ with attraction domain including the region $g_0/f_0 \approx -1/3$, which should be the relevant one for the ^3He superfluid transition.

In order to investigate the RG flow in the region $g_0 < 0$, we apply the same analysis of Refs. 36 and 46. We refer to these references for details. We resum the perturbative series

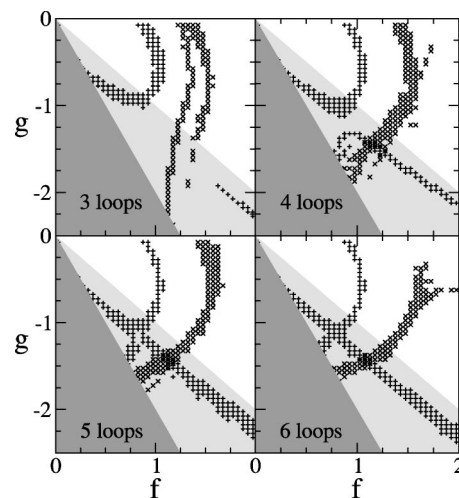


FIG. 1. Zeros of the β functions in the f - g plane in the MZM scheme. Pluses and crosses correspond to zeroes of β_f and β_g , respectively. The FP coordinates are: $f^* = 1.08(12)$, $g^* = -1.46(10)$. Different colors are used to mark the regions: (1) $f+g > 0$, (2) $f+g < 0$ and $f+\frac{1}{2}g > 0$, (3) $f+\frac{1}{2}g < 0$.

by means of the conformal-mapping method⁴² that takes into account the large-order behavior of the perturbative series. To determine the systematic errors, we vary two different parameters, b and α (see Refs. 36 and 46 for definitions), in the analysis. We also apply this method for those values of f and g for which the series are not Borel summable but still satisfy $f+g/2 > 0$. As already discussed, although in this case the sequence of approximations given by the conformal-mapping method is only asymptotic, it should provide reasonable estimates, since we are taking into account the leading large-order behavior. In Fig. 1 we report our results for the zeroes of the β functions, obtained from the analysis of the ℓ -loop series, $\ell=3,4,5,6$. For each β function we consider 18 different approximants of $\beta(f, g)/f$ with $b=3,6, \dots, 18$ and $\alpha=0,2,4$ and we determine the lines in the (f, g) plane on which they vanish. Then, we divide the domain $0 \leq f \leq 2$ and $-2.5 \leq g \leq 0$ into 40×50 rectangles, marking those in which at least *three* approximants of each β function vanish. No evidence for a FP appears at three loops, consistently with Ref. 25. As the number of ℓ loops increases, a new stable FP—quite stable with respect to ℓ —clearly appears. Such zeroes appear in 9%, 22%, 59%, and 89% of the cases we consider for $\ell=3,4,5,6$. Clearly, the new zero is increasingly stable as ℓ increases. At six loops we obtain a robust evidence for a FP at

$$f^* = 1.08(12), \quad g^* = -1.46(10), \quad (9)$$

where the error bars correspond to two standard deviations. All zeros of the approximants with $3 \leq b \leq 18$ and $2 \leq \alpha \leq 4$ lie within half of the reported confidence interval and only a few with $\alpha=0$ give significantly different estimates for f^* and g^* . Moreover, for this smaller class of approximants a stable FP appears in 100% (73%) of the cases for $\ell=6$ ($\ell=5$). Estimate (9) is stable with respect to the number of loops. For $\ell=5$ we obtain $f^* = 1.12(18)$, $g^* = -1.47(18)$. Notice that the FP belongs to the region in which the series

are not Borel summable, but the FP values still satisfy $f^* + g^*/2 > 0$. Therefore, our resummations should be reliable: the stability of the results with respect to ℓ supports this expectation. The FP (9) is stable, since the eigenvalues of the corresponding stability matrix have positive real part. Our estimate is $\omega_{\pm} = 1.03(29+15) \pm i0.72(11+10)$, where the first error takes into account the variation of the estimate with α and b , while the second one takes into account the error on f^* and g^* . Note that the eigenvalues are apparently complex, suggesting that the FP is actually a focus, as it happens for the chiral FP.³⁷ Using the RG functions η_{ψ} and η_t , we have generated perturbative series with FP value corresponding to ν , η , γ , and β . Correspondingly, we obtain the following estimates of the critical exponents:

$$\begin{aligned} \nu &= 0.59(3+1), & \eta &= 0.079(2+5), & \gamma &= 1.14(6+2), \\ & & & & & (10) \\ \beta &= 0.32(2+1), \end{aligned}$$

where, as before, the first error gives the dependence on α and b and the second is related to the uncertainty on the FP coordinates. Consistent results are obtained from the analysis of the perturbative series for $1/\nu$, η/f^2 , f^2/η , $1/\gamma$, and $1/\beta$. The RG relations among the different critical exponents are also well verified.

