Time-domain electromagnetic energy in a frequency-dispersive left-handed medium

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From Maxwell's equations and the Poynting theorem, the time-domain electric and magnetic energy densities are generally defined in the frequency-dispersive media based on the conservation of energy. As a consequence, a general definition of electric and magnetic energy is proposed. Comparing with existing formulations of electric and magnetic energy in frequency-dispersive media, the new definition is more reasonable and is valid in any case. Using the new definition and staring from the equation of motion, we have shown rigorously that the total energy density and the individual electric and magnetic energy densities are always positive in a realistic artificial left-handed medium (LHM) [R. A. Shelby, D. R. Smith, and S. Schultz, Science **292**, 77 (2001)], which obeys actually the Lorentz medium model, although such a LHM has negative permittivity and negative permeability simultaneously in a certain frequency range. We have also shown that the conservation of energy is not violated in LHM. The earlier conclusions can be easily extended to the Drude medium model and the cold plasma medium model. Through an exact analysis of a one-dimensional transient current source radiating in LHM, numerical results are given to demonstrate that the work done by source, the power flowing outwards a surface, and the electric and magnetic energy stored in a volume are all positive in the time domain.

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I. INTRODUCTION

Since Vesalago introduced a concept of left-handed medium (LHM) (Ref.1) in 1968, the study of LHM has not received much attention until recently when the artificial LHM was realized using periodic structures.[2–6] Due to its negative permittivity and negative permeability, the directions of the electric field, the magnetic field, and the propagation vector in such a medium form a left-handed system. Comparing with the conventional right-handed medium, there are many unusual physical properties occurred in LHM. For example, the wave-propagation direction in LHM is opposite to the energy-flow direction, the electromagnetic (EM) wave propagates towards the source in LHM due to its negative index of refraction, a LHM slab bends the refractive wave to the same side of the incident wave, and a flat lossless LHM slab can be made as a perfect lens, etc. Recently, intensive study of LHM has been made in the theory, experiments, and potential applications.[2–19]

In this paper, we will investigate the time-domain electric and magnetic energy stored in a frequency-dispersive LHM. Because the permittivity and permeability are simultaneously negative in a certain frequency band, it is a big concern whether the electric and magnetic energy densities are positive or negative in such a LHM. In the artificial LHM which was realized by Shelby *et al.*[6] using a two-dimensional array of repeated unit cells of conducting rods and split ring resonators, the permittivity and permeability have the following forms:

$$\epsilon(\omega) = \epsilon_0\left(1 - \frac{\omega_{ep}^2}{\omega^2 - \omega_{eo}^2 + i\omega\gamma_e}\right), \tag{1}$$

$$\mu(\omega) = \mu_0\left(1 - \frac{F\omega_{mp}^2}{\omega^2 - \omega_{mo}^2 + i\omega\gamma_m}\right), \tag{2}$$

which are actually coincident with the Lorentz medium model.[20–22] When $\omega_{eo}=\omega_{mo}=0$, the earlier model will reduce to the Drude medium model[8,21]

$$\epsilon(\omega) = \epsilon_0\left[1 - \frac{\omega_{ep}^2}{\omega(\omega + i\gamma_e)}\right], \tag{3}$$

$$\mu(\omega) = \mu_0\left[1 - \frac{F\omega_{mp}^2}{\omega(\omega + i\gamma_m)}\right]. \tag{4}$$

Furthermore, when $\gamma_e=\gamma_m=0$, the Drude medium model will reduce to the cold plasma medium model

$$\epsilon(\omega) = \epsilon_0\left(1 - \frac{\omega_{ep}^2}{\omega^2}\right), \quad \mu(\omega) = \mu_0\left(1 - \frac{F\omega_{mp}^2}{\omega^2}\right). \tag{5}$$

The earlier three models are usually used in LHM, which are obviously frequency dispersive. It is well known that the time-domain electric and magnetic energy densities in a frequency nondispersive medium are defined as[22,23,29]

$$w_e = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E}, \quad w_m = \tfrac{1}{2}\mathbf{B}\cdot\mathbf{H}. \tag{6}$$

Apparently, the electric and magnetic energy would be negative if the permittivity and permeability were negative, as they could be in dispersive media.[20] In frequency dispersive media, however, Eq. (6) becomes invalid and modified versions of electric and magnetic energy densities have been proposed for lossless dispersive media.[22–28] Generally, the time-averaged energy densities in time harmonic forms are written as

$$\langle w_e \rangle = \frac{1}{4}\frac{\partial[\omega\epsilon(\omega)]}{\partial\omega}|\tilde{\mathbf{E}}|^2, \quad \langle w_m \rangle = \frac{1}{4}\frac{\partial[\omega\mu(\omega)]}{\partial\omega}|\tilde{\mathbf{H}}|^2. \quad (7)$$

The earlier definition provides a correct way to describe the electric and magnetic energy densities in lossless dispersive media. It has been shown that both $\langle w_e \rangle$ and $\langle w_m \rangle$ are positive for the cold plasma medium,[20] which is lossless. We can also obtain the same conclusion for lossless Drude and Lorentz medium models by substituting Eqs. (1)–(4) into Eq. (7) with $\gamma_e = \gamma_m = 0$.

