Phase transitions of the mean-field Potts glass model in a field

Terufumi Yokota
Advanced Science Research Center, Japan Atomic Energy Research Institute, Tokai, Ibaraki 319-1195, Japan
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Phase transitions in the infinite-range p-state Potts glass model are studied in the presence of a field. The phase transition line into the transverse ordered phase is a straight line in a field-temperature phase diagram for small field and small effective ferromagnetic interaction. This is in contrast with the Gabay-Toulouse line for the vector spin-glass model. Replica symmetry breaking pattern is discussed near the phase transition line. Replica symmetry for the longitudinal order parameter is weakly broken at a higher temperature than the transition line to the transverse ordered phase for $p \gtrsim 3.2$. For $p \gtrsim 4.6$, the transition to the transverse ordered phase reveals discontinuity in the order parameter just as the $p>4$ case without a field.

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I. INTRODUCTION

The p-state Potts glass model has been studied for a long time as an extension of the Ising spin-glass model. After the introduction of the mean-field theory of the Potts glass model,[1,2] self-consistent description of the glass phase within the Parisi's replica theory was established.[3,4] The nature of the glass phase is rather different from the one for the mean-field theory of the Ising spin-glass model. A discontinuous transition with a jump in the Edward-Anderson order parameter but without latent heat appears for $p>4$. Moreover there is a dynamical transition temperature that is higher than the static grass transition temperature.[4,5] This feature is expected to be related to a better understanding of the structural glass.[5] The model has been studied also as a model of orientational glasses.[6]

Most of the previous works on the Potts glass model concentrated on the zero field case. Although a short description can be found in Ref. 7, a more detailed description as for the vector spin-glass model[8,9] would be helpful to see the nature of the Potts glass phase in a field.

In this paper, the mean-field Potts glass model in a field is studied near the glass transition temperature. Because we are interested in the glass phase, the effective ferromagnetic interaction that is inevitable for $p>2$ (Ref. 1) is set to zero at the continuous glass transition temperature without field. In Sec. II, replica mean-field theory is formulated near the glass transition temperature for a small field. The replica symmetric solution and its stability are discussed in Sec. III. The phase transition line to the transverse glass phase is shown to be straight in the field-temperature phase diagram for a small field. This differs from the vector spin-glass case.[10] Replica symmetry in the high temperature phase in which the transverse glass order parameter is zero is broken at some temperature higher than the transverse freezing temperature for $p \gtrsim 3.2$ (Ref. 11). The longitudinal order parameter function in this phase is given in Sec. IV. Because the breaking of the replica symmetry is rather weak for the longitudinal order, the replica symmetry breaking of the transverse freezing is studied in the replica symmetric approximation for the longitudinal order. This is considered in Sec. V. A summary of results is presented in Sec. VI.

II. THE MEAN-FIELD p-STATE POTTS GLASS MODEL

The Hamiltonian of the mean-field Potts glass model is

$$\mathcal{H} = -\sum_{\langle ij\rangle} J_{ij}(p\delta_{\sigma_i\sigma_j}-1) - \sum_i \sum_{\lambda=1}^{p} h^{\lambda}(p\delta_{\sigma_i\lambda}-1), \qquad (1)$$

where the sum of the first term is over all distinct pairs. The quenched random bond J_{ij} has a mean value J_0 and variance J^2/N with N number of sites. The uniform field h^{λ} is applied and δ is the Kronecker's delta function. The Potts variable σ_i takes one of the integer values from 1 to p. If the model is considered as a representative of the orientational glasses, the field may correspond to a stress field.[12]