The exponents reported in Eq. (10) follow the usual nomenclature of magnetic systems. However, while ν is indeed the correlation-length exponent in ${}^3\text{He}$, η , γ , and β are not accessible in ${}^3\text{He}$ and, in particular, are not related to the magnetic properties of the superfluid. Indeed, in ${}^3\text{He}$ an external magnetic field \mathbf{H} gives rise to a coupling of the form⁴⁷

$$i \sum_{abc} \epsilon_{abc} H_a \psi_b^* \psi_c. \quad (11)$$

Therefore, the magnetic exponents are obtained by computing the RG dimension of the operator

$$C_a = i \epsilon_{abc} \psi_b^* \psi_c. \quad (12)$$

Such an operator was discussed at length in Refs. 19, 20, and 41 in the context of the frustrated chiral models and its associated RG dimension was computed at the chiral FP with $g > 0$. In the present case the analysis of the six-loop series of Ref. 41 at the FP [Eq. (9)] gives

$$\eta_H = 2.72(3+4), \quad \gamma_H = -0.34(2+3), \quad \beta_H = 1.05(4+3), \quad (13)$$

where the two errors are related to the dependence on α and b and to the uncertainty on the FP coordinates. We have also resummed the series corresponding to $1/\eta_H$, $1/\beta_H$, and $1/\gamma_H$. In the first two cases, the results are in perfect agreement with the estimates [reported in Eq. (13)]; only for γ_H do we find a significant difference. Note, however, that scaling relations give $\gamma_H = \nu(2 - \eta_H) = -0.42(5)$ and $\gamma_H = 3\nu - 2\beta_H = -0.33(18)$, and thus confirm the direct estimate (13).

In Fig. 2 we show the RG flow in the (f, g) plane for several values of the Hamiltonian parameters f_0 and g_0 and for a given approximant of the two β functions (other approximants give similar results). We consider values in the stability region $f_0 > 0$ and $f_0 + g_0 > 0$ and restrict the calcula-

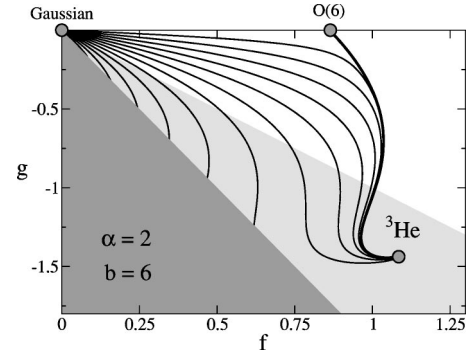


FIG. 2. RG flow in the f - g plane in the MZM scheme. It is obtained by using the approximant ($b=6$, $\alpha=2$) for $\beta_f(f, g)/f$ and $\beta_g(f, g)/f$. The thick line is related to the limiting RG trajectory obtained for $g_0 \rightarrow 0^-$. The other trajectories reaching the FP correspond to (from top to bottom) $g_0/f_0 = -0.085$, -0.172 , -0.261 , -0.354 , -0.454 , -0.561 . Different colors are used to mark the regions: (1) $f+g > 0$, (2) $f+g < 0$ and $f+\frac{1}{2}g > 0$, (3) $f+\frac{1}{2}g < 0$.

tion to the region of interest $g_0 < 0$. We refer to Ref. 48 for details on the determination of the RG trajectories and their relation with the quartic parameters f_0 and g_0 of the Hamiltonian, which are implicitly given by Eqs. (A1) and (A2). In this respect we note that nothing prevents RG trajectories corresponding to f_0 and g_0 in the stability region from flowing toward regions where the renormalized couplings f and g do not satisfy $f > 0$ and $f+g > 0$, and, in particular, toward the fixed point $f^* = 1.08(12)$, $g^* = -1.46(10)$. The conditions $f \geq 0$ and $f+g \geq 0$ are relevant only for the Borel summability of the expansion, cf. Eq. (7). All trajectories corresponding to $-0.65 \leq g_0/f_0 < 0$ lie completely in the region $f+\frac{1}{2}g > 0$ in which resummations should be reliable and are attracted by the stable FP. Therefore, the attraction domain of the FP (9) safely includes the region $g_0/f_0 \approx -1/3$, which should be the relevant one for the normal-to-planar transition in ${}^3\text{He}$. In Fig. 2 we also report (thick line) the RG trajectory that starts at the O(6) unstable FP and that can be obtained as the limit $g_0 \rightarrow 0^-$ of the trajectories starting at the Gaussian FP. Indeed, in this limit the flow first runs close to the $g=0$ axis and then follows the thick line. This special trajectory is relevant for the analysis of the crossover behavior between the O(6) FP and the stable FP with $g < 0$; see, e.g., Ref. 48.