When the frequency dispersive medium is lossy, however, Eq. (7) becomes improper because the corresponding time-averaged energy densities are complex. For the Lorentz medium model, we have

$$\langle w_e \rangle = \frac{\epsilon_0}{4}\left[1 + \frac{\omega_{ep}^2(\omega^2+\omega_{eo}^2)}{(\omega^2-\omega_{eo}^2+i\omega\gamma_e)^2}\right]|\tilde{\mathbf{E}}|^2, \quad (8)$$

$$\langle w_m \rangle = \frac{\mu_0}{4}\left[1 + \frac{F\omega_{mp}^2(\omega^2+\omega_{mo}^2)}{(\omega^2-\omega_{mo}^2+i\omega\gamma_m)^2}\right]|\tilde{\mathbf{H}}|^2. \quad (9)$$

In order to obtain real energy densities, a direct extension to Eq. (7) is taking the real part to yield [Ref. 20, Eq. (30)]:

$$\langle w_e \rangle = \frac{1}{4}\Re e\left[\frac{\partial[\omega\epsilon(\omega)]}{\partial\omega}\right]|\tilde{\mathbf{E}}|^2, \quad (10)$$

$$\langle w_m \rangle = \frac{1}{4}\Re e\left[\frac{\partial[\omega\mu(\omega)]}{\partial\omega}\right]|\tilde{\mathbf{H}}|^2. \quad (11)$$

By doing so, the time-averaged energy densities are found to be positive in most cases and to be negative near the resonant frequency.[20] The same conclusion was achieved by using a more complicated definition to the electric energy density [Ref. 29, Eq. (22)].

In a recent work done by Ziolkowski, the time-domain energy densities were redefined [Ref. 20, Eqs. (39)–(40)], which were split into field energy densities and dipole energy densities as

$$w_e^{\text{field}} = \tfrac{1}{2}\epsilon_0|\mathbf{E}|^2, \; w_m^{\text{field}} = \tfrac{1}{2}\mu_0|\mathbf{H}|^2, \quad (12)$$

$$w_e^{\text{dipole}} = \tfrac{1}{2}\mathbf{P}\cdot\mathbf{E}, \; w_m^{\text{dipole}} = \tfrac{1}{2}\mu_0\mathbf{M}\cdot\mathbf{H}, \quad (13)$$

where $\mathbf{P}$ is the polarization vector and $\mathbf{M}$ is the magnetization vector. Using such definitions, it was found that the averaged field energy is always positive but the averaged dipole energy could become negative above the resonant frequency.[20] Actually, the definitions (12) and (13) are completely equivalent to the traditional definition (6) after considering the simple relations among $\mathbf{D}$, $\mathbf{E}$, $\mathbf{P}$ and $\mathbf{B}$, $\mathbf{H}$, $\mathbf{M}$.

In view of the nonphysical negative energy from the earlier considerations, new definitions of time-domain electric and magnetic energy densities are proposed in the frequency-dispersive media based on the Maxwell's equations and conservation theorem of energy. Such definitions are more general and reasonable, which are valid to both lossless and lossy media. Using the new definitions, we will give a rigorous proof that the energy and energy densities are always positive for the Lorentz medium model, Drude medium model, and the cold plasma media. Numerical examples are given to illustrate the earlier conclusions.

II. ENERGY DENSITY AND ENERGY IN A FREQUENCY DISPERSIVE MEDIUM

From the Maxwell's equations which are valid in both frequency dispersive and nondispersive media, one easily obtains the Poynting's theorem in the time domain as[30]

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} - \mathbf{E}\cdot\mathbf{J}_s, \quad (14)$$

where $\mathbf{J}_s$ is the excitation electric current density. Integrating the earlier equation over a volume V containing the source $\mathbf{J}_s$, we obtain

$$-\int_V \mathbf{E}\cdot\mathbf{J}_s dV = \int_S (\mathbf{E}\times\mathbf{H})\cdot\hat{n}dS + \int_V \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}dV + \int_V \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H}dV, \quad (15)$$

in which S is the closed surface of the volume V, and $\hat{n}$ is an outward unit vector normal to S. Equations (14) and (15) are the well-known Poynting theorem in which the quantity $\mathbf{S} = \mathbf{E}\times\mathbf{H}$ (W/m^2) is known as the Poynting vector. The earlier equation implies a conservation of energy

