$su(N,N)$ algebra and constants of motion for bosonic mean-field exciton equations

Z. S. Yang, N. H. Kwong, and R. Binder
Optical Sciences Center, The University of Arizona, Tucson, Arizona 85721, USA
(Received 21 March 2004; published 15 November 2004)

The ultrafast (picosecond) coherent dynamics of exciton systems in semiconductors can be approximately described by bosonic mean-field equations. These equations are nonlinear and therefore difficult to solve analytically. It is thus important to study the general dynamical properties of these equations, such as the underlying symmetry and corresponding conservation laws. It is shown in this paper that, for an N-species exciton system (e.g., heavy-hole and light-hole excitons), a mean-field Hamiltonian (including the coupling to external fields and fermionic corrections) can be formulated which is a member of the $su(N,N)$ algebra. As a consequence, the equations of motion for the center-of-mass momentum dependent exciton distribution and the coherent biexciton amplitude can be cast into a form similar to that of the optical Bloch vector in two-level atoms that belong to the algebra $su(2)$ [or, more generally, N-level atoms with algebra $su(N)$]. It is shown that the analog to the Bloch sphere in N-level atoms is an unbounded hypersurface (generalized hyperboloid) that constrains the motion of the exciton distribution and coherent biexciton amplitude. Further constants of motions that constrain the motion on the hypersurface are found from an $su(N,N)$ generalization to the Hioe-Eberly method in $su(N)$ systems (N-level atoms) [F. Hioe and J. Eberly, Phys. Rev. Lett. **47**, 838 (1981)].

DOI: 10.1103/PhysRevB.70.195319 PACS number(s): 78.67.De, 71.35.Gg, 78.47.+p

I. INTRODUCTION

Nonlinear semiconductor optics based on excitations near the fundamental band gap offers invaluable information on many-body effects and particle correlations (for recent reviews, see, e.g., Refs. 1–4). While excitation in the interband continuum (i.e., transitions from a valence band high into a conduction band) probes mainly electron-hole plasma effects, optical excitations of the lowest bound excitons (usually the 1*s* excitons) probe many-body effects of the correlated exciton system. One of the excitation regimes that has been under intense study recently is the so-called $\chi^{(3)}$ regime, in which the optical nonlinear response is treated up to the third order in the light field amplitudes involved in the optical excitation and the probing of the semiconductor response (see, e.g., Refs. 5–16 for theoretical details).

In addition to fermionic theories, which treat the exciton properly in terms of its electron and hole constituents, there exists a wide variety of studies treating the excitons as bosonic particles (see, e.g., Refs. 17–22). Fermionic theories are more fundamental, but they are often very difficult to evaluate, especially in cases where excitons dominate the nonlinear response. This happens, for example, when the optical light field is in resonance with the lowest exciton state and the exciton population is kept at low densities and low temperature. In cases like these, bosonic theories are advantageous, because they can give simple and transparent physical interpretations for the observed optical nonlinearities.

In order to discuss excitonic optical nonlinearities, one needs to specify the geometry of the system and the time scale under consideration. In this paper, we are interested in the exciton dynamics in a thin semiconductor quantum well. The light field is assumed to be in normal incidence, which means that the optically excited excitons (or, more precisely, excitonic interband coherences) have zero in-plane center-of-mass momentum. Moreover, we only deal with the ultrafast time scale, on which dephasing and relaxation processes can be ignored. (See, for example, Refs. 23–25.) For a sample at low temperature, excited in the low-density regime, the corresponding times are typically on the order of several picoseconds (or even tens of picoseconds).

Under these conditions, it is reasonable to assume that the dynamics of the correlated exciton system can well be described by a mean-field Hamiltonian, i.e., a Hamiltonian that treats the correlations in a mean-field description and rules out incoherent exciton-exciton scattering and relaxation processes. Mean-field Hamiltonians have been used in many different areas of physics to extract important information on the system's dynamics. One of the most prominent examples is that of plasma screening in electron or electron-hole plasmas which can be described by a mean-field Hamiltonian with self-consistently computed screened Coulomb potential (Ehrenreich-Cohen method) (for a textbook discussion see, for example, Ref. 26, p. 142). The mean-field Hamiltonian is constructed in such a way that the equations of motion of the operators under consideration obtained from this Hamiltonian are identical to those obtained from the full Hamiltonian followed by a Hartree-Fock factorization.

In the present context we are dealing with an N-exciton system (where N denotes the number of different exciton species, for example $N=4$ if we have heavy-hole and light-hole excitons and both are twofold spin degenerate). We will determine an appropriate mean-field Hamiltonian which is used to determine the equations of motion of bosonic (exciton) operators. It includes optical excitation, exciton-exciton interaction, and fermionic corrections that describe phase-space blocking in the exciton-photon coupling. In particular, we will derive the mean-field Heisenberg equations for the exciton distribution function (i.e., the distribution as function of center-of-mass in-plane momentum of the excitons), and the coherent biexciton amplitude. While the equations of motion of these two functions are difficult to solve, we will be able to derive all constants of motion associated with the dynamical symmetry of the mean-field Hamiltonian.

In two-level atoms (and also in N-level atoms), constants of motion are conveniently discussed in terms of the Bloch sphere and restrictions for the Bloch vector moving on the Bloch sphere (see Ref. 27 for an extensive discussion of the two-level atom). In this case, the underlying symmetry group that can explain the constants of motion is $SU(N)$. The Hamiltonian of the N-level atom can be expressed in terms of the group generators of $SU(N)$, and the dynamical variables of the system can be associated with these generators. Hioe and Eberly have shown how to exploit the properties of the $su(N)$ algebra in the formulation of the equations of motion of the dynamical variables of an N-level system,[28,29] and how one can obtain conservation laws (constants of motion) corresponding to $SU(N)$.

In our case, described by the N-exciton mean-field Hamiltonian, the underlying symmetry group is not $SU(N)$. It has been shown by Huang[30] that, in the case of a single-species weakly interacting Bose system, the underlying symmetry group is $SU(1,1)$. This group is well known in the context of quantum optics, and a generalized $SU(1,1)$ Bloch vector has been formulated by Dattoli *et al.*[31]

It is therefore natural for us to ask whether our mean-field Hamiltonian, which is a Hamiltonian for an N-species exciton system and includes, besides exciton-exciton interaction, effects of optical excitation as well as fermionic corrections, can be associated with the symmetry group $SU(N,N)$. If so (and we will show that this is indeed possible), one can derive a generalized Bloch vector and relate it to the expectation values of the dynamical variables of interest (in this case the exciton momentum distribution and the biexciton amplitude). We will show that, similar to the case of a two-level atom, one can define a generalized Bloch vector and generalized Bloch hypersurface (not a sphere in this case) which yield immediately information about constraints on the dynamics of the dynamical variables. Moreover, we will show that the method employed by Hioe and Eberly[28] to determine higher constants of motion based on $SU(N)$ can be generalized to $SU(N,N)$, even though the generalization will have important mathematical differences compared to the Hioe-Eberly method.

This paper is organized as follows: in Sec. II, we briefly review the basic concepts of the groups $SU(2)$ and $SU(1,1)$ [including generalizations to $SU(N)$ and $SU(M,N)$] and their algebras; in Sec. III, we study the structure of the effective mean-field Hamiltonian of a general spatially homogeneous exciton system and relate it to the $su(N,N)$ algebra. This $su(N,N)$ dynamical algebra allows us to write the mean-field equations in two equivalent forms. One is a generalized Bloch vector form, and the other is a matrix equation that is formally analogous to the Liouville equation for a density matrix (but we stress that the analogy is only formal and only useful for the derivation of the desired constants of motion). In Sec. IV, we use the ("density matrix") equation obtained in Sec. III to study higher constants of motion associated with the $su(N,N)$ symmetry. A brief summary is given in Sec. V.

II. $SU(2)$ AND $SU(1,1)$ GROUPS, ALGEBRAS, AND THEIR REALIZATIONS IN PHYSICAL SYSTEMS

Most of the general mathematical properties of the groups $SU(2)$ and, to a lesser degree $SU(1,1)$, are by now text book knowledge (see, e.g., Ref. 32). However, we believe it is useful for the purpose of this paper to start out with a brief review of these groups [and their generalizations $SU(N)$ and $SU(M,N)$] as well as their algebras. The short review of the mathematical properties of $SU(N)$ and $SU(M,N)$ will be limited to aspects relevant for the study of exciton dynamics discussed in the latter chapters of this paper. We will also review some known aspects of the relationship between the groups $SU(N)$ and $SU(M,N)$ and their algebras and physical systems, notably that between $SU(2)$ and the two-level atom. This will help lay the foundation of the discussion of the application of $SU(N,N)$ to a bosonic description of excitons.