It would be convenient to rewrite the Hamiltonian (1) using the simplex representation[13] as follows:

$$\mathcal{H} = -\sum_{\langle ij\rangle}\sum_{a=1}^{p-1} J_{ij}S_{ia}S_{ja} - \sum_i \sum_{a=1}^{p-1} h_a S_{ia}, \qquad (2)$$

where the spin $\mathbf{S_i}$ can be one of the $(p-1)$-dimensional vectors $\{\mathbf{e}^{\lambda}\}\ \lambda=1,\ldots,p$. These vectors satisfy the following relations:

$$\sum_{a=1}^{p-1} e_a^{\lambda} e_a^{\lambda'} = p\delta_{\lambda\lambda'} - 1, \qquad (3)$$

$$\sum_{\lambda=1}^{p} e_a^{\lambda} e_b^{\lambda} = p\delta_{ab}, \qquad (4)$$

and

$$\sum_{\lambda=1}^{p} e_a^{\lambda} = 0. \qquad (5)$$

An explicit form of the vectors may be

$$e_a^{\lambda} = \sqrt{\frac{p(p-a)}{p+1-a}} \times \begin{cases} 0, & \lambda < a, \\ 1, & \lambda = a, \\ -\dfrac{1}{p-a}, & \lambda > a. \end{cases} \qquad (6)$$

The relation of the field in (2) to that in (1) becomes

$$h_a = \sum_{\lambda=1}^{p} h^\lambda e_a^\lambda. \tag{7}$$

Taking $\lambda=1$ as the direction of the uniform field in (1), $a=1$ is the direction of the uniform field in (2) from the explicit form (7). The model is not symmetric under the exchange $h_a \leftrightarrow -h_a$. We study the case of $h_a > 0$ in the following unless otherwise stated because richness of the phase structure appears more explicitly for $h_a > 0$.

Using the replica method to take the random average for J_{ij}, the free energy per site f is given by

$$-\beta f = \lim_{N\to\infty}\lim_{n\to 0}\frac{1}{nN}(\langle Z^n\rangle_J - 1), \tag{8}$$

where $\beta = 1/k_B T$ with k_B Boltzmann's constant and T the temperature. Z is the partition function, n is the number of the replica, and $\langle\cdots\rangle_J$ represents the disorder average. The replicated partition function averaged over the random bond distribution becomes

$$\langle Z^n\rangle_J = \mathrm{Tr}_{\,\mathbf{S}_i^\alpha=\{\mathbf{e}^\lambda\}}\exp\Bigg[\frac{\mu}{2N}\sum_{\langle ij\rangle}\sum_{\alpha\neq\beta}(\mathbf{S}_i^\alpha\cdot\mathbf{S}_j^\alpha)(\mathbf{S}_i^\beta\cdot\mathbf{S}_j^\beta) + \frac{\beta\hat{J}}{N}\sum_{\langle ij\rangle}\sum_{\alpha}\mathbf{S}_i^\alpha\cdot\mathbf{S}_j^\alpha + \frac{\mu}{2N}\sum_{\langle ij\rangle}(p-1)n + \beta\sum_i\sum_a h_a S_{ia}^\alpha\Bigg], \tag{9}$$

where $\mu \equiv \beta^2 J^2$ and $\beta\hat{J} \equiv \beta J_0 + (\mu/2)(p-2)$. It should be noted that effective ferromagnetic interaction $\hat{J}$ grows as temperature decreases for $p>2$ (Ref. 1). Introducing variables $q_{ab}^{\alpha\beta}$ and m_a^α, the free energy becomes

$$-\beta f = \lim_{N\to\infty}\lim_{n\to 0}\frac{1}{nN}\Bigg(\prod_{\alpha\neq\beta}\prod_{a,b}\left(\frac{\mu N}{4\pi}\right)^{1/2}\int_{-\infty}^{\infty}dq_{ab}^{\alpha\beta} \times\prod_{\alpha}\prod_{a}\left(\frac{\beta\hat{J}N}{2\pi}\right)^{1/2}\int_{-\infty}^{\infty}dm_a^\alpha \times\exp\Bigg\{\frac{\mu N}{4}(p-1)n - \frac{\mu N}{4}\sum_{\alpha\neq\beta}\sum_{a,b}(q_{ab}^{\alpha\beta})^2 - \frac{\beta\hat{J}N}{2}\sum_{a,\alpha}(m_a^\alpha)^2 + \ln\Bigg[\mathrm{Tr}_{\,S_a^\alpha=\{e_a^\lambda\}}\exp\Bigg(\frac{\mu}{2}\sum_{\alpha\neq\beta}S_a^\alpha S_b^\alpha q_{ab}^{\alpha\beta} + \beta\hat{J}\sum_\alpha S_a^\alpha m_a^\alpha + \beta\sum_\alpha S_a^\alpha h_a\Bigg)\Bigg]^N\Bigg\} - 1\Bigg). \tag{10}$$