$$P_s = P + \frac{\partial}{\partial t}(W_e + W_m), \quad (16)$$

i.e., the total work done by the source current inside V equals the power flowing outwards the surface S plus the change of electromagnetic energy stored in V per unit time. In the earlier equation

$$P = \int_S (\mathbf{E}\times\mathbf{H})\cdot\hat{n}dS \quad (17)$$

is the total power flowing outwards the surface S,

$$P_s = -\int_V \mathbf{E}\cdot\mathbf{J}_s dV \quad (18)$$

is the total work done by the source current $\mathbf{J}_s$ inside V, and

$$W_{e,m} = \int_V w_{e,m} dV \quad (19)$$

is the electromagnetic energy stored inside V. Here, w_e and w_m are the electric and magnetic energy densities, which are generally defined by

$$\frac{\partial w_e}{\partial t} = \frac{\partial\mathbf{D}}{\partial t}\cdot\mathbf{E}, \quad \frac{\partial w_m}{\partial t} = \frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H}. \quad (20)$$

When the medium is frequency nondispersive, the constitutive relations give $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$. Then Eq. (20) reduces to the traditional definitions shown in Eq. (6). Obviously, both the electric and magnetic energy densities are negative for the frequency nondispersive LHM, which does not occur in nature or is nonphysical.

In the frequency dispersive medium, Eq. (20) gives general definitions of electric and magnetic energy densities as

$$w_e = \int_{-\infty}^{t} dt' \frac{\partial \mathbf{D}}{\partial t'} \cdot \mathbf{E}, \tag{21}$$

$$w_m = \int_{-\infty}^{t} dt' \frac{\partial \mathbf{B}}{\partial t'} \cdot \mathbf{H}. \tag{22}$$

Hence, the total electric and magnetic energy stored in the volume V is further written as

$$W_e = \int_{-\infty}^{t} dt' \int_V dV \frac{\partial \mathbf{D}}{\partial t'} \cdot \mathbf{E}, \tag{23}$$

$$W_m = \int_{-\infty}^{t} dt' \int_V dV \frac{\partial \mathbf{B}}{\partial t'} \cdot \mathbf{H}. \tag{24}$$

The earlier definitions are valid to all media, frequency dispersive or nondispersive, lossy or lossless.

Now we consider the behavior of energy densities defined earlier in the Lorentz medium model, Drude medium model, and the cold plasma medium.

In classical electrodynamic theory, a dispersive and absorbing medium can be represented by a collection of damped and noninteracting electrons of displacement $\mathbf{r}$, mass m, and effective charge q.[31,32] Under the action of an electric field, the equation of motion is expressed as

$$q\mathbf{E} = m\left(\frac{\partial^2 \mathbf{r}}{\partial t^2} + \gamma_e \frac{\partial \mathbf{r}}{\partial t} + \omega_{eo}^2 \mathbf{r}\right), \tag{25}$$

where γ_e is the damping constant, which is usually much smaller than the binding or resonant frequency ω_{eo}. Suppose that there are N electrons. Then the polarization vector is given by

$$\mathbf{P} = Nq\mathbf{r}. \tag{26}$$

Hence, we easily obtain a relation between $\mathbf{P}$ and $\mathbf{E}$ from Eqs. (25) and (26) as

$$\frac{\partial^2 \mathbf{P}}{\partial t^2} + \gamma_e \frac{\partial \mathbf{P}}{\partial t} + \omega_{eo}^2 \mathbf{P} = \epsilon_0 \omega_{ep}^2 \mathbf{E}, \tag{27}$$

in which

$$\omega_{ep}^2 = \frac{Nq^2}{m\epsilon_0}. \tag{28}$$

In the frequency dispersive medium, the electric displacement $\mathbf{D}$ has a simple relationship with the electric field $\mathbf{E}$ and the polarization vector $\mathbf{P}$:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}. \tag{29}$$

Under the time-harmonic excitation with $e^{-i\omega t}$, we have $\partial/\partial t = -i\omega$. Hence, Eqs. (27) and (29) easily give

$$\mathbf{D} = \epsilon_0 \left(1 - \frac{\omega_{ep}^2}{\omega^2 - \omega_{eo}^2 + i\omega\gamma_e}\right)\mathbf{E} \tag{30}$$

in the frequency domain. As a consequence, the equivalent permittivity in the frequency domain is written as Eq. (1), which obeys the Lorentz medium model.[6,20,21]

Similarly, the magnetization vector $\mathbf{M}$ in the artificial LHM which is composed of two-dimensional array of repeated unit cells of conducting rods and split ring resonators is written as[3,32]

$$\frac{\partial^2 \mathbf{M}}{\partial t^2} + \gamma_m \frac{\partial \mathbf{M}}{\partial t} + \omega_{mo}^2 \mathbf{M} = F\omega_{mp}^2 \mathbf{H}, \tag{31}$$

where ω_{mo} is the resonance frequency of the magnetic dipole, γ_m is the damping frequency, and F characterizes the strength of the interaction between the dipole and the magnetic field. Then the magnetic flux $\mathbf{B}$ is expressed as in terms of $\mathbf{M}$ and $\mathbf{H}$:

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}. \tag{32}$$

Under the time-harmonic excitation with $e^{-i\omega t}$, Eqs. (31) and (32) easily give the equivalent permeability in the frequency domain, as shown in Eq. (2), which also obeys the Lorentz medium model[6,20,21].