The group $SU(2)$ is the group of all two-dimensional unitary matrices with unit determinant

$$U^{(2)\dagger}U^{(2)} = I, \quad \det\{U^{(2)}\} = 1 \tag{1}$$

or more explicitly

$$U^{(2)} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1, \tag{2}$$

where α and β are complex numbers. We use the superscript $^{(2)}$ in our discussion of $SU(2)$, while the superscript $^{(11)}$ will be used below in the discussion of $SU(1,1)$. The three generators of $SU(2)$ can be chosen as

$$L_j^{(2)} = \sigma_j, \quad j = 1,2,3, \tag{3}$$

where σ_j's are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

The matrices $\{L_j^{(2)}, j=1,2,3\}$ obey the commutation relations

$$[L_h^{(2)}, L_j^{(2)}] = 2i\sum_{l=1}^{3} \xi_{hjl}^{(2)} L_l^{(2)}, \tag{5}$$

where the $\xi_{hjl}^{(2)}$ are antisymmetric under the interchange of any two subscripts and $\xi_{123}^{(2)}=1$. The $L_j^{(2)}$'s form a basis of the $su(2)$ algebra, whose elements (the linear superpositions of $L_j^{(2)}$'s with arbitrary real coefficients) are the two-dimensional traceless Hermitian matrices, i.e.,

$$L^{(2)\dagger} = L^{(2)}, \quad \mathrm{Tr}\{L^{(2)}\} = 0. \tag{6}$$

Similarly, the group $SU(1,1)$ is the group of all two-dimensional matrices satisfying[33]

$$U^{(11)\dagger}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}U^{(11)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \det\{U^{(11)}\} = 1 \tag{7}$$

or more explicitly

$$U^{(11)} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \quad |\alpha|^2 - |\beta|^2 = 1. \tag{8}$$

The three generators of the group $SU(1,1)$ can be chosen in terms of the Pauli matrices as

$$L_j^{(11)} = \eta_j^{(11)}\sigma_j, \quad \eta_1^{(11)} = i, \quad \eta_2^{(11)} = i, \quad \eta_3^{(11)} = 1 \tag{9}$$

with the commutation relations

$$[L_j^{(11)}, L_h^{(11)}] = 2i\sum_{l=1}^{3} \xi_{jhl}^{(2)} \frac{\eta_j^{(11)}\ \eta_h^{(11)}}{\eta_l^{(11)}} L_l^{(11)}. \tag{10}$$

The $SU(1,1)$ generators in Eq. (9) form a basis of the $su(1,1)$ algebra, whose elements (the linear superpositions of $L_j^{(11)}$'s with arbitrary real coefficients) are two-dimensional matrices with the following properties:

$$L^{(11)\dagger}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} L^{(11)}, \quad \mathrm{Tr}\{L^{(11)}\} = 0. \tag{11}$$

The similarity between definitions given in Eqs. (1) and (2) and those in Eqs. (7) and (8) suggests that we can define both groups/algebras in a unified way:

$$\text{Group:} \begin{cases} U^\dagger G U = G \\ \det\{U\} = 1, \end{cases} \quad \text{Algebra:} \begin{cases} L^\dagger G = GL \\ \mathrm{Tr}\{L\} = 0 \end{cases} \tag{12}$$

with

$$G = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{for } SU(2) \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{for } SU(1,1). \end{cases} \tag{13}$$

If we consider the group members as operators acting in a two-dimensional linear space, they conserve the "lengths" of vectors

$$\ell \equiv \sum_{j=1}^{2} G_{ij} x_i^* x_j = \mathbf{x}^\dagger G \mathbf{x}, \tag{14}$$

where $\mathbf{x}$ is any two-dimensional column vector and G can be viewed as the "metric matrix."

The definition given by Eq. (12) can be naturally generalized to the $SU(M,N)$ [with $SU(M) \equiv SU(M,0)$] group/algebra, i.e., the $SU(M,N)$ group/algebra is defined as the set of all $(M+N)\times(M+N)$ matrices U/L satisfying (12) with

$$G = \begin{pmatrix} I_{M\times M} & 0 \\ 0 & -I_{N\times N} \end{pmatrix}, \tag{15}$$

where I is the unit matrix.

The definition given by Eq. (12) for the $SU(M,N)$ group/algebra is rather abstract, and a more explicit form of this group/algebra is usually desirable in practical applications. For the purpose of this paper, we write down an explicit form of one set of $SU(N,N)$ group generators which form a basis of the $su(N,N)$ algebra,

$$\vec{\bar{L}} = (\overbrace{A_{(12)}, A_{(13)}, \ldots, A_{(1,2N)}, A_{(23)}, \ldots, A_{(2N-1,2N)}}^{N(2N-1)}, \underbrace{B_{(12)}, B_{(13)}, \ldots, B_{(1,2N)}, B_{(23)}, \ldots, B_{(2N-1,2N)}}_{N(2N-1)}, \underbrace{C_{(1)}, \ldots, C_{(2N-1)}}_{2N-1}), \tag{16}$$

where

$$(A_{(mn)})_{\mu\nu} = \eta_{mn}^A(\delta_{\mu m}\delta_{\nu n} + \delta_{\mu n}\delta_{\nu m}), \quad 1 \leqslant m < n \leqslant 2N,$$

$$(B_{(mn)})_{\mu\nu} = \eta_{mn}^B(-i)(\delta_{\mu m}\delta_{\nu n} - \delta_{\mu n}\delta_{\nu m}), \quad 1 \leqslant m < n \leqslant 2N,$$

$$(C_{(r)})_{\mu\nu} = \sqrt{\frac{2}{r(r+1)}}\delta_{\mu\nu}(\delta_{\mu 1} + \delta_{\mu 2} + \cdots + \delta_{\mu r} - r\delta_{\mu\ r+1}), \quad 1 \leqslant r \leqslant 2N-1, \tag{17}$$

$$\eta_{mn}^A = \eta_{mn}^B = i^{[\Theta(N-m)-\Theta(N-n)]/2}, \quad \Theta(m) \equiv \begin{cases} 1 & m \geqslant 0 \\ -1 & m < 0. \end{cases} \tag{18}$$

The commutation relations of $\vec{\bar{L}}$ are

$$[\bar{L}_h, \bar{L}_j] = 2i\sum_{l=1}^{(2N)^2-1} \xi_{hjl}\frac{\eta_h\ \eta_j}{\eta_l}\bar{L}_l, \tag{19}$$

where ξ_{hjl} are antisymmetric under the interchange of any two subscripts and

$$\boldsymbol{\eta} = (\underbrace{\eta_{12}^A, \eta_{13}^A, \ldots, \eta_{1,2N}^A, \eta_{23}^A, \ldots, \eta_{2N-1,2N}^A}_{N(2N-1)}, \underbrace{\eta_{12}^B, \eta_{13}^B, \ldots, \eta_{1,2N}^B, \eta_{23}^B, \ldots, \eta_{2N-1,2N}^B}_{N(2N-1)}, \underbrace{1, \ldots, 1}_{2N-1}). \tag{20}$$

The $\bar{L}_i$'s defined in Eq. (16) and the unit matrix $\bar{I}$ satisfy the following orthogonality relations:

$$\mathrm{Tr}\{\bar{L}_i^\dagger \bar{L}_j\} = 2\delta_{ij}, \quad \mathrm{Tr}\{\bar{L}_i^\dagger \bar{I}\} = \mathrm{Tr}\{\bar{I}^\dagger \bar{L}_i\} = 0, \quad \mathrm{Tr}\{\bar{I}^\dagger \bar{I}\} = 2N, \tag{21}$$

which will be used to prove Eq. (63) at the end of the following section.

We now turn to the physical application of the $su(2)$ and $su(1,1)$ groups/algebras. We start from a two-level quantum system, where the Hilbert space is spanned by two orthonormal states $|1\rangle$ and $|2\rangle$. The four operators $|i\rangle\langle j|$ $(i,j=1,2)$ form a complete basis for the linear operators of the system in the sense that any linear operator of the system can be written as a linear superposition (with complex coefficients) of these four operators. Equivalently, one can use the basis

$$\hat{L}_1^{(2)} = \{|1\rangle\langle 2| + |2\rangle\langle 1|\}, \quad \hat{L}_2^{(2)} = -i\{|1\rangle\langle 2| - |2\rangle\langle 1|\},$$

$$\hat{L}_3^{(2)} = |1\rangle\langle 1| - |2\rangle\langle 2|, \quad \hat{\aleph}^{(2)} = |1\rangle\langle 1| + |2\rangle\langle 2|, \qquad (22)$$

which has the advantage of revealing the $su(2)$ dynamical structure of the system. To see this, we first note that the $\{\hat{L}_j^{(2)}; j=1,2,3\}$ in Eq. (22) satisfy the same commutation relations as the generators of the $SU(2)$ group [see Eq. (5)]. Thus they form a representation of the set of generators, Eq. (3), and are a basis of the $su(2)$ algebra. Moreover, the $\hat{L}_j^{(2)}$'s commute with $\hat{\aleph}^{(2)}$:

$$[\hat{L}_j^{(2)}, \hat{\aleph}^{(2)}] = 0. \qquad (23)$$

We have already mentioned that, because of the completeness of the basis states $|1\rangle$ and $|2\rangle$, any linear operator of the system, including the Hamiltonian, can be written as linear superposition of the $\hat{L}_j^{(2)}$'s and $\hat{\aleph}^{(2)}$:

$$\hat{O} = \sum_{j=1}^{3} o_j \hat{L}_j^{(2)} + o_0 \hat{\aleph}^{(2)}, \quad \hat{H} = \sum_{j=1}^{3} h_j \hat{L}_j^{(2)} + h_0 \hat{\aleph}^{(2)}. \qquad (24)$$