It is interesting to note that the estimates of the critical exponents [Eq. (10)] are close to those associated with the chiral FP at $f^* = -0.3(1)$, $g^* = 1.5(1)$, i.e.,³⁶ $\nu = 0.55(3)$ and $\eta = 0.10(1)$. The reason is that the functions $\beta_{f,g}(f, g)$ and $\eta_{\psi,t}(f, g)$ have the approximate symmetry

$$g \rightarrow \hat{g} = -g, \quad f \rightarrow \hat{f} = f + g, \quad (14)$$

i.e., satisfy $\eta_{\psi,t}(f, g) \approx \eta_{\psi,t}(f+g, -g)$, $\beta_f(f, g) \approx \beta_f(f+g, -g) + \beta_g(f+g, -g)$, and $\beta_g(f, g) \approx -\beta_g(f+g, -g)$. Perturbatively, the violations of these relations are much smaller than the RG functions themselves and appear at three loops in the β functions and at five loops in $\eta_{\psi,t}$. Moreover, this symmetry is satisfied by the constant $A(f, g)$, cf. Eq. (7), that controls the large-order behavior. This means that the presence of the chiral FP³⁶ $f_{\text{ch}}^* \approx -0.3$, $g_{\text{ch}}^* \approx 1.5$, implies the presence of an-

other FP for $f^* \approx f_{\text{ch}}^* + g_{\text{ch}}^* \approx 1.2$ and $g^* \approx -g_{\text{ch}}^* \approx -1.5$, in substantial agreement with Eq. (9), with approximately the same exponents [Eq. (10)]. Note that the approximate symmetry (14) does not hold for $\eta_H(f, g)$ and thus the exponents η_H , β_H , and γ_H differ significantly from the corresponding exponents at the chiral FP computed in Ref. 41.

In conclusion, the analysis of six-loop series in the MZM scheme provides a rather robust evidence for a stable FP with attraction domain in the region $g_0 < 0$, and therefore for the existence of a universality class associated with the normal-to-planar superfluid transition in ^3He . This result is at variance with earlier perturbative studies based on ϵ expansion²⁴ and lower order (three-loop) calculations within the same scheme,²⁵ and also a nonperturbative study of the RG flow of the effective average action in the lowest order of the derivative expansion.²⁶

III. MASSLESS MINIMAL-SUBTRACTION SCHEME

In this perturbative approach one considers the massless critical theory in dimensional regularization and renormalizes it in the minimal-subtraction (MS) scheme.⁴⁹ In the standard ϵ -expansion scheme⁵⁰ the FPs, i.e., the common zeroes of the β functions, are determined perturbatively by expanding in powers of ϵ . Once the expansion of the FP values f^* and g^* is available, exponents are obtained by expanding the RG functions at the FP in powers of ϵ . The MS scheme without ϵ expansion⁴⁰ is strictly related. The RG functions $\beta_{f,g}$ and $\eta_{\psi,t}$ are the MS functions. However, ϵ is no longer considered as a small quantity but it is set to its physical value, i.e., in three dimensions one simply sets $\epsilon=1$. Then, the FP values f^* , g^* are determined from the common zeroes of the resummed β functions. Critical exponents are determined by evaluating the resummed RG functions η_ψ and η_t at f^* and g^* . Notice that f^* and g^* have nothing to do with the FP values obtained within the MZM scheme, since f and g indicate different quantities in the two schemes.

The β functions and the RG functions associated with the standard exponents have been computed to five loops in Ref. 34 for generic values of N . Moreover, we computed the five-loop series of the RG dimension of the operator (12) associated with the magnetic field in ^3He . In Appendix B we report the series for the relevant case $N=3$.

The series are resummed by using the conformal-mapping method. Semiclassical arguments allow us to compute the large-order behavior of the perturbative series. We obtain the same formulas reported in Sec. II, cf. Eqs. (5) and (7), with the only difference that $a=1/2$. We should mention that in this approach the series are essentially four dimensional. Thus, they may be affected by renormalons which make the expansion non-Borel summable for any f and g and are not detected by a semiclassical analysis, see, e.g., Ref. 51. This problem should also affect the MS series of the $O(N)$ -symmetric theories. However, the good agreement between the results of their analysis assuming Borel summability⁴⁰ and the estimates obtained by other methods indicates that renormalon effects are either very small or absent (note that, as shown in Ref. 52, this may occur in some renormalization schemes). For example, the analysis⁴⁰ of the