Now we consider the electric and magnetic energy densities defined in Eqs. (21) and (22) in time domain. For the electric energy density, we have

$$\frac{\partial w_e}{\partial t} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \cdot \mathbf{E} + \frac{\partial \mathbf{P}}{\partial t} \cdot \mathbf{E} = \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 |\mathbf{E}|^2\right) + \frac{\partial \mathbf{P}}{\partial t} \cdot \mathbf{E}. \tag{33}$$

Substituting Eq. (27) into Eq. (33) and replacing the second $\mathbf{E}$ in the right-hand side, we easily obtain

$$\frac{\partial w_e}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}\epsilon_0 |\mathbf{E}|^2 + \frac{1}{2\epsilon_0 \omega_{ep}^2}\left(\left|\frac{\partial \mathbf{P}}{\partial t}\right|^2 + \omega_{eo}^2 |\mathbf{P}|^2\right)\right] + \frac{\gamma_e}{\epsilon_0 \omega_{ep}^2}\left|\frac{\partial \mathbf{P}}{\partial t}\right|^2. \tag{34}$$

Hence, we easily obtain the electric energy density as

$$w_e = \frac{1}{2}\epsilon_0 |\mathbf{E}|^2 + \frac{1}{2\epsilon_0 \omega_{ep}^2}\left(\left|\frac{\partial \mathbf{P}}{\partial t}\right|^2 + \omega_{eo}^2 |\mathbf{P}|^2\right) + \frac{\gamma_e}{\epsilon_0 \omega_{ep}^2}\int_{-\infty}^{t} dt' \left|\frac{\partial \mathbf{P}}{\partial t'}\right|^2. \tag{35}$$

Similarly, the magnetic energy density can be expressed as

$$w_m = \frac{1}{2}\mu_0 |\mathbf{H}|^2 + \frac{\mu_0}{2F\omega_{mp}^2}\left(\left|\frac{\partial \mathbf{M}}{\partial t}\right|^2 + \omega_{mo}^2 |\mathbf{M}|^2\right) + \frac{\mu_0 \gamma_m}{F\omega_{mp}^2}\int_{-\infty}^{t} dt' \left|\frac{\partial \mathbf{M}}{\partial t'}\right|^2. \tag{36}$$

From Eqs. (35) and (36), we clearly see that both the electric and magnetic energy densities are positive in the Lorentz medium model although the permittivity and permeability may be simultaneously negative in a certain frequency band. No matter how large is the frequency band and no matter the medium is lossless or lossy, the energy densities are always positive. Hence, the electric and magnetic energy stored in any volume is always positive. Because the Drude medium model is a special case of Lorentz medium model when $\omega_{eo}=\omega_{mo}=0$, and the cold plasma medium model is a special case of Drude medium model when $\gamma_e=\gamma_m=0$, the earlier conclusions are valid to both Drude and cold plasma medium models either.

Actually, the last terms in Eqs. (35) and (36), which correspond to the imaginary parts of the permittivity and permeability in the frequency domain, represent the heat energy due to the loss. Hence, the conservation of energy described in Eq. (16) can be rewritten in an alternative form as

$$P_s = P + (P_{\sigma e} + P_{\sigma m}) + \frac{\partial}{\partial t}(W_e^n + W_m^n), \tag{37}$$

where P_s and P are defined in Eqs. (18) and (17), $P_{\sigma e}$ and $P_{\sigma m}$ denote the heat energy per unit time due to the electric and magnetic losses in the permittivity and permeability, which are given by

$$P_{\sigma e} = \int_V \frac{\gamma_e}{\epsilon_0 \omega_{ep}^2}\left|\frac{\partial \mathbf{P}}{\partial t}\right|^2 dV, \tag{38}$$

$$P_{\sigma m} = \int_V \frac{\mu_0 \gamma_m}{F\omega_{mp}^2}\left|\frac{\partial \mathbf{M}}{\partial t}\right|^2 dV, \tag{39}$$

and W_e^n and W_m^n are new-defined electric and magnetic energy stored inside V with energy densities

$$w_e^n = \frac{1}{2}\epsilon_0|\mathbf{E}|^2 + \frac{1}{2\epsilon_0\omega_{ep}^2}\left(\left|\frac{\partial \mathbf{P}}{\partial t}\right|^2 + \omega_{eo}^2|\mathbf{P}|^2\right), \tag{40}$$

$$w_m^n = \frac{1}{2}\mu_0|\mathbf{H}|^2 + \frac{\mu_0}{2F\omega_{mp}^2}\left(\left|\frac{\partial \mathbf{M}}{\partial t}\right|^2 + \omega_{mo}^2|\mathbf{M}|^2\right). \tag{41}$$