The free energy is given by the extremal condition

$$-\beta f = \lim_{n\to 0}\frac{1}{n}\mathrm{ext}(A), \tag{11}$$

where A is given by

$$A = \frac{\mu}{4}(p-1)n - \frac{\mu}{4}\sum_{\alpha\neq\beta}\sum_{a,b}(q_{ab}^{\alpha\beta})^2 - \frac{1}{2}\beta\hat{J}\sum_{a,\alpha}(m_a^\alpha)^2 + \ln\mathrm{Tr}_{\,S_a^\alpha=\{e_a^\lambda\}} \times\exp\Bigg[\frac{\mu}{2}\sum_{\alpha\neq\beta}S_a^\alpha S_b^\alpha q_{ab}^{\alpha\beta} + \beta\hat{J}\sum_{a,\alpha}S_a^\alpha m_a^\alpha + \beta\sum_\alpha S_a^\alpha h_a\Bigg]. \tag{12}$$

The variables $q_{ab}^{\alpha\beta}$ and m_a^α represent glass and ferromagnetic order parameters, respectively. Expanding the exponential keeping terms to fourth order in the order parameters, the trace can be evaluated. After re-exponentiating, A is written as

$$\begin{aligned}
A \simeq{}& \frac{\mu}{4}(p-1)n - \frac{\mu}{4}\sum_{\alpha\neq\beta}\sum_{a,b}(q_{ab}^{\alpha\beta})^2 - \frac{1}{2}\beta\hat{J}\sum_{a,\alpha}(m_a^\alpha)^2 + \frac{\beta^2}{2}\sum_\alpha\sum_a(\hat{J}z_a^\alpha)^2 + \frac{\mu^2}{4}\sum_{\alpha\neq\beta}\sum_{a,b}(q_{ab}^{\alpha\beta})^2 \\
&+ \frac{\beta^3}{6}\sum_\alpha \hat{J}z_a^\alpha\hat{J}z_b^\alpha\hat{J}z_c^\alpha\frac{v_{abc}}{p} + \frac{\beta^2\mu}{2}\sum_{\alpha\neq\beta}\hat{J}z_a^\alpha\hat{J}z_b^\alpha q_{ab}^{\alpha\beta} + \frac{\beta\mu^2}{2}\sum_{\alpha\neq\beta}\hat{J}z_a^\alpha q_{bd}^{\alpha\beta}q_{cd}^{\alpha\beta}\frac{v_{abc}}{p} + \frac{\mu^3}{12}\sum_{\alpha\neq\beta}q_{ad}^{\alpha\beta}q_{be}^{\alpha\beta}q_{cf}^{\alpha\beta}\frac{v_{abc}v_{def}}{p^2} \\
&+ \frac{u^3}{6}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}q_{ab}^{\alpha\beta}q_{bc}^{\beta\gamma}q_{ca}^{\gamma\alpha} + \frac{\beta^4}{24}\sum_\alpha\left(\frac{v_{abcd}}{p} - 3\delta_{ab}\delta_{cd}\right)\hat{J}z_a^\alpha\hat{J}z_b^\alpha\hat{J}z_c^\alpha\hat{J}z_d^\alpha + \sum_{\alpha\neq\beta}\hat{J}z_a^\alpha\hat{J}z_b^\alpha\hat{J}z_d^\beta q_{cd}^{\alpha\beta}\frac{v_{abc}}{p} \\
&+ \frac{\beta^2\mu^2}{4}\sum_{\alpha\neq\beta}\left(\frac{v_{abcd}}{p} - \delta_{ab}\delta_{cd}\right)\hat{J}z_a^\alpha\hat{J}z_b^\alpha q_{ce}^{\alpha\beta}q_{de}^{\alpha\beta} + \frac{\beta^2\mu^2}{4}\sum_{\alpha\neq\beta}\hat{J}z_a^\alpha\hat{J}z_d^\beta q_{be}^{\alpha\beta}q_{cf}^{\alpha\beta}\frac{v_{abc}v_{def}}{p^2} + \frac{\beta^2u^2}{2}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}\hat{J}z_a^\alpha\hat{J}z_c^\gamma q_{ab}^{\alpha\beta}q_{bc}^{\beta\gamma} \\
&+ \frac{\beta\mu^3}{6}\sum_{\alpha\neq\beta}\hat{J}z_a^\alpha q_{be}^{\alpha\beta}q_{cf}^{\alpha\beta}q_{dg}^{\alpha\beta}\frac{v_{abcd}v_{efg}}{p^2} + \frac{\beta\mu^3}{2}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}\hat{J}z_a^\alpha q_{bd}^{\alpha\beta}q_{de}^{\beta\gamma}q_{ec}^{\gamma\alpha}\frac{v_{abc}}{p}
\end{aligned}$$