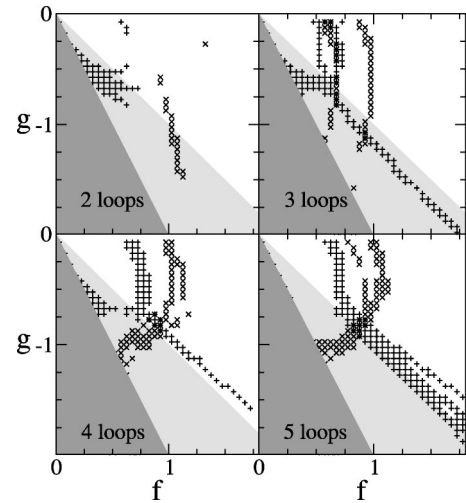


FIG. 3. Zeros of the β functions in the f - g plane in the $\overline{\text{MS}}$ scheme. Pluses and crosses correspond to zeroes of β_f and β_g , respectively. The FP coordinates are: $f^*=0.89(16)$, $g^*=-0.90(24)$. Different colors are used to mark the regions: (1) $f+g > 0$, (2) $f+g < 0$ and $f+\frac{1}{2}g > 0$, (3) $f+\frac{1}{2}g < 0$.

five-loop $\overline{\text{MS}}$ series (using the conformal-mapping method and the semiclassical large-order behavior) gives $\nu=0.629(5)$ for the Ising model and $\nu=0.667(5)$ for the XY model, which are in good agreement with the most precise estimates obtained by lattice techniques, $\nu=0.630\,12(16)$ (Ref. 53) and $\nu=0.630\,20(12)$ (Ref. 54) for the Ising model, and $\nu=0.671\,55(27)$ (Ref. 55) for the XY universality class. On the basis of these results, we will assume renormalon effects to be negligible in the analysis of the two-variable series of the theory at hand.

As in the MZM scheme, two unstable FPs can be easily identified, the Gaussian FP and the $O(6)$ FP along the f axis. A stable FP is found in the region $g > 0$, Ref. 38, at $f^*=0.0(1)$ and $g^*=0.9(1)$, with critical exponents $\nu=0.61(6)$ and $\eta=0.09(1)$, in reasonable agreement with the results of the MZM scheme.

Also in this case there is an approximate symmetry $g \rightarrow \hat{g} = -g$ and $f \rightarrow \hat{f} = f+g$, which is violated at three loops in the β functions and at five loops in the standard critical exponents, and which is satisfied by the large-order behavior. Thus, we expect a stable FP in the region $g < 0$ at $f^* \approx 0.9$, $g^* \approx -0.9$, with exponents approximately equal to those of the chiral FP at $g > 0$.

In order to find the zeroes of the β functions, we first resummed the series for the functions $\beta_f(f, g)/f$ and $\beta_g(f, g)/f$. For each β function we considered several approximants corresponding to α and b in the range $0 \leq \alpha \leq 4$ and $3 \leq b \leq 18$, as in the MZM case. In Fig. 3 we show the zeroes of the β functions in the region $g < 0$. A common zero is clearly observed at $f^* \approx 0.9$ and $g^* \approx -0.9$. In order to give an estimate of the FP, we considered all independent combinations for the two β functions. Most combinations, approximately 92% and 49% for $\ell=5$ and 4, respectively, have a common zero in the region $g < -0.3$, leading to the estimate

$$f^* = 0.89(16), \quad g^* = -0.90(24), \quad (15)$$

where we have again reported two standard deviations as error. As before, essentially all zeroes lie in half of the quoted interval—only a few corresponding to approximants with $\alpha=0$ differ significantly. Estimates (15) are also stable with respect to the number of loops and in good agreement with those obtained by using the approximate symmetry. We also performed a different analysis, in which we resummed $B_f(f,g)/f^2$ and $B_g(f,g)/f^2$, cf. Eq. (B4), obtaining consistent results. Notice that in this scheme the stable FP is at the boundary of the region in which perturbative expansions are Borel summable, since $f^* + g^* \approx 0$. This gives us further confidence on the reliability of the final results.

For the stability eigenvalues, we find that in approximately 33% of the cases the eigenvalues turn out to be real, while in 67% of the cases they are complex. If we average the complex ones we obtain $\omega_{\pm} = 1.2(4+4) \pm i0.8(2+4)$; real eigenvalues give instead $\omega_1 = 0.9(8+7)$ and $\omega_2 = 5(5+1)$. Note that the real results fluctuate wildly, indicating that they are probably not reliable. On the other hand, the complex estimate is more stable and is in good agreement with the MZM result.

Critical exponents are obtained as before. For the standard exponents we find

$$\begin{aligned} \nu &= 0.63(3+5), \quad \eta = 0.086(13+11), \\ \gamma &= 1.20(5+10), \quad \beta = 0.34(2+2), \end{aligned} \quad (16)$$

while for the magnetic (chiral) exponents we find

$$\eta_H = 2.63(5+14), \quad \gamma_H = -0.34(4+7), \quad \beta_H = 1.11(4+8). \quad (17)$$

The error is given as a sum of two terms, related, respectively, to the variation with α and b (we used $0 \leq \alpha \leq 4$ and $3 \leq b \leq 18$) and to the uncertainty of the FP coordinates. Estimates (16) and (17) are in good agreement with the MZM ones: in all cases the difference is significantly smaller than the quoted errors, which may be an indication that our error estimates are rather conservative.