Equation (37) clearly states a conservation of energy: the total work done by the source current inside V is the sum of the power flowing outwards the surface S, the heat energy per unit time inside V produced by the electromagnetic losses, and the change of electromagnetic energy stored in V per unit time. From the earlier equations, it is obvious that $P_{\sigma e}\geqslant 0$, $P_{\sigma m}\geqslant 0$, $W_e^n\geqslant 0$, and $W_m^n\geqslant 0$ in any cases for the Lorentz medium model, including the artificial LHM.[6]

III. NUMERICAL RESULTS

In order to reveal the physical property of work, power, and energy in LHM, we consider a one-dimensional problem where the source is a current sheet on an infinite plane, as shown in Fig. 1. Here, the current density is a Gaussian pulse defined by

$$\mathbf{J}_s(z,t) = \hat{y}J_0\delta(z)t_0 f(t-t_0), \tag{42}$$

in which

$$f(t) = -2\alpha^2 t e^{-\alpha^2 t^2}.$$

The Gaussian pulse has also a Gaussian-distributed frequency spectrum

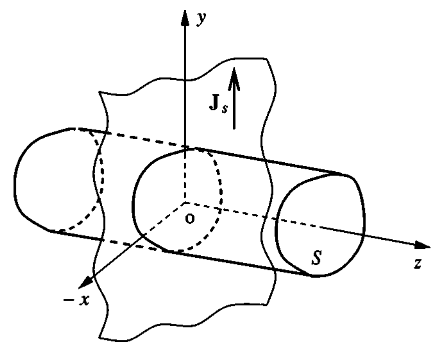

FIG. 1. A uniform current sheet extending to infinity in the x and y directions in a LHM. A cylinder with unit area in the cross section lies along $z\in[-z_0,z_0]$.

$$A(\omega) = -i\frac{\sqrt{\pi}}{\alpha}J_0\omega t_0 e^{-\omega^2/4\alpha^2}e^{i\omega t_0}. \tag{43}$$

In the frequency domain, the electric and magnetic fields have closed-form solutions as

$$\tilde{\mathbf{E}}(z,\omega) = -\hat{y}\frac{\mu_r}{2n}\eta_0 A(\omega)e^{ink|z|}, \tag{44}$$

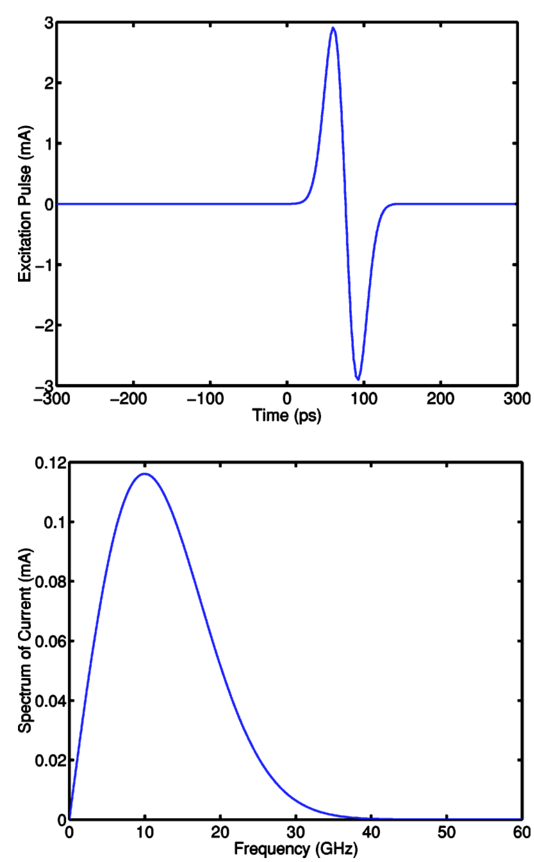

FIG. 2. The excitation Gaussian pulse and its frequency spectrum. (a) The Gaussian pulse. (b) The frequency spectrum.