In other words, any linear operator of the system is a superposition of the $\hat{L}_j^{(2)}$'s and $\hat{\aleph}^{(2)}$ with complex coefficients, and the (Hermitian) Hamiltonian is a superposition of the $\hat{L}_j^{(2)}$'s and $\hat{\aleph}^{(2)}$ with real coefficients. Equation (23) means that for the most general Hamiltonian of a two-level system, which is a linear combination of the $\hat{L}_j^{(2)}$ and $\hat{\aleph}^{(2)}$ with real coefficients, the $\hat{\aleph}^{(2)}$ part does not contribute to the equations of motion, because it commutes with any linear operator of the system. $\hat{\aleph}^{(2)}$ itself is a constant of motion

$$\frac{d}{dt}\hat{\aleph}^{(2)} = 0. \qquad (25)$$

Since the operators $\hat{L}_j^{(2)}$ defined in Eq. (22) also satisfy the same commutation relations as those given in Eq. (5), they form an operator representation of the basis of the $su(2)$ algebra. Now, other than a term proportional to $\hat{\aleph}^{(2)}$, which commutes with any linear operator of the system, the Hamiltonian is a linear combination with real coefficients of the three $su(2)$ generators. Thus we conclude that the dynamical algebra of the two-level atomic system is $su(2)$.

Equation (22) is the operator representation of $su(2)$ algebra in the first quantization picture. Going to the second quantization picture, we replace the Dirac states by creation and annihilation operator (which can either be bosonic or fermionic):

$$\hat{L}_1^{(2)\prime} = \hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1, \quad \hat{L}_2^{(2)\prime} = (-i)\{\hat{b}_1^\dagger \hat{b}_2 - \hat{b}_2^\dagger \hat{b}_1\},$$

$$\hat{L}_3^{(2)\prime} = \hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2, \quad \hat{\aleph}^{(2)\prime} = \hat{b}_1^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2. \qquad (26)$$

In this way, we construct an operator representation $\{\hat{L}_j^{(2)\prime}\}$ for the $su(2)$ algebra in Fock space, all of which commute with $\hat{\aleph}^{(2)\prime}$. Note that, although we start from a two-level representation and then pass to the second quantization picture, the operators in Eq. (26) are actually defined in a higher-dimensional Hilbert space (Fock space). This operator representation has its application in the Hartree-Fock theory of electron-hole systems in semiconductors, which can be treated as a coupled ensemble of $su(2)$ subsystems analogous to two-level atoms: in the Hartree-Fock theory, for each center-of-mass momentum **k** the corresponding "two levels" of an electron are the conduction and valence bands.

The $SU(1,1)$ case is more complicated. We note that the two-dimensional matrix representation (i.e., the original definition) in Eq. (8) of the $SU(1,1)$ group is *not* unitary. Actually, because of the noncompactness of the group $SU(1,1)$, there is no finite-dimensional unitary representation for it;[33] as a consequence there is no finite-dimensional Hermitian representation for the $su(1,1)$ algebra. Since in standard quantum mechanics, the Hamiltonian is Hermitian, we do not expect the dynamical algebra of a finite-dimensional quantum system to be $su(1,1)$. However, in the infinite-dimensional bosonic Fock space, we do have a unitary/Hermitian representation for the $su(1,1)$ group /algebra. For example, in quantum optics it is well known that there are two-mode electromagnetic fields whose Hamiltonian, written in terms of bosonic creation and annihilation operators, is a linear combination of the $su(1,1)$ generators[34]

$$\hat{L}_1^{(11)} = i\{\hat{b}_1^\dagger \hat{b}_2^\dagger - \hat{b}_2 \hat{b}_1\}, \quad \hat{L}_2^{(11)} = \hat{b}_1^\dagger \hat{b}_2^\dagger + \hat{b}_2 \hat{b}_1,$$

$$\hat{L}_3^{(11)} = \hat{b}_1^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2 + 1, \quad \hat{\aleph}^{(11)} = \hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2. \qquad (27)$$

Again, the $\hat{L}_j^{(11)}$'s defined in these equations all commute with $\hat{\aleph}^{(11)}$, and satisfy the commutation relations of the $su(1,1)$ generators (10), thus giving a *Hermitian* operator representation of the $su(1,1)$ algebra.

The above discussions can also be generalized to $su(N)$ and $su(M,N)$, respectively. For the purpose of this paper, we only write down the operator representation of the $su(N,N)$ generators in Fock space. Given two sets of bosonic creation and annihilation operators $\{\hat{a}_u, \hat{a}_u^\dagger\}, \{\hat{c}_u, \hat{c}_u^\dagger\}$ $(u=1,2,\ldots,N)$ that fulfill the commutation relations

$$[\hat{a}_u, \hat{a}_{u'}^\dagger] = \delta_{uu'}, \quad [\hat{c}_u, \hat{c}_{u'}^\dagger] = \delta_{uu'}, \quad \text{others} = 0, \qquad (28)$$

we can construct the operator representation for the basis of the $su(N,N)$ algebra, for which we already have given a matrix representation [see Eq. (16)]. The operator representation can be written as

$$\vec{\hat{L}} = (\overbrace{\hat{A}_{(12)}, \hat{A}_{(13)}, \ldots, \hat{A}_{(1,2N)}, \hat{A}_{(23)}, \ldots, \hat{A}_{(2N-1,2N)}}^{N(2N-1)},$$

$$\underbrace{\hat{B}_{(12)}, \hat{B}_{(13)}, \ldots, \hat{B}_{(1,2N)}, \hat{B}_{(23)}, \ldots, \hat{B}_{(2N-1,2N)}}_{N(2N-1)},$$

$$\underbrace{\hat{C}_{(1)}, \ldots, \hat{C}_{(2N-1)}}_{2N-1}) \qquad (29)$$

with

$$\hat{A}_{(mn)} = \eta^{A}_{mn}(\hat{P}_{(mn)} + \hat{P}_{(nm)}), \quad 1 \le m < n \le 2N,$$

$$\hat{B}_{(mn)} = \eta^{B}_{mn}(-i)(\hat{P}_{(mn)} - \hat{P}_{(nm)}), \quad 1 \le m < n \le 2N,$$

$$\hat{C}_{(r)} = \sqrt{\frac{2}{r(r+1)}}(\hat{P}_{(11)} + \hat{P}_{(22)} + \cdots + \hat{P}_{(rr)} - r\hat{P}_{(r+1\ r+1)}),$$
$$1 \le r \le 2N-1, \tag{30}$$

where $\eta^{A/B}_{mn}$ is defined in Eq. (18) and

$$\hat{P}_{(mn)} = \begin{cases} \hat{a}^{\dagger}_{m}\hat{a}_{n}, & 1 \le m \le N, \quad 1 \le n \le N \\ \hat{a}^{\dagger}_{m}\hat{c}^{\dagger}_{n-N}, & 1 \le m \le N, \quad N+1 \le n \le 2N \\ -\hat{c}_{m-N}\hat{a}_{n}, & N+1 \le m \le 2N, \quad 1 \le n \le N \\ -\hat{c}_{m-N}\hat{c}^{\dagger}_{n-N}, & N+1 \le m \le 2N, \quad N+1 \le n \le 2N. \end{cases} \tag{31}$$

One can prove that the operators in Eq. (29) satisfy the same commutation relations as the matrices in Eq. (16):

$$[\hat{L}_h, \hat{L}_j] = 2i \sum_{l=1}^{(2N)^2-1} \xi_{hjl} \frac{\eta_h\ \eta_j}{\eta_l} \hat{L}_l. \tag{32}$$

Thus the operators in Eq. (29) are actually the operator representation of the elements in Eq. (16) and form a basis of the $su(N,N)$ algebra in Fock space. Note that the operators in Eq. (29) are Hermitian because of the unitarity of the operator representation of the $SU(N,N)$ group. Also all the operators in Eq. (29) commute with

$$\hat{\aleph} = \sum_{m=1}^{2N} \hat{P}_{(mm)} + N = \sum_{j=1}^{N} [\hat{a}^{\dagger}_{j}\hat{a}_{j} - \hat{c}^{\dagger}_{j}\hat{c}_{j}]. \tag{33}$$

We will use this representation when discussing the dynamical algebra of the N-species exciton mean-field equations in the following section.

III. THE $su(N,N)$ DYNAMICAL STRUCTURE OF THE BOSONIC MEAN-FIELD EQUATIONS FOR EXCITONS

In this section, we show that the effective mean-field Hamiltonian of an N-species exciton system in a semiconductor quantum well is a sum of sub-Hamiltonians with $su(N,N)$ dynamical algebras. This leads to the equations of the generalized "Bloch vector" and "density matrix" of the system, both of which are equivalent to the mean-field equations obtained directly from the original Hamiltonian. They can be used to study the conserved quantities associated with the $su(N,N)$ algebras, as will be discussed in the following section.