In Fig. 4 we show the RG flow in the quartic-coupling f, g plane for several values of f_0 and g_0 in the region $f_0 > 0$, $f_0 + g_0 > 0$, and $g_0 < 0$. We report the results for a single approximant for both β functions. Others give qualitatively similar results. All trajectories corresponding to $-0.9 \lesssim g_0/f_0 < 0$ lie in the region $f + \frac{1}{2}g > 0$ and apparently flow towards the stable FP. Therefore, the attraction domain of the FP (15) safely includes the region $g_0/f_0 \approx -1/3$, which should be relevant for the normal-to-planar transition in ^3He .

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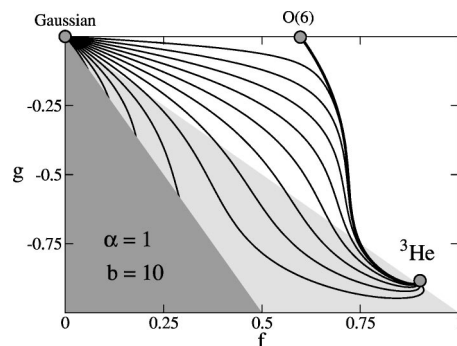


FIG. 4. RG flow in the f - g plane in the $\overline{\text{MS}}$ scheme. It is obtained by using the approximant ($b=10$, $\alpha=1$) for $\beta_f(f,g)/f$ and $\beta_g(f,g)/f$. The thick line represents the RG trajectory for $g_0 \rightarrow 0^-$. The other trajectories reaching the FP correspond to (from top to bottom) $g_0/f_0 = -0.085, -0.172, -0.261, -0.354, -0.454, -0.561, -0.679, -0.810$. Different colors are used to mark the regions: (1) $f+g > 0$, (2) $f+g < 0$ and $f + \frac{1}{2}g > 0$, (3) $f + \frac{1}{2}g < 0$.

APPENDIX A: SIX-LOOP SERIES OF MASSIVE ZERO-MOMENTUM SCHEME

The theory is renormalized by introducing a set of zero-momentum renormalization conditions involving the two- and four-point correlation functions, i.e.

$$\Gamma_{ai}^{(2)}(p) = \delta_{ai} Z_{\psi}^{-1} [m^2 + p^2 + O(p^4)], \quad (A1)$$

$$\Gamma_{abij}^{(4)}(0) = m Z_{\psi}^{-2} \frac{32\pi}{9} \left[f \frac{1}{2} (\delta_{ai} \delta_{bj} + \delta_{aj} \delta_{bi}) + g \delta_{ab} \delta_{ij} \right], \quad (A2)$$

where $\Gamma^{(2)}$ and $\Gamma^{(4)}$ are, respectively, the Fourier transform of two- and four-point one-particle irreducible correlation functions. In addition, one introduces the function $Z_i(f,g)$ that is defined by the relation $\Gamma_{ai}^{(1,2)}(0) = \delta_{ai} Z_i^{-1}$, where $\Gamma^{(1,2)}$ is the one-particle irreducible two-point function with an insertion of $\psi^* \cdot \psi$. The FPs of the theory are given by the common zeroes of the Callan–Symanzik β -functions

$$\beta_f(f,g) = m \frac{\partial f}{\partial m} \Big|_{f_0, g_0}, \quad \beta_g(f,g) = m \frac{\partial g}{\partial m} \Big|_{f_0, g_0}. \quad (A3)$$

The RG functions η_{ψ} and η_t associated with the critical exponents are defined by

$$\eta_{\psi,t}(f,g) = \frac{\partial \ln Z_{\psi,t}}{\partial \ln m} \Big|_{f_0, g_0}. \quad (A4)$$

Their relation with the critical exponents is given by

$$\eta = \eta_{\psi}(f^*, g^*), \quad \nu = [2 + \eta_t(f^*, g^*) - \eta_{\psi}(f^*, g^*)]^{-1}. \quad (A5)$$