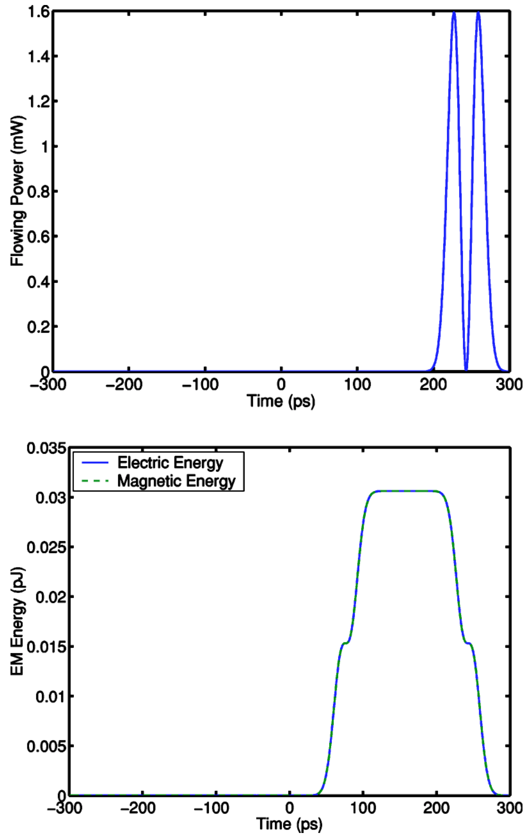

FIG. 3. The total power flowing outwards S and the electromagnetic energy stored inside V in free space. (a) Power flow. (b) Electromagnetic energy.

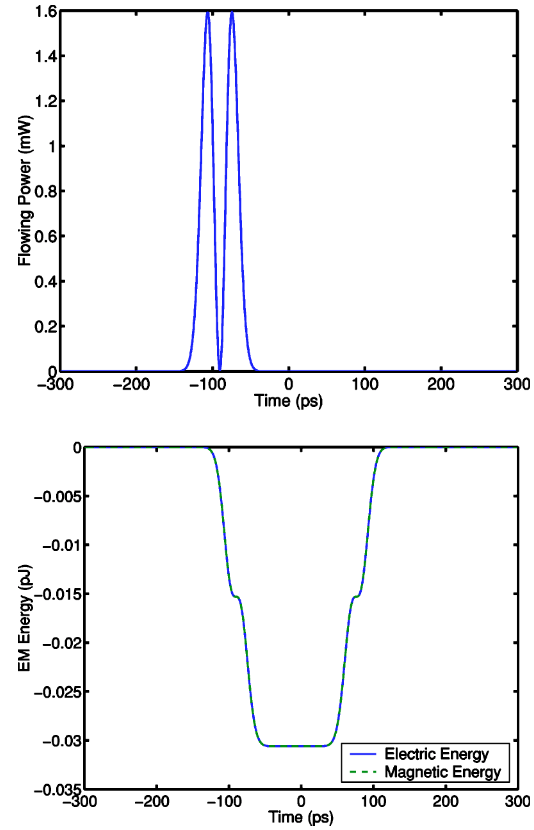

FIG. 4. The total power flowing outwards S and the electromagnetic energy stored inside V in the frequency nondispersive LHM when $\epsilon_r = \mu_r = -1$. (a) Power flow. (b) Electromagnetic energy.

$$\tilde{\mathbf{H}}(z,\omega) = \hat{x}\frac{1}{2}\mathrm{sign}(z)A(\omega)e^{ink|z|}, \tag{45}$$

where $n = \sqrt{\mu_r \epsilon_r}$ is the index of refraction, μ_r and ϵ_r are the relative permeability and permittivity, k is the wave number in free space, $\eta_0 = 120\pi$ (ohms) is the wave impedance in free space, and sign(z) is a sign function which is 1 for $z > 0$ and -1 for $z < 0$. Furthermore, the frequency-domain electric and magnetic fluxes are given by $\tilde{\mathbf{D}} = \epsilon_0 \epsilon_r \tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}} = \mu_0 \mu_r \tilde{\mathbf{H}}$. Then, the time-domain electromagnetic fields are generally obtained through the Fourier transform

$$\mathbf{F}(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\mathbf{F}}(z,\omega)e^{-i\omega t}d\omega, \tag{46}$$

in which $\mathbf{F} = \mathbf{E}, \mathbf{D}, \mathbf{H}$, and $\mathbf{B}$. When the LHM is frequency nondispersive, the earlier fields have closed-form solutions.

Now we consider a cylinder with volume V and surface S which is perpendicular to the current sheet, as shown in Fig. 1. The cylinder has a length of $2z_0$ and a unit area in the cross section, which lies along $z \in [-z_0, z_0]$. Hence, the total work P_s done by the source inside the cylinder, the power P flowing outward the cylinder, and the electromagnetic energy stored inside the cylinder can be easily computed. In a homogeneous and frequency nondispersive LHM where $\epsilon_r < 0$, $\mu_r < 0$, and $n < 0$, the work, power, and energy are expressed as

$$P_s = \frac{\mu_r}{2n}\eta_0 J_0^2 t_0^2 f^2(t - t_0), \tag{47}$$

$$P = \frac{\mu_r}{2n}\eta_0 J_0^2 t_0^2 f^2(t - t_0 - nz_0/c), \tag{48}$$

$$W_e = W_m = \frac{\mu_r}{4}\mu_0 J_0^2 t_0^2 \int_0^{z_0} f^2(t - t_0 - nz/c)dz, \tag{49}$$

where c is the light speed in free space, and the integral in Eq. (49) has a closed form as series expansion. From Eqs. (47)–(49), we clearly see that the source work and the power flow are both positive, and the EM energy is **negative** in such a frequency nondispersive LHM. Although the energy is negative, the conservation of energy is satisfied.