As mentioned in the Introduction, the excitons are treated as exact bosons with the exchange effects of fermionic constituents being included in the Hamiltonian (i.e., in the boson-boson interaction and boson-photon coupling). The bosonic Hamiltonian has been designed in a way that it yields the same nonlinear interband polarization in the third-order regime as a more rigorous fermionic theory which is approximated to account only for 1s excitons. This way of constructing the bosonic exciton Hamiltonian yields explicit expressions for the interaction and coupling matrix elements. However, the discussion in this paper does not depend on the details of the matrix elements in the Hamiltonian. Therefore, we will base our discussion on a Hamiltonian of the following general form

$$\hat{H} = \hat{H}^{(1)} + \hat{H}^{(2)} + \hat{H}^{(3)} + \hat{H}^{(4)},$$

$$\hat{H}^{(2)} = \sum_{j,\mathbf{k}_1} \varepsilon_j(\mathbf{k}_1)\hat{b}^{\dagger}_{j}(\mathbf{k}_1)\hat{b}_j(\mathbf{k}_1),$$

$$\hat{H}^{(4)} = \frac{1}{2}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q},\alpha} \{V_{j_1 j_2 j'_2 j'_1}(\mathbf{k}+\mathbf{q},\mathbf{k}')\hat{b}^{\dagger}_{j_1}(\mathbf{k}_1+\mathbf{q})$$
$$\times \hat{b}^{\dagger}_{j_2}(\mathbf{k}_2-\mathbf{q})\hat{b}'_{j_2}(\mathbf{k}_2)\hat{b}'_{j_1}(\mathbf{k}_1),$$

$$\hat{H}^{(1)} = \sum_{j} \{\Omega_j(t)\hat{b}^{\dagger}_{j}(\mathbf{0}) + \Omega^{*}_{j}(t)\hat{b}_j(\mathbf{0})\},$$

$$\hat{H}^{(3)} = \sum_{\mathbf{k}_1,\mathbf{q},\alpha} \{J_{j_1 j_2 j'_2 j'_1}(\mu_{j_2 j_1}\mathbf{k}_1+\mathbf{q},\mu_{j'_2 j'_1}\mathbf{k}_1)$$
$$\times \hat{b}^{\dagger}_{j_1}(\mathbf{k}_1+\mathbf{q})\hat{b}^{\dagger}_{j_2}(-\mathbf{q})\Omega_{j'_2}(t)\hat{b}_{j'_1}(\mathbf{k}_1) + \text{H.c.}\},$$

$$\mathbf{k} \equiv \mu_{j_2 j_1}\mathbf{k}_1 - \mu_{j_1 j_2}\mathbf{k}_2, \quad \mathbf{k}' \equiv \mu_{j'_2 j'_1}\mathbf{k}_1 - \mu_{j'_1 j'_2}\mathbf{k}_2,$$

$$\mu_{jj'} \equiv \frac{m_j}{m_j + m_{j'}}, \quad \alpha \equiv \{j_1, j_2, j'_1, j'_2\}. \tag{34}$$

Here, m_j is mass of the exciton of species j, and $\hat{b}$ and $\hat{b}^{\dagger}$ are bosonic exciton annihilation and creation operators, respectively:

$$[\hat{b}_j(\mathbf{k}), \hat{b}^{\dagger}_{j'}(\mathbf{k}')] = \delta_{jj'}\delta_{\mathbf{kk}'},$$

$$[\hat{b}_j(\mathbf{k}), \hat{b}_{j'}(\mathbf{k}')] = [\hat{b}^{\dagger}_{j}(\mathbf{k}), \hat{b}^{\dagger}_{j'}(\mathbf{k}')] = 0. \tag{35}$$

In Hamiltonian (34), $\hat{H}_1$ is the dipole coupling to the light field, $\hat{H}_2$ is the kinetic energy, $\hat{H}_3$ is the exchange correction to $\hat{H}_1$, and $\hat{H}_4$ is the boson-boson interaction. Furthermore, ε is the single-particle kinetic energy, V is the two-particle interaction matrix, $\Omega/\hbar$ is the Rabi frequency of the light field, and J is the third-order correction of the exciton-photon coupling coefficient related to the four-exciton wave-function overlap matrix. The single exciton states are labeled by in-plane wave vectors or momenta ($\mathbf{k}$'s and $\mathbf{q}$'s) and internal quantum numbers (j's); the zero momenta in $\hat{H}_1$ reflects the fact that we are dealing with normal-incidence irradiation of the quantum well, i.e., the light creates only excitons with zero in-plane momentum.

The dynamical variables of the system are constructed to be consistent with (in-plane) momentum conservation

$$b_j(t) \equiv \langle \hat{b}_j(\mathbf{k}=\mathbf{0})\rangle,$$

$$F_{j_1j_2}(\mathbf{k},t) \equiv \langle \hat{b}^\dagger_{j_2}(\mathbf{k})\hat{b}_{j_1}(\mathbf{k})\rangle - \langle \hat{b}^\dagger_{j_2}(\mathbf{0})\rangle\langle \hat{b}_{j_1}(\mathbf{0})\rangle \delta_{\mathbf{k0}},$$

$$F^a_{j_1j_2}(\mathbf{k},t) \equiv \langle \hat{b}_{j_2}(-\mathbf{k})\hat{b}_{j_1}(\mathbf{k})\rangle - \langle \hat{b}_{j_2}(\mathbf{0})\rangle\langle \hat{b}_{j_1}(\mathbf{0})\rangle \delta_{\mathbf{k0}}, \quad (36)$$

where $b(t)$ is related to the coherent exciton amplititude (interband polarization), F is related to the optically inactive exciton density, and F^a is related to the coherent biexciton amplitude. The mean-field equations for these variables can be obtained by using the Heisenberg equation ($i\hbar\partial/\partial t\hat{O} = [\hat{O},\hat{H}]$ for any operator $\hat{O}$) and the random-phase approximation, as illustrated in Ref. 26: first calculate the commutators on the right-hand sides of the Heisenberg equations for $\hat{b}_j(\mathbf{k}=\mathbf{0})$, $\hat{b}^\dagger_{j_2}(\mathbf{k})\hat{b}_{j_1}(\mathbf{k})$, and $\hat{b}_{j_2}(-\mathbf{k})\hat{b}_{j_1}(\mathbf{k})$, then take the expectation values of the equations and factorize the right-hand sides into products of the variables defined in Eq. (36), consistent with the momentum conservation. However, it can be equivalently treated in another way: we first linearize the Hamiltonian, Eq. (34), into products of the variables in Eq. (36) and one or two creation/annihilation operators. [The momentum conservation requires that these operators can only be $\hat{b}_j(\mathbf{k}=\mathbf{0}), \hat{b}^\dagger_{j_2}(\mathbf{k})\hat{b}_{j_1}(\mathbf{k}), \hat{b}_{j_2}(-\mathbf{k})\hat{b}_{j_1}(\mathbf{k})$.] Then we put this linearized Hamiltonian in the Heisenberg equations and take the expectation values of these equations. These two ways of obtaining the mean-field equations are equivalent, but the latter can give more insight into the symmetry properties and related conservation laws.

In this paper we focus on the equations for $F_{j_1j_2}(\mathbf{k})$ and $F^a_{j_1j_2}(\mathbf{k})$ with nonzero $\mathbf{k}$, since, as we will see later, the main point of this paper does not concern the equation of b_j. In other words, b_j can be formally taken as an external parameter. Among all the terms in the linearized Hamiltonian, only those containing two operators contribute to commutators in the Heisenberg equations for $\hat{b}^\dagger_{j_2}(\mathbf{k})\hat{b}_{j_1}(\mathbf{k})$ and $\hat{b}_{j_2}(-\mathbf{k})\hat{b}_{j_1}(\mathbf{k})$, since any single operator term is proportional to $\hat{b}_j(\mathbf{k}=\mathbf{0})$ which must commute with operators with nonzero momenta. This fact, together with the arguments on the linearized Hamiltonian in last paragraph, we have the following effective Hamiltonian for the nonzero momentum excitons:

$$\begin{aligned}\hat{H}_{eff} &= \sum_{j_1j_2,\mathbf{k}} \Lambda_{j_1j_2}(\mathbf{k},t)\hat{b}^\dagger_{j_1}(\mathbf{k})\hat{b}_{j_2}(\mathbf{k}) \\ &\quad + \sum_{j_1j_2,\mathbf{k}} \{\Lambda^{a\prime *}_{j_1j_2}(\mathbf{k},t)\hat{b}_{j_1}(\mathbf{k})\hat{b}_{j_2}(-\mathbf{k}) + \text{H.c.}\} \\ &= \sum_{j_1j_2,(\mathbf{k})'} \{\Lambda_{j_1j_2}(\mathbf{k},t)\hat{b}^\dagger_{j_1}(\mathbf{k})\hat{b}_{j_2}(\mathbf{k}) \\ &\quad + \Lambda_{j_1j_2}(-\mathbf{k},t)\hat{b}^\dagger_{j_1}(-\mathbf{k})\hat{b}_{j_2}(-\mathbf{k}) \\ &\quad + \Lambda^{a*}_{j_1j_2}(\mathbf{k},t)\hat{b}_{j_1}(\mathbf{k})\hat{b}_{j_2}(-\mathbf{k}) \\ &\quad + \Lambda^{a}_{j_1j_2}(\mathbf{k},t)\hat{b}^\dagger_{j_1}(\mathbf{k})\hat{b}^\dagger_{j_2}(-\mathbf{k})\}, \end{aligned} \quad (37)$$