The RG functions have been computed to six loops in Ref. 36. Here we report the series for $N=3$

$$\begin{aligned}
 \beta_f = & -f + \frac{1}{9}(14f^2 + 8fg + 8g^2) - \frac{16}{2187}(109f^3 + 100f^2g \\
 & + 177fg^2 + 90g^3) + (0.792\,654f^4 + 1.122\,06f^3g \\
 & + 2.632\,35f^2g^2 + 2.085\,29fg^3 + 0.127\,631g^4) \\
 & - (1.021\,68f^5 + 1.894\,78f^4g + 5.187\,72f^3g^2 \\
 & + 5.768\,31f^2g^3 + 2.119\,92fg^4 + 0.243\,881g^5) \\
 & + (1.566\,86f^6 + 3.646\,24f^5g + 11.4687f^4g^2 \\
 & + 16.3586f^3g^3 + 10.9169f^2g^4 + 3.753\,87fg^5 \\
 & + 0.480\,925g^6) - (2.762\,92f^7 + 7.702\,53f^6g \\
 & + 26.9939f^5g^2 + 46.7321f^4g^3 + 44.5897f^3g^4 \\
 & + 25.4728f^2g^5 + 7.243\,62fg^6 + 0.637\,996g^7), \quad (\text{A6})
 \end{aligned}$$

$$\begin{aligned}
 \beta_g = & -g + \frac{2}{3}g(2f + g) - \frac{16}{2187}g(127f^2 + 127fg + 6g^2) \\
 & + g(0.926\,499f^3 + 1.389\,75f^2g + 0.898\,551fg^2 \\
 & + 0.216\,645g^3) - g(1.318\,84f^4 + 2.637\,69f^3g \\
 & + 2.874\,66f^2g^2 + 1.556\,75fg^3 + 0.179\,409g^4) \\
 & + g(2.108\,66f^5 + 5.271\,64f^4g + 8.032\,32f^3g^2 \\
 & + 6.775\,66f^2g^3 + 2.294\,11fg^4 + 0.193\,466g^5) \\
 & - g(3.9354f^6 + 11.8062f^5g + 22.6695f^4g^2 \\
 & + 25.6626f^3g^3 + 14.855f^2g^4 + 3.991\,77fg^5 \\
 & + 0.406\,903g^6), \quad (\text{A7})
 \end{aligned}$$

$$\begin{aligned}
 \eta_\psi = & \frac{32}{2187}(2f^2 + 2fg + 3g^2) + (0.003\,792\,33f^3 \\
 & + 0.005\,688\,49f^2g + 0.008\,939\,06fg^2 + 0.003\,521\,45g^3) \\
 & + (0.008\,731\,36f^4 + 0.017\,4627f^3g + 0.039\,5962f^2g^2 \\
 & + 0.030\,8648fg^3 + 0.003\,463\,84g^4) - (0.002\,7221f^5 \\
 & + 0.006\,805\,25f^4g + 0.022\,6066f^3g^2 + 0.027\,1046f^2g^3 \\
 & + 0.008\,572\,77fg^4 + 0.000\,012\,0125g^5) \\
 & + (0.009\,541\,81f^6 + 0.028\,6254f^5g + 0.090\,213\,3f^4g^2 \\
 & + 0.132\,717f^3g^3 + 0.095\,859\,5f^2g^4 + 0.034\,245fg^5 \\
 & + 0.004\,131\,89g^6), \quad (\text{A8})
 \end{aligned}$$

$$\begin{aligned}
 \eta_t = & -\frac{1}{9}(8f + 4g) + \frac{8}{81}(2f^2 + 2fg + 3g^2) - (0.206\,554f^3 \\
 & + 0.309\,831f^2g + 0.467\,379fg^2 + 0.182\,051g^3) \\
 & + (0.194\,175f^4 + 0.388\,35f^3g + 0.930\,785f^2g^2 \\
 & + 0.736\,61fg^3 + 0.029\,92g^4) - (0.291\,118f^5 \\
 & + 0.727\,795f^4g + 1.887\,48f^3g^2 + 2.103\,43f^2g^3 \\
 & + 0.863\,41fg^4 + 0.105\,499g^5) + (0.444\,686f^6 \\
 & + 1.334\,06f^5g + 4.191\,37f^4g^2 + 6.159\,31f^3g^3
 \end{aligned}$$

$$+ 4.384\,35f^2g^4 + 1.527fg^5 + 0.176\,483g^6). \quad (\text{A9})$$

The critical exponents associated with a real magnetic external field are determined from the RG dimension of the operator C_a , cf. Eq. (12). In order to compute it, we define the renormalization function $Z_c(f, g)$ from the one-particle irreducible two-point function $\Gamma^{(c,2)}$ with an insertion of the operator C_a , i.e.