Next we study the electromagnetic energy stored in LHM through numerical experiments. In the following numerical experiments, the Gaussian pulse and its frequency spectrum are illustrated in Fig. 2, where $J_0 = 1$ mA, $\alpha = 10\pi\sqrt{2}$ GHz, and $t_0 = 76.3079$ ps. For such parameters, the maximum spectrum is located at 10 GHz, and the spectrum at 68 GHz goes down to 10^{-8}.

For easy comparison, we first consider the radiation of the Gaussian pulse in free space. Figure 3 illustrates the power flowing outwards S and the energy stored inside V when $z_0 = 50$ mm. From Fig. 3, we observe that both the power and

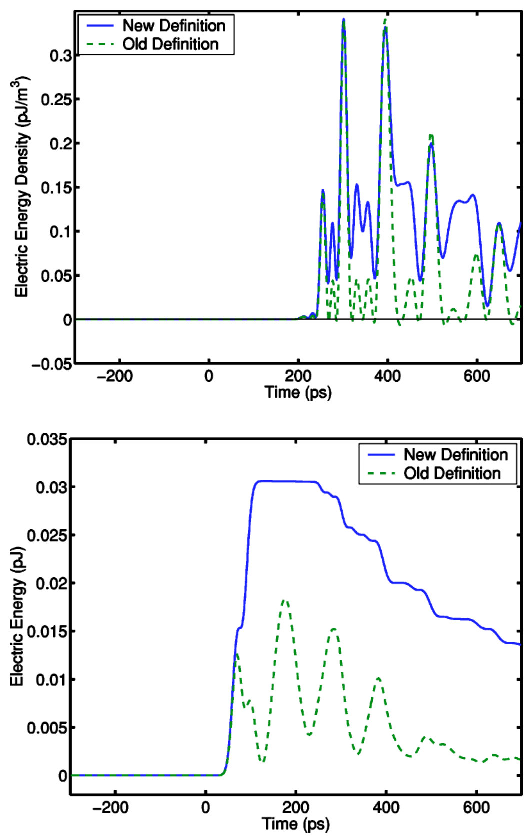

FIG. 5. The energy density and the stored energy in a realistic LHM with a bandwidth of 2 GHz ($f_{eo}=f_{mo}=10$ GHz, $f_{ep}=f_{mp}=12$ GHz, and $\gamma=100$ MHz). (a) Energy density. (b) Stored energy.

energy are positive and causal in free space. In the frequency nondispersive LHM, the power flow and stored energy are illustrated in Fig. 4 when $z_0=50$ mm, where $\varepsilon_r=\mu_r=-1$.

From Fig. 4, we clearly see that the stored energy in such a frequency nondispersive LHM is negative, which is nonphysical. Although the power is positive, both the power flow and energy occur before the excitation of current source. This is obviously a violation of causality. Hence,

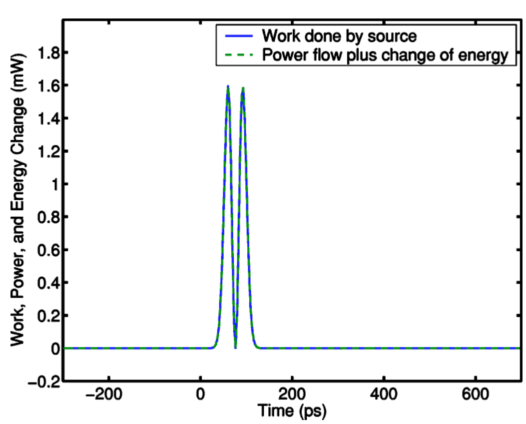

FIG. 6. Comparison of the work done by the source P_s (solid line) and the power plus the change of energy per unit time $P+\partial(W_e+W_m)/\partial t$ (dashed line) in a realistic LHM with the bandwidth of 2 GHz.