$$+\frac{\beta\mu^3}{2}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}\hat{J}z_a^{\alpha}q_{ab}^{\alpha\beta}q_{ce}^{\beta\gamma}q_{de}^{\beta\gamma}\frac{v_{bcd}}{p}+\frac{\mu^4}{48}\sum_{\alpha\neq\beta}\left(\frac{v_{abcd}v_{efgh}}{p^2}-3\delta_{ab}\delta_{cd}\delta_{ef}\delta_{gh}\right)q_{ae}^{\alpha\beta}q_{bf}^{\alpha\beta}q_{cg}^{\alpha\beta}q_{dh}^{\alpha\beta}$$
$$+\frac{\mu^4}{4}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}q_{ab}^{\alpha\beta}q_{ce}^{\beta\gamma}q_{df}^{\beta\gamma}q_{ga}^{\gamma\alpha}\frac{v_{bcd}v_{efg}}{p^2}+\frac{\mu^4}{8}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}\left(\frac{v_{bcde}}{p}-\delta_{bc}\delta_{de}\right)q_{ab}^{\alpha\beta}q_{ac}^{\alpha\beta}q_{df}^{\beta\gamma}q_{ef}^{\beta\gamma}$$
$$+\frac{\mu^4}{8}\sum_{\substack{\alpha\beta\gamma\delta\\ \text{all different}}}q_{ab}^{\alpha\beta}q_{bc}^{\beta\gamma}q_{cd}^{\gamma\delta}q_{da}^{\delta\alpha}, \tag{13}$$

where

$$v_{abc\cdots}\equiv\sum_{\lambda=1}^{p}e_a^{\lambda}e_b^{\lambda}e_c^{\lambda}\cdots. \tag{14}$$

As the direction of the uniform field is $a=1$, the glass order parameter may be written as

$$q_{ab}^{\alpha\beta}=q_{\mathrm{L}}^{\alpha\beta}\delta_{a1}\delta_{b1}+(\delta_{ab}-\delta_{a1}\delta_{b1})q_{\mathrm{T}}^{\alpha\beta}. \tag{15}$$

For the ferromagnetic order parameter, we introduce z_a^{α} as

$$\hat{J}z_a^{\alpha}\equiv(\hat{J}m_a^{\alpha}+h_a)\delta_{a1}\equiv(\hat{J}m^{\alpha}+h)\delta_{a1}=\hat{J}z^{\alpha}. \tag{16}$$