$$\Gamma^{(c,2)}(0)_{aij} = Z_c^{-1} T_{aij}, \quad (\text{A10})$$

where T_{aij} is an appropriate tensor normalized so that $Z_c = 1$ at tree level. The RG function

$$\eta_c(f, g) = \left. \frac{\partial \ln Z_c}{\partial \ln m} \right|_{f_0, g_0} \quad (\text{A11})$$

was computed to six-loop in Ref. 41. For the three-component case one obtains

$$\begin{aligned}
 \eta_c = & -\frac{2}{9}(f - 2g) + \frac{8}{243}(3f^2 - 2fg - 3g^2) - (0.042\,6744f^3 \\
 & - 0.108\,535f^2g - 0.051\,2203fg^2 + 0.077\,7653g^3) \\
 & + (0.064\,6127f^4 - 0.034\,0279f^3g - 0.002\,364\,81f^2g^2 \\
 & + 0.027\,5227fg^3 - 0.025\,1600g^4) - (0.071\,3318f^5 \\
 & - 0.0868\,275f^4g - 0.0504\,380f^3g^2 + 0.004\,250\,84f^2g^3 \\
 & - 0.072\,1756fg^4 - 0.012\,0727g^5) + (0.125\,344f^6 \\
 & - 0.039\,4498f^5g + 0.076\,3944f^4g^2 + 0.055\,0844f^3g^3 \\
 & - 0.194\,565f^2g^4 - 0.063\,2578fg^5 + 0.015\,2157g^6).
 \end{aligned}$$

The critical exponents η_H , γ_H , and β_H are obtained by

$$\eta_H = 1 - 2\eta_c + 2\eta, \quad \gamma_H = \nu(2 - \eta_H), \quad \beta_H = \nu(1 - \eta_c + \eta), \quad (\text{A12})$$

where $\eta_c \equiv \eta_c(f^*, g^*)$.

APPENDIX B: FIVE-LOOP SERIES IN MINIMAL-SUBTRACTION SCHEME

In the $\overline{\text{MS}}$ scheme one sets

$$\psi = [Z_\psi(f, g)]^{1/2} \psi_R, \quad (\text{B1})$$

$$f_0 = A_d \mu^\epsilon F(f, g),$$

$$g_0 = A_d \mu^\epsilon G(f, g),$$

where the renormalization functions Z_ψ , F , and G are determined from the divergent part of the two- and four-point correlation functions computed in dimensional regularization. They are normalized so that $Z_\psi(f, g) \approx 1$, $F(f, g) \approx f$, and $G(f, g) \approx g$ at tree level. Here A_d is a d -dependent constant given by $A_d = 2^d \pi^{d/2} \Gamma(d/2)/3$. We also introduce a mass renormalization constant $Z_t(f, g)$ by requiring $Z_t \Gamma^{(1,2)}$ to be finite when expressed in terms of f and g . Here $\Gamma^{(1,2)}$ is the two-point function with an insertion of $\psi^* \cdot \psi$. Once the renormalization constants are determined, one computes the β functions from

$$\beta_f(f, g) = \mu \left. \frac{\partial f}{\partial \mu} \right|_{f_0, g_0}, \quad \beta_g(f, g) = \mu \left. \frac{\partial g}{\partial \mu} \right|_{f_0, g_0} \quad (\text{B2})$$

and the critical RG functions η_ψ and η_t from

$$\eta_{\psi, t}(f, g) = \left. \frac{\partial \log Z_{\psi, t}}{\partial \log \mu} \right|_{f_0, g_0}. \quad (\text{B3})$$

Their relation with the critical exponents is given in Eq. (A5). The β functions have a simple dependence on d

$$\beta_f(f, g) = (d-4)f + B_f(f, g), \quad \beta_g(f, g) = (d-4)g + B_g(f, g), \quad (\text{B4})$$

where the functions $B_f(f, g)$ and $B_g(f, g)$ —as well as the RG functions $\eta_{\psi, t}(f, g)$ —are independent of d .

In the following we report the five-loop series³⁴ for $N=3$:

$$\begin{aligned} B_f = & \frac{1}{3}(7f^2 + 4fg + 4g^2) - \frac{1}{9}(24f^3 + 22f^2g + 39fg^2 + 20g^3) \\ & + (9.07446f^4 + 12.5703f^3g + 29.4275f^2g^2 + 23.8424fg^3 \\ & + 2.41477g^4) - (46.7683f^5 + 85.6528f^4g + 227.316f^3g^2 \\ & + 256.217f^2g^3 + 110.777fg^4 + 16.3103g^5) + (296.166f^6 \\ & + 677.379f^5g + 2026.83f^4g^2 + 2898.93f^3g^3 \\ & + 2096.65f^2g^4 + 791.170fg^5 + 109.827g^6), \end{aligned} \quad (\text{B5})$$

$$\begin{aligned} B_g = & g \left[(2f + g) - \frac{1}{9}(28f^2 + 28fg + g^2) + (11.1573f^3 \right. \\ & + 16.7359f^2g + 9.75736fg^2 + 2.06019g^3) - (62.5357f^4 \\ & + 125.071f^3g + 123.278f^2g^2 + 60.7252fg^3 + 10.5496g^4) \\ & + (422.235f^5 + 1055.59f^4g + 1458.99f^3g^2 \\ & \left. + 1132.71f^2g^3 + 411.714fg^4 + 52.1814g^5) \right], \end{aligned} \quad (\text{B6})$$