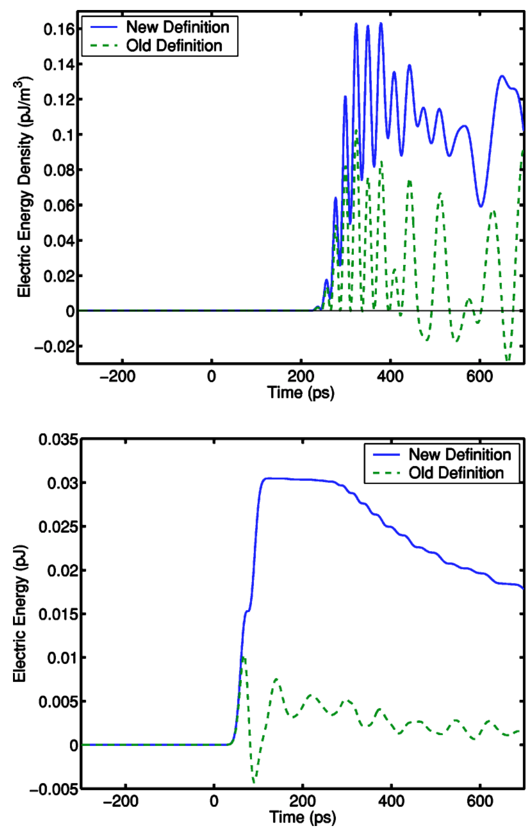

FIG. 7. The energy density (upper) and the stored energy (lower) in the realistic LHM with a wider bandwidth of 5 GHz ($f_{eo}=f_{mo}=8$ GHz, $f_{ep}=f_{mp}=13$ GHz, and $\gamma=100$ MHz). (a) Energy density. (b) Stored energy.

such a frequency nondispersive LHM breaks the basic physical laws.

However, a frequency nondispersive LHM does not occur in nature. In an artificial LHM which obeys the Lorentz medium model,[6] where the permittivity and permeability are shown in Eqs. (1) and (2), the energy density and the stored energy at $z_0=50$ mm are illustrated in Fig. 5 when $f_{eo}=f_{mo}=10$ GHz, $f_{ep}=f_{mp}=12$ GHz, and $\gamma=100$ MHz. Here, double precisions are used to make the numerical integration as accurate as possible, and the integration error is set to 10^{-8} in computing the electromagnetic fields. Because the electric energy and magnetic energy are completely the same, only the electric energy is given, which has been computed using the new definition and the old definition given by Eq. (6) or Eqs. (12) and (13) for easy comparison.

From Fig. 5, we clearly observe that both the energy density and energy are causal. In this numerical experiment, the LHM has a bandwidth of 2 GHz where the permittivity and permeability are simultaneously negative. Using the old definition, the energy density is slightly negative in some small time periods, as shown in Fig. 5(a). From the new definition based on the conservation of energy, however, the energy density is always positive. As a consequence, the energy stored inside V is always positive, as shown in Fig. 5(b).

In order to verify the conservation of energy, we have computed the work done by the source current in the volume V, the power flowing outwards the surface S, and the change of energy stored inside V per unit time using the new definition. Figure 6 illustrates the comparison of the work P_s (see the solid line) and the power plus the change of energy per unit time $P+\partial(W_e+W_m)/\partial t$ (see the dashed line). Clearly, they are nearly the same and the tiny difference is due to the integration error. The earlier numerical experiment verifies the conservation of energy. Hence, the basic physical laws are not violated at all in the realistic artificial LHM.

To verify the earlier conclusions further, we consider the artificial LHM with a wider bandwidths of 5 GHz ($f_{eo}=f_{mo}=8$ GHz, $f_{ep}=f_{mp}=13$ GHz, and $\gamma=100$ GHz). For such a case, the electric energy density at $z_0=50$ mm and the electric energy stored inside the volume are illustrated in Fig. 7. When the LHM has a wider bandwidth, the energy density from the traditional definition has stronger negative parts which produces a negative energy stored inside V in a time period, as shown in Fig. 7. Using the new definition, however, both the energy density and the total energy stored inside V are always positive even in the wideband LHM with 5 GHz.

From Figs. 5 and 7, we also observe that the total energy stored inside V computed by the new definition is quite similar for different bandwidths. This obeys the principle of energy stability. Using the old definition, however, the total energy is quite different. Similar to Fig. 6, the conservation of energy is also verified for the wider bandwidth. Hence, the basic physical laws are not violated at all in the realistic artificial LHM.

IV. CONCLUSIONS

The traditional definition of electromagnetic energy is only valid to frequency nondispersive media, and the formulations proposed by Landau and Lifshitz and others[20–28] for dispersive media are only valid to lossless media. For general frequency dispersive media, new definitions of electric and magnetic energy densities are proposed based on the Maxwell's equations and the conservation theorem of energy. Using the new definitions and the equation of motion, we have shown rigorously that the electric and magnetic energy and energy densities are always positive in the Lorentz medium model, Drude medium model, and cold plasma medium although the permittivity and permeability could be simultaneously negative in a certain frequency band, just like a LHM. We have also shown that the conservation of energy is not violated in such a LHM. Numerical results are given to illustrate the earlier conclusions.

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