where $(\mathbf{k})'$ means that the summation is restricted to the upper half of the $\mathbf{k}$ "plane" (i.e., $\{\mathbf{k}|k_x \geqslant 0\}$), and

$$\begin{aligned}\Lambda_{j_1j_2}(\mathbf{k},t) &= \varepsilon_{j_1}(\mathbf{k})\delta_{j_1j_2} + \sum_{\mathbf{q}\ j\ j'} \{V_{j_1j'jj_2}(\mu_{j'j_1}\mathbf{k} - \mu_{j_1j'}\mathbf{q}, \mu_{jj_2}\mathbf{k} \\ &\quad - \mu_{j_2j}\mathbf{q}) + V_{j_1j'j_2j}(\mu_{j'j_1}\mathbf{k} - \mu_{j_1j'}\mathbf{q}, \mu_{j_2j}\mathbf{q} - \mu_{jj_2}\mathbf{k})\} \\ &\quad \times F_{jj'}(\mathbf{q},t) + \sum_{j\ j'} \{V_{j_1j'jj_2}(\mu_{j'j_1}\mathbf{k}, \mu_{jj_2}\mathbf{k}) \\ &\quad + V_{j_1j'j_2j}(\mu_{j'j_1}\mathbf{k}, -\mu_{j_2j}\mathbf{k})\} b^*_{j'}(t) b_j(t) \\ &\quad + 2\sum_{j\ j'} \{J_{j_1j'jj_2}(\mu_{j'j_1}\mathbf{k}, \mu_{jj_2}\mathbf{k}) b^*_{j'}(t)\Omega_j(t) \\ &\quad + J^*_{j_2j'jj_1}(\mu_{j'j_2}\mathbf{k}, \mu_{jj_1}\mathbf{k}) b_{j'}(t)\Omega^*_j(t)\},\end{aligned}$$

$$\begin{aligned}\Lambda^{a\prime}_{j_1j_2}(\mathbf{k},t) = \frac{1}{2}\Bigg\{ &\sum_{\mathbf{q}} V_{j_1j_2j'j}(\mathbf{k},\mathbf{q}) F^a_{jj'}(\mathbf{q},t) \\ &+ \sum_{jj'} V_{j_1j_2j'j}(\mathbf{k},\mathbf{0}) b_j(t) b'_j(t) \\ &+ 2\sum_{jj'} J_{j_1j_2j'j}(\mathbf{k},\mathbf{0})\Omega_{j'}(t) b_j(t)\Bigg\},\end{aligned}$$

$$\Lambda^a_{j_1j_2}(\mathbf{k},t) = \Lambda^{a\prime}_{j_1j_2}(\mathbf{k},t) + \Lambda^{a\prime}_{j_1j_2}(-\mathbf{k},t). \quad (38)$$

This leads to the following mean-field equations for F and F^a

$$i\hbar\frac{\partial}{\partial t}F(\mathbf{k},t) = [\Lambda(\mathbf{k},t),F(\mathbf{k},t)] + \Lambda^a(\mathbf{k},t)F^{a\dagger}(\mathbf{k},t) - F^a(\mathbf{k},t)\Lambda^{a\dagger}(\mathbf{k},t),$$

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}F^a(\mathbf{k},t) &= \Lambda(\mathbf{k},t)F^a(\mathbf{k},t) + F^a(\mathbf{k},t)\Lambda^T(-\mathbf{k},t) \\ &\quad + \Lambda^a(\mathbf{k},t)F^T(-\mathbf{k},t) + F(\mathbf{k},t)\Lambda^a(\mathbf{k},t) + \Lambda^a(\mathbf{k},t),\end{aligned} \quad (39)$$

where F, F^a, Λ and Λ^a are matrices with elements defined in Eqs. (36) and (38). The superscripts † and T denote the Hermitian conjugate and transposition, respectively. From the definitions of these matrices we have

$$F^\dagger(\mathbf{k},t) = F(\mathbf{k},t), \quad \Lambda^\dagger(\mathbf{k},t) = \Lambda(\mathbf{k},t),$$

$$F^{a\ T}(-\mathbf{k},t) = F^a(\mathbf{k},t), \quad \Lambda^{a\ T}(-\mathbf{k},t) = \Lambda^a(\mathbf{k},t). \quad (40)$$

The Hamiltonian given in Eq. (37) is a sum of sub-Hamiltonians of different $\mathbf{k}$ pairs

$$\hat{H}_{eff} = \sum_{(\mathbf{k})'} \hat{H}_{eff}(\mathbf{k})$$

$$\begin{aligned}\hat{H}_{eff}(\mathbf{k}) &= \sum_{j_1j_2} \{\Lambda_{j_1j_2}(\mathbf{k},t)\hat{b}^\dagger_{j_1}(\mathbf{k})\hat{b}_{j_2}(\mathbf{k}) + \Lambda_{j_1j_2}(-\mathbf{k},t)\hat{b}^\dagger_{j_1}(-\mathbf{k}) \\ &\quad \times \hat{b}_{j_2}(-\mathbf{k}) + \Lambda^{a*}_{j_1j_2}(\mathbf{k},t)\hat{b}_{j_1}(\mathbf{k})\hat{b}_{j_2}(-\mathbf{k}) \\ &\quad + \Lambda^a_{j_1j_2}(\mathbf{k},t)\hat{b}^\dagger_{j_1}(\mathbf{k})\hat{b}^\dagger_{j_2}(-\mathbf{k})\}.\end{aligned} \quad (41)$$

Each $\hat{H}_{eff}(\mathbf{k})$ in Eq. (41) is a linear combination of the following operators:

$$\hat{b}^\dagger_{j_1}(\mathbf{k})\hat{b}_{j_2}(\mathbf{k}), \quad \hat{b}^\dagger_{j_1}(-\mathbf{k})\hat{b}_{j_2}(-\mathbf{k}),$$

$$\hat{b}_{j_2}(-\mathbf{k})\hat{b}_{j_2}(\mathbf{k}), \quad \hat{b}^\dagger_{j_1}(\mathbf{k})\hat{b}^\dagger_{j_2}(-\mathbf{k}). \tag{42}$$

With the substitution

$$\hat{a}_j \equiv \hat{b}_j(\mathbf{k}), \quad \hat{c}_j \equiv \hat{b}_j(-\mathbf{k}) \tag{43}$$

it can be shown that, *for each* **k** *pair*, every operator in Eq. (42) is linear superposition of the $su(N,N)$ generators in Eq. (29) and $\hat{\aleph}$ in Eq. (33) (plus an additive term proportional to the identity operator). Therefore, the Hamiltonian given in Eq. (41) can be written as

$$\hat{H}_{eff}(\mathbf{k}) = \frac{\hbar}{2}\sum_{j=1}^{(2N)^2-1}\Pi_j(\mathbf{k},t)\hat{L}_j(\mathbf{k}) + \frac{\hbar}{2}\Pi_0(\mathbf{k},t)\hat{\aleph}(\mathbf{k}) + \gamma(\mathbf{k},t)\hat{I}. \tag{44}$$

The Hermiticity of the Hamiltonian requires that all the coefficients (Π_j,Π_0,γ) in Eq. (44) are real. Moreover, since $\hat{\aleph}(\mathbf{k})$ and c-number $\gamma(\mathbf{k},t)$ commute with all $\hat{L}_j(\mathbf{k})$'s, the last two terms in Hamiltonian (44) do not contribute to the dynamical equations of the operators in Eq. (42) and can therefore be ignored. The effective mean-field Hamiltonian is now

$$\hat{H}_{eff}(\mathbf{k}) = \frac{\hbar}{2}\sum_{j=1}^{(2N)^2-1}\Pi_j(\mathbf{k},t)\hat{L}_j(\mathbf{k}). \tag{45}$$

It is obvious that this Hamiltonian has the dynamical algebra of $su(N,N)$.