$$\begin{aligned} \eta_\psi = & \frac{1}{18}(2f^2 + 2fg + 3g^2) - \frac{1}{216}(14f^3 + 21f^2g + 33fg^2 \\ & + 13g^3) + \frac{5}{2592}(86f^4 + 172f^3g + 474f^2g^2 + 388fg^3 \\ & - 9g^4) - (0.709208f^5 + 1.77302f^4g + 4.78026f^3g^2 \\ & + 5.39737f^2g^3 + 2.31434fg^4 + 0.325646g^5), \end{aligned} \quad (\text{B7})$$

$$\begin{aligned} \eta_t = & -\frac{2}{3}(2f + g) + \frac{1}{3}(2f^2 + 2fg + 3g^2) - \frac{1}{108}(208f^3 + 312f^2g \\ & + 486fg^2 + 191g^3) + (7.00217f^4 + 14.0043f^3g \\ & + 29.2911f^2g^2 + 22.2890fg^3 + 3.64149g^4) - (33.6900f^5 \\ & + 84.2251f^4g + 207.371f^3g^2 + 226.832f^2g^3 \\ & + 108.339fg^4 + 19.1618g^5). \end{aligned} \quad (\text{B8})$$

We also computed the $\overline{\text{MS}}$ expansion of the RG dimension of the operator C_a , cf. Eq. (12). For this purpose, we computed the renormalization constant $Z_c(f, g)$ by requiring $Z_c \Gamma^{(c,2)}$ to be finite when expressed in terms of f and g , where $\Gamma^{(c,2)}$ is the one-particle irreducible two-point function with an insertion of the operator C_a . Then, one defines the corresponding RG function

$$\eta_c(f, g) = \left. \frac{\partial \log Z_c}{\partial \log \mu} \right|_{f_0, g_0}. \quad (\text{B9})$$

The resulting series for the three-component case is given by

$$\begin{aligned} \eta_c = & -\frac{1}{3}(f - 2g) + \frac{1}{9}(3f^2 - 2fg - 3g^2) - \frac{5}{216}(22f^3 - 42f^2g \\ & - 27fg^2 + 26g^3) + (1.93436f^4 - 2.50866f^3g \\ & - 2.13390f^2g^2 + 0.910908fg^3 + 0.285515g^4) \\ & - (8.73201f^5 - 9.61852f^4g - 9.00230f^3g^2 \\ & + 1.92888f^2g^3 - 0.333945fg^4 - 0.810730g^5). \end{aligned} \quad (\text{B10})$$

The critical exponents η_H , γ_H and β_H are computed by using relations (A12). The expansion of η_c was computed for generic values of N . We verified that $\eta_c(f, 0)$ coincides with the expansion of the RG function η_T , related to the insertion of the spin-2 operator in the two-point function, in the $O(2N)$ theory obtained by setting $g=0$ (perturbative five-loop series for $\phi_T = (2 + \eta_T - \eta)\nu$ were computed in Ref. 45). Moreover, for $N=2$ one can prove and check the relation $\eta_c(2x/3, -x/3) = \eta_t(2x/3, -x/3)$, and $\eta_{t, \phi}(2x/3, -x/3) = \eta_{t, \phi}^{XY}$ where $\eta_{t, \phi}^{XY}$ are the RG functions of the XY ($O(2)$ -symmetric) theory, which can be found in Ref. 56.

*Corresponding author. E-mail address: vicari@df.unipi.it

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- $$\mathcal{H} = \int d^d x \left\{ \frac{1}{2} \sum_a [(\partial_\mu \phi_a)^2 + r \phi_a^2] + \frac{1}{4!} u_0 \left(\sum_a \phi_a^2 \right)^2 + \frac{1}{4!} v_0 \sum_{a,b} [(\phi_a \cdot \phi_b)^2 - \phi_a^2 \phi_b^2] \right\},$$
- where $u_0 = \frac{3}{2}(f_0 + g_0)$ and $v_0 = 3g_0$.
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$$i\alpha \sum_{abc} H_a (AA^\dagger)_{bc} \epsilon_{abc} + \beta \sum_{ab} H_a H_b (AA^\dagger)_{ab}.$$

In the presence of dipolar interactions $A_{ai} = \epsilon_{aib} \psi_b$, so that we obtain

$$i\alpha \sum_{abc} H_a \psi_b^* \psi_c \epsilon_{abc} + \beta \sum_{ab} H_a H_b (\psi_a^* \psi_b - \delta_{ab} |\psi|^2).$$

An additional term is also possible, $H^2 \text{Tr} AA^\dagger$, which is found, however, to be negligibly small (Ref. 9). Of course, for $H \rightarrow 0$, only the linear term is relevant.

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