In the remainder of this section we cast the mean-field equation (39) for F and F^a into two other equivalent forms, the generalized Bloch vector and the generalized density matrix equations. Both of them will be found useful in identifying conservation laws inherent in the mean-field equations (39). This is in complete analogy to the case of the N-level atom, where Hioe and Eberly derived conservation laws from the Bloch vector and the Liouville equations.[28]

We begin with deriving the Bloch vector equation. We will define the generalized Bloch vector, analogous to the definition in Ref. 28 for the atomic N-level system. For each $\vec{k}$-pair, the set of $\hat{L}_j(\mathbf{k})$ and $\hat{\aleph}(\mathbf{k})$ obtained from substituting Eq. (43) into Eqs. (29)–(31) satisfy the same commutation relations as those in Eq. (32):

$$[\hat{L}_h(\mathbf{k}),\hat{L}_j(\mathbf{k})] = 2i\sum_{l=1}^{(2N)^2-1}\xi_{hjl}\frac{\eta_h\ \eta_j}{\eta_l}\hat{L}_l(\mathbf{k}), \quad [\hat{L}_j(\mathbf{k}),\hat{\aleph}(\mathbf{k})] = 0. \tag{46}$$

This implies

$$[\hat{L}_j(\mathbf{k}),\hat{H}_{eff}(\mathbf{k})] = i\hbar\sum_{l,h=1}^{(2N)^2-1}\xi_{jhl}\frac{\eta_j\eta_h}{\eta_l}\Pi_h(\mathbf{k},t)\hat{L}_l(\mathbf{k}), \tag{47}$$

$$[\hat{\aleph}(\mathbf{k}),\hat{H}_{eff}(\mathbf{k})] = 0. \tag{48}$$

The mean-field equations for $\hat{\mathbf{L}}$ and $\hat{\aleph}$ are then easily obtained by using the Heisenberg equations:

$$\frac{\partial}{\partial t}\langle\hat{\aleph}(\mathbf{k})\rangle = 0, \tag{49}$$

$$\frac{\partial}{\partial t}\langle\hat{L}_j(\mathbf{k})\rangle = \sum_{l,h=1}^{(2N)^2-1}\xi_{jhl}\Pi_h(\mathbf{k},t)\frac{\eta_j\ \eta_h}{\eta_l}\langle\hat{L}_l(\mathbf{k})\rangle. \tag{50}$$

To get a completely antisymmetric equation, we define

$$\hat{S}_j(\mathbf{k}) = \frac{\hat{L}_j(\mathbf{k})}{\eta_j}, \quad \Gamma_j(\mathbf{k},t) = \eta_j\Pi_j(\mathbf{k},t). \tag{51}$$

Substituting Eq. (51) into Eq. (50) leads to

$$\frac{\partial}{\partial t}\langle\hat{S}_j(\mathbf{k})\rangle = \sum_{l,h=1}^{(2N)^2-1}\xi_{jhl}\Gamma_h(\mathbf{k},t)\langle\hat{S}_l(\mathbf{k})\rangle. \tag{52}$$

Equation (52) [together with Eq. (49)] is the antisymmetric form of the mean-field equations for F and F^a. It is the bosonic counterpart of Bloch vector equations in the atomic N-level case (for example, in the two-level case it is the equation that has the familiar "torque" structure $d\vec{s}/dt = \vec{\gamma}\times\vec{s}$, which is crucial to the understanding of the time evolution of the system[28]). Equation (52) allows us to introduce the concept of the generalized Bloch vector $\langle\vec{\hat{S}}(\mathbf{k})\rangle$ in the case of excitonic mean-field dynamics. We note that the fermionic counterpart of the semiconductor generalization of the Bloch vector has been discussed in the context of the Hartree-Fock approximation (semiconductor Bloch equations) (see, for example, Refs. 35 and 36). There, the dynamics of the generalized Bloch vector is governed by a torque equation analogous to that of a two-level atom with the familiar $SU(2)$ as the underlying symmetry group.

One conservation laws follows, in an obvious way, immediately from Eq. (49) [indeed, Eq. (49) states the conservation of the quantity $\langle\hat{\aleph}(\mathbf{k})\rangle$]. Another conservation law can be obtained from the antisymmetry of Eq. (52), which yields

$$\frac{\partial}{\partial t}\sum_{i=1}^{(2N)^2-1}\langle\hat{S}_i(\mathbf{k})\rangle\langle\hat{S}_i(\mathbf{k})\rangle = 0. \tag{53}$$

Equation (53) says that the generalized Bloch vector $\langle\vec{\hat{S}}(\mathbf{k})\rangle$ moves on a hypersurface

$$\sum_{i=1}^{(2N)^2-1}\langle\hat{S}_i(\mathbf{k})\rangle\langle\hat{S}_i(\mathbf{k})\rangle = \text{const}. \tag{54}$$

The properties of this hypersurface are very different from those in the atomic N-level case, because some components of $\langle\vec{\hat{S}}(\mathbf{k})\rangle$ are imaginary. Instead of a hypersphere as in the atomic N-level case, the hypersurface where the generalized Bloch vector for excitons, $\langle\vec{\hat{S}}(\mathbf{k})\rangle$, moves is *unbounded*. We will discuss the structure of this surface more specifically at the end of the following section.

Equations (49) and (53) are the two lowest order conserved quantities associated with the $su(N,N)$ symmetry. In order to obtain higher order constants of motion, we make use of the $2N\times 2N$ density matrix formalism analogous to that of the N-level atomic system $(su(N))$.[28] To this end, we note that the $\hat{L}_j$'s in Eq. (16) and the L_j's in Eq. (29) are both representations for the $su(N,N)$ algebra. The former is a representation in the infinite-dimensional Fock space while the latter is a representation in a $2N$-dimensional space. For reasons that will become apparent at the end of this section, we can *define* a matrix (which we will call "quasi-Hamiltonian") in the following way. We replace the $\hat{L}_j$'s in the original Hamiltonian [Eq. (45)] by their counterparts of $2N\times 2N$ matrices. For practical purposes, namely to make the explicit form of this quasi-Hamitonian simpler [see Eq. (64) in the following section], we also add a term proportional to the unit matrix $\overline{I}$. We then have the definition

$$\overline{M}_H(\mathbf{k}) = \frac{\hbar}{2}\sum_{j=1}^{(2N)^2-1}\Pi_j(\mathbf{k},t)\overline{L_j} + \frac{\hbar}{2}\Pi_0(\mathbf{k},t)\overline{I}. \tag{55}$$

Clearly, $\overline{M}_H(\mathbf{k})$, just as the $\overline{L}_j$'s, is a $2N\times 2N$ matrix. This quasi-Hamiltonian will play an important role in the $2N\times 2N$ density matrix equation we are looking for. Indeed, we will show now that the Heisenberg equations [Eq. (50)] can be cast into a form similar to a Liouville equation, in which $\overline{M}_H(\mathbf{k})$ plays the role of the Hamiltonian.

The commutation between $\overline{L}_j$'s and $\overline{M}_H$ are the same as those given in Eq. (47)

$$[\overline{L}_j,\overline{M}_H(\mathbf{k})] = i\hbar\sum_{l,h=1}^{(2N)^2-1}\xi_{jhl}\frac{\eta_j\eta_h}{\eta_l}\Pi_h(\mathbf{k},t)\overline{L}_l. \tag{56}$$

We can now use Eq. (50) [together with Eqs. (55) and (56)] to derive the $2N\times 2N$ generalized density matrix equation. Multiplying both sides of Eq. (50) by $\overline{L}_j/\eta_j^2$ and summing over the index j, we have

$$\begin{aligned}\frac{\partial}{\partial t}\sum_{j=1}^{(2N)^2-1}\frac{\langle\hat{L}_j(\mathbf{k})\rangle\overline{L_j}}{\eta_j^2} &= \sum_{j,l,h=1}^{(2N)^2-1}\xi_{jhl}\Pi_h(\mathbf{k},t)\frac{\eta_h}{\eta_j\eta_l}\langle\hat{L}_l(\mathbf{k})\rangle\overline{L_j}\\ &= \sum_{l=1}^{(2N)^2-1}\frac{\langle\hat{L}_l(\mathbf{k})\rangle}{\eta_l^2}\sum_{j,h=1}^{(2N)^2-1}(-\xi_{lhj})\\ &\quad\times\Pi_h(\mathbf{k},t)\frac{\eta_l\ \eta_h}{\eta_j}\overline{L_j}\\ &= \sum_{l=1}^{(2N)^2-1}\frac{\langle\hat{L}_l(\mathbf{k})\rangle}{\eta_l^2}\frac{1}{i\hbar}[\overline{M}_H(\mathbf{k}),\overline{L_l}]\\ &= \frac{1}{i\hbar}\left[\overline{M}_H(\mathbf{k}),\ \sum_{l=1}^{(2N)^2-1}\frac{\langle\hat{L}_l(\mathbf{k})\rangle\overline{L_l}}{\eta_l^2}\right].\end{aligned} \tag{57}$$

We now define $\overline{R_0}(\mathbf{k})=\Sigma_{j=1}^{(2N)^2-1}(\langle\hat{L}_j(\mathbf{k})\rangle\overline{L_j}/\eta_j^2)$ which simplifies Eq. (57) to

$$i\hbar\frac{\partial}{\partial t}\overline{R_0}(\mathbf{k}) = [\overline{M}_H(\mathbf{k}),\overline{R_0}(\mathbf{k})]. \tag{58}$$

It is interesting to note that this equation has the form of the Liouville equation for the density matrix

$$i\hbar\frac{d}{dt}\overline{\rho} = [\overline{H},\overline{\rho}] \tag{59}$$

if we formally make the correspondence $\overline{M}_H\leftrightarrow\overline{H}$, $\overline{R}_0\leftrightarrow\overline{\rho}$. However, neither $\overline{R}_0$ nor $\overline{M}_H$ has the standard interpretation as that in the quantum mechanics. Actually, unlike $\overline{H}$ and $\overline{\rho}$ in the density matrix (or Liouville) equation, neither $\overline{R}_0$ nor $\overline{M}_H$ is Hermitian because the $2N\times 2N$ representation of the $su(N,N)$ algebra $(\overline{L}_j)$ is not Hermitian. Thus, Eq. (58) does not define a complete quantum mechanical problem. It is merely a convenient way to rewrite the mean-field equations of motion for the exciton momentum distribution and biexciton amplitude which were orginially given by Eq. (39).

We have derived Eq. (58) from Eq. (50). Conversely, one can also obtain Eq. (50) starting from Eq. (58), thus we conclude that Eq. (58) and Eq. (50) are equivalent to each other.

However, only *together with* Eq. (49) is Eq. (50) [or Eq. (58)] equivalent to the original mean-field equations Eqs. (39) for F and F^a, and it is more convenient to have an equation [having the same form as Eq. (58)] which *itself* is equivalent to Eqs. (39). To this end, we define another density matrix

$$\overline{R}(\mathbf{k}) = \frac{1}{2}\left\{\overline{R_0}(\mathbf{k}) + \left(\frac{\langle\hat{\aleph}(\mathbf{k})\rangle}{N} - 1\right)\overline{I}\right\}. \tag{60}$$

The factor $\frac{1}{2}$ and the constant -1 in this definition are only for practical purposes to simplify the explicit form of $\overline{R}$ [see Eq. (64) in the following section]. Equations (58) and (49) immediately suggest that

$$i\hbar\frac{\partial}{\partial t}\overline{R}(\mathbf{k}) = [\overline{M}_H(\mathbf{k}),\overline{R}(\mathbf{k})]. \tag{61}$$

Although Eq. (61) looks almost the same as Eq. (58), Eq. (61) itself is equivalent to Eqs. (50) and (49). Using the definition for $\overline{R}(\mathbf{k})$ [Eq. (60)] and $\overline{M}_H(\mathbf{k})$ [Eq. (55)], one can prove this by showing that one can obtain Eqs. (50) and (49) starting from Eq. (61). Thus Eq. (61) is equivalent to Eqs. (39).

An immediate result from Eq. (61) is that the trace of $\overline{R}^n$, for any $\mathbf{k}$ and for non-negative integer n, is a constant of motion

$$\frac{\partial}{\partial t}\mathrm{Tr}\{\overline{R^n}\} = 0. \tag{62}$$

This should clarify the reasons for casting the Heisenberg equations of motion into a form analogous to a Liouville equation [i.e., Eq. (61)]. In particular, Eq. (62) suggests up to

$2N$ constants of motion. This is because for $n>2N$ there should be no more independent constants of motion. We will express these constants of motion in terms of physical quantities in the following section.

Using Eq. (21) one can easily show

$$\mathrm{Tr}\{\overline{R}\} = \langle\hat{\aleph}\rangle - N, \quad \mathrm{Tr}\{\overline{R}^2\} = \frac{1}{2}\sum_{i=1}^{(2N)^2-1}\langle\hat{S}_i\rangle\langle\hat{S}_i\rangle + \frac{1}{2N}(\langle\hat{\aleph}\rangle - N)^2. \tag{63}$$

Substituting Eq. (63) into Eq. (62) leads exactly to Eqs. (49) and (53), the two lowest-order constants of motion.

IV. CONSTANTS OF MOTION OF THE MEAN-FIELD EXCITON EQUATIONS ASSOCIATED WITH THE $su(N,N)$ DYNAMICAL ALGEBRA

In this section, we use Eq. (62) to get the explicit forms of some higher order conserved quantities. Up to now, the quantities in Eq. (62) have been defined in an abstract way. Now we will write them down directly in terms of the exciton momentum distribution F and the coherent biexcitation amplitude F^a. Using the definitions of $\overline{R}$, $\overline{M}_H$, F, F^a, Γ, and Γ^a, we obtain the quasi-density matrix and quasi-Hamiltonian matrix entering Eq. (61) as

$$\overline{R}(\mathbf{k}) = \begin{pmatrix} F(\mathbf{k}) & -F^a(\mathbf{k}) \\ F^{a\dagger}(\mathbf{k}) & -F^T(-\mathbf{k}) - 1 \end{pmatrix},$$

$$\overline{M}_H(\mathbf{k}) = \begin{pmatrix} \Lambda(\mathbf{k}) & \Lambda^a(\mathbf{k}) \\ -\Lambda^{a\dagger}(\mathbf{k}) & -\Lambda^T(-\mathbf{k}) \end{pmatrix}. \tag{64}$$

Equation (64) expresses the density matrix and Hamiltonian in terms of the physical quantities. It helps us to make the physical contents of the constants of motion more clear. With Eq. (64) one can also easily check that Eqs. (39) and (61) are equivalent.

We now consider the constants of motion given by Eq. (62). For $n=1$ we have

$$\frac{\partial}{\partial t}\mathrm{Tr}\{\overline{R}(\mathbf{k},t)\} = \frac{\partial}{\partial t}[\mathrm{Tr}\{F(\mathbf{k},t) - F(-\mathbf{k},t)\} - N] = \frac{\partial}{\partial t}\langle\hat{\aleph}(\mathbf{k},t)\rangle = 0. \tag{65}$$

For $n=2$ we have

$$\overline{R}^2(\mathbf{k}) = \begin{pmatrix} F^2(\mathbf{k}) - F^a(\mathbf{k})F^{a\dagger}(\mathbf{k}) & -F(\mathbf{k})F^a(\mathbf{k}) + F^a(\mathbf{k})F^T(-\mathbf{k}) + F^a(\mathbf{k}) \\ F^{a\dagger}(\mathbf{k})F(\mathbf{k}) - F^T(-\mathbf{k})F^{a\dagger}(\mathbf{k}) - F^{a\dagger}(\mathbf{k}) & -F^{a\dagger}(\mathbf{k})F^a(\mathbf{k}) + (F^T(-\mathbf{k}) + I)^2 \end{pmatrix}. \tag{66}$$

Therefore,

$$\frac{\partial}{\partial t}\mathrm{Tr}\{\overline{R}^2(\mathbf{k},t)\} = \frac{\partial}{\partial t}\mathrm{Tr}\{F(\mathbf{k},t)^2 + [F(-\mathbf{k},t) + I]^2 - 2F^{a\dagger}(\mathbf{k},t)F^a(\mathbf{k},t)\} = 0. \tag{67}$$

Note that the above two constants of motion come directly from the $su(N,N)$ symmetry, e.g., the $n=1$ case holds, whether the system has spatial inversion symmetry or not. If the system has spatial reflection invariance

$$F(\mathbf{k}) = F(-\mathbf{k}), \quad F^a(\mathbf{k}) = F^a(-\mathbf{k}) \tag{68}$$

then the $n=1$ case is trivial. For $n=2$, Eq. (67) is in this case simplified to

$$\frac{\partial}{\partial t}\mathrm{Tr}\{F(\mathbf{k},t)(F(\mathbf{k},t) + I) - F^{a\dagger}(\mathbf{k},t)F^a(\mathbf{k},t)\} = 0. \tag{69}$$

The counterpart of this constant of motion is familiar in atomic two-level systems as well as fermionic cases, such as the electron-hole Hartree-Fock equations, also known as semiconductor Bloch equations (see, for example, Refs. 35 and 36 and pp. 323 and 391 of Ref. 37).

For simplicity, we will assume spatial inversion symmetry, Eq. (68), when discussing the higher-order conserved quantities. For $n=3$, the case is again trivial in sense that the second-order constant of motion together with the reflection symmetry gives $\mathrm{Tr}\{\overline{R}^3\}=\text{const}$ (without the inversion symmetry this is not true and the third-order constant of motion is independent of the first two). For $n=4$ a somewhat lengthy but straightforward algebra leads to

$$\frac{\partial}{\partial t}\mathrm{Tr}\{[F(\mathbf{k},t)(F(\mathbf{k},t) + I) - F^a(\mathbf{k},t)F^{a\dagger}(\mathbf{k},t)]^2 - 2F(\mathbf{k},t)F^a(\mathbf{k},t)[F^{a\dagger}(\mathbf{k},t)F(\mathbf{k},t) - F^T(\mathbf{k},t)F^{a\dagger}(\mathbf{k},t)]\} = 0. \tag{70}$$

Similar steps can be used to get all the other higher-order conserved quantities. Note again that only $n\leqslant 2N$ give independent constants of motion.

As an example, we now treat the case where we have only one species of excitons, i.e. $N=1$, in more detail. Equation (69) is then reduced to

$$[1 + f(\mathbf{k},t)]f(\mathbf{k},t) - |f^a(\mathbf{k},t)|^2 = \text{const (independent of time } t), \tag{71}$$

where f and f^a are simplified notations for F_{11} and F^a_{11}, respectively. The counterpart of Eq. (71) in the electron-hole Hartree-Fock system (semiconductor Bloch equations) is well known (again, see, for example, p. 391 of Ref. 37):

$$n_e(\mathbf{k},t)(1 - n_e(\mathbf{k},t)) - |P(\mathbf{k},t)|^2 = \text{const}, \tag{72}$$

where n_e and P are the electron occupation number and interband polarization, respectively. Equation (72) is essentially the same as Eq. (2.37) in Ref. 27 of a two-level atom system

$$u^2(t) + v^2(t) + w^2(t) = 1 \tag{73}$$

if we make the correspondence

$$w \leftrightarrow 2n_e(\mathbf{k}) - 1, \quad u + iv \leftrightarrow 2P(\mathbf{k}) \tag{74}$$

which, as already mentioned, implies that the Hartree-Fock electron-hole system is a coupled ensemble of subsystems (for different **k**'s) analogous to atomic two-level systems.

The sign difference in Eqs. (71) and (72) makes the behaviors of the quantities in these two equations qualitatively different. To see this, we take $(\text{Re}f^a, \text{Im}f^a, f)$ or $(\text{Re}P, \text{Im}P, n_e)$ as the three components of the (x,y,z) coordinates in an abstract three-dimensional space. Then, Eq. (71) says that $(\text{Re}f^a, \text{Im}f^a, f)$ lies on an unbounded hyperboloid, while Eq. (72) says that $(\text{Re}P, \text{Im}P, n_e)$ is restricted to a sphere (which is well known as the "Bloch sphere"). Figure 1 illustrates this qualitative difference graphically. It comes from the qualitatively different dynamical algebras of the two systems: $su(2)$ in the case of the electron-hole system and $su(1,1)$ in the exciton case.

While formally the motion of the generalized excitonic Bloch vector is on an unbounded surface, one should note that physical reasons put a restrictrion on the actual motion on this surface. The underlying assumption of coherent ultrafast exciton dynamics governed by the mean-field equations of motion restricts the validity of the model to relatively small exciton densities, which in turn limits the dynamics of the excitonic Bloch vector to lie within a limited region on the unbounded hyperboloid.

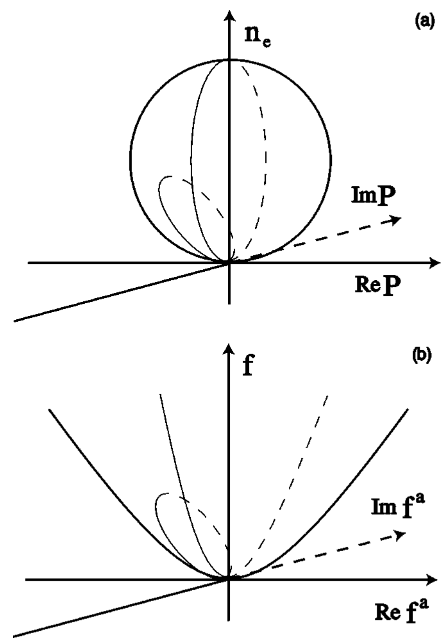

FIG. 1. (a) Schematic of the conventional Bloch sphere restricting the motion of the Bloch vector of two-level atoms and fermionic systems described by the (real) population n_f and the complex polarization P to a sphere [$su(2)$ algebra]. (b) The excitonic analog to the Bloch sphere (a hyperboloid) restricting the motion of the generalized Bloch vector of an excitonic system described by the (real) excitonic population f and the complex two-exciton amplitude f^a to a hyperboloid [$su(1,1)$ algebra].

V. SUMMARY

In summary, we point out that the momentum-conserving exciton mean-field equations, including the coupling to external fields and fermionic corrections, have the dynamical structure $su(N,N)$. We have shown that one can define a non-real generalized Bloch vector and a non-Hermitian density matrix description, which allowed us to investigate the symmetry properties and related conservation laws of the system. An explicit method to obtain all the constants of motion associated with the $su(N,N)$ algebra has been given. The qualitatively different behavior of the generalized Bloch vector of the exciton system, compared to that of the Bloch vector of the electron-hole system, is thus explained by the difference between the dynamical algebras of the two systems. To conclude this paper, we remark that the same method we use in this paper to get the conserved quantities resulting from the dynamical algebra can also be applied to multiband electron-hole systems, of which the dynamical algebra is $su(N)$.

ACKNOWLEDGMENTS

This work was supported by DARPA, ONR, JSOP, and COEDIP.

[1] A. L. Smirl, in *Semiconductor Quantum Electronics: From Quantum Physics to Smart Devices*, edited by A. Miller, M. Ebrahimzadeh, and D. M. Finlayson (Institute of Physics Publishing, London, 1999), pp. 25–94.
[2] T. Meier and S. Koch, *Ultrafast Physical Processes in Semiconductors*, Semiconductors and Semimetals, Vol. 67 (Academic, New York, 2001), pp. 231–313.
[3] D. Chemla and J. Shah, Nature (London) **411**, 549 (2001).
[4] W. Schäfer and M. Wegener, *Semiconductor Optics and Transport Phenomena* (Springer, Berlin, 2002).
[5] V. M. Axt and A. Stahl, Z. Phys. B: Condens. Matter **93**, 195 (1994).
[6] V. M. Axt and A. Stahl, Z. Phys. B: Condens. Matter **93**, 205 (1994).
[7] M. Z. Maialle and L. J. Sham, Surf. Sci. **305**, 256 (1994).
[8] M. Lindberg, Y. Z. Hu, R. Binder, and S. W. Koch, Phys. Rev. B **50**, 18 060 (1994).
[9] K. Victor, V. Axt, and A. Stahl, Phys. Rev. B **51**, 14 164 (1995).

[10] T. Östreich, K. Schönhammer, and L. J. Sham, Phys. Rev. Lett. **74**, 4698 (1995).
[11] T. Östreich, K. Schönhammer, and L. Sham, Phys. Rev. B **58**, 12 920 (1998).
[12] V. M. Axt, K. Victor, and T. Kuhn, Phys. Status Solidi B **206**, 189 (1998).
[13] V. M. Axt and S. Mukamel, J. Nonlinear Opt. Phys. Mater. **101**, 1 (1998).
[14] N. H. Kwong and R. Binder, Phys. Rev. B **61**, 8341 (2000).
[15] S. W. Koch, M. Kira, and T. Meier, J. Opt. B: Quantum Semiclassical Opt. **3**, R29 (2001).
[16] R. Takayama, Eur. Phys. J. B **25**, 445 (2002).
[17] E. Hanamura, J. Phys. Soc. Jpn. **37**, 1545 (1974).
[18] E. Hanamura and H. Haug, Phys. Rep., Phys. Lett. **33**, 209 (1977).
[19] S. Moskalenko and D. W. Snoke, *Bose-Einstein Condensation of Excitons and Biexcitons and Coherent Nonlinear Optics with Excitons* (Cambridge University Press, Cambridge, 2000).
[20] G. Rochat, Phys. Rev. B **61**, 13 856 (2000).
[21] M. Combescot and O. Betbeder-Matibet, Europhys. Lett. **58**, 87 (2002).
[22] S. Okumura and T. Ogawa, Phys. Rev. B **65**, 035105 (2002).
[23] O. Schmitt, L. Banyai, and H. Haug, Phys. Rev. B **60**, 16 506 (1999).
[24] C. Piermarocchi, Phys. Rev. B **53**, 15 834 (1996).
[25] M. Kira, W. Hoyer, T. Stroucken, and S. Koch, Phys. Rev. Lett. **87**, 176401 (2001).
[26] H. Haug and S. W. Koch, *Quantum Theory of the Optical and Electronic Properties of Semiconductors*, 2nd ed. (World Scientific, Singapore, 1993).
[27] L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Dover Publications, New York, 1975).
[28] F. Hioe and J. Eberly, Phys. Rev. Lett. **47**, 838 (1981).
[29] F. Hioe and J. Eberly, Phys. Rev. A **25**, 2168 (1982).
[30] H.-B. Huang, Phys. Rev. B **47**, 14 348 (1993).
[31] G. Dattoli, A. Dipace, and A. Torre, Phys. Rev. A **33**, 4387 (1986).
[32] J. F. Cornwell, *Group Theory in Physics* (Academic, London, 1984).
[33] L. Clavelli and P. Ramond, Phys. Rev. D **2**, 973 (1970).
[34] C. C. Gerry, Phys. Rev. A **31**, 2721 (1985).
[35] S. Glutsch and R. Zimmermann, Phys. Rev. B **45**, 5857 (1992).
[36] T. Östreich and K. Schönhammer, Z. Phys. B: Condens. Matter **91**, 189 (1993).
[37] R. Binder and S. Koch, Prog. Quantum Electron. **19**, 307 (1995).