Assuming replica symmetry for the ferromagnetic order parameter as $m^{\alpha}=m$, A can be written in terms of $q_{\mathrm{L}}^{\alpha\beta}$, $q_{\mathrm{T}}^{\alpha\beta}$, and z as follows:

$$A\simeq\frac{\mu}{4}(p-1)n+\frac{\mu(\mu-1)}{4}\sum_{\alpha\neq\beta}[(q_{\mathrm{L}}^{\alpha\beta})^2+(p-2)(q_{\mathrm{T}}^{\alpha\beta})^2]+\frac{n\beta}{2}[\beta(\hat{J}z)^2-\hat{J}m^2]+\frac{n\beta^3}{6}(\hat{J}z)^3\frac{p-2}{\sqrt{p-1}}+\frac{\beta^2\mu}{2}(\hat{J}z)^2\sum_{\alpha\neq\beta}q_{\mathrm{L}}^{\alpha\beta}$$
$$+\frac{\beta\mu^2}{2}\hat{J}z\sum_{\alpha\neq\beta}[(q_{\mathrm{L}}^{\alpha\beta})^2-(q_{\mathrm{T}}^{\alpha\beta})^2]\frac{p-2}{\sqrt{p-1}}+\frac{u^3}{12}\sum_{\alpha\neq\beta}\left[\frac{(p-2)^2}{p-1}(q_{\mathrm{L}}^{\alpha\beta})^3+3\frac{p-2}{p-1}q_{\mathrm{L}}^{\alpha\beta}(q_{\mathrm{T}}^{\alpha\beta})^2+\frac{p(p-2)(p-3)}{p-1}(q_{\mathrm{T}}^{\alpha\beta})^3\right]$$
$$+\frac{\mu^3}{6}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}[q_{\mathrm{L}}^{\alpha\beta}q_{\mathrm{L}}^{\beta\gamma}q_{\mathrm{L}}^{\gamma\alpha}+(p-2)q_{\mathrm{T}}^{\alpha\beta}q_{\mathrm{T}}^{\beta\gamma}q_{\mathrm{T}}^{\gamma\alpha}]+\frac{n\beta^4}{24}\frac{p^2-6p+6}{p-1}(\hat{J}z)^4+\frac{\beta^3\mu}{4}\sum_{\alpha\neq\beta}(\hat{J}z)^3q_{\mathrm{L}}^{\alpha\beta}\frac{p-2}{\sqrt{p-1}}$$
$$+\frac{\beta^2\mu^2}{4}\sum_{\alpha\neq\beta}(\hat{J}z)^2\left[\frac{2(p-2)^2}{p-1}(q_{\mathrm{L}}^{\alpha\beta})^2-\frac{(p-2)(p-3)}{p-1}(q_{\mathrm{T}}^{\alpha\beta})^2\right]+\frac{\beta^2\mu^2}{2}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}(\hat{J}z)^2q_{\mathrm{L}}^{\alpha\beta}q_{\mathrm{L}}^{\beta\gamma}$$
$$+\frac{\beta\mu^3}{6}\sum_{\alpha\neq\beta}\hat{J}z\left[\frac{(p-2)(p^2-3p+3)}{(p-1)^{3/2}}(q_{\mathrm{L}}^{\alpha\beta})^3-3\frac{p-2}{(p-1)^{3/2}}q_{\mathrm{L}}^{\alpha\beta}(q_{\mathrm{T}}^{\alpha\beta})^2-\frac{p(p-2)(p-3)}{(p-1)^{3/2}}(q_{\mathrm{T}}^{\alpha\beta})^3\right]$$
$$+\frac{\beta\mu^3}{2}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}\hat{J}z\left[\frac{p-2}{\sqrt{p-1}}q_{\mathrm{L}}^{\alpha\beta}q_{\mathrm{L}}^{\beta\gamma}q_{\mathrm{L}}^{\gamma\alpha}+\frac{p-2}{\sqrt{p-1}}q_{\mathrm{L}}^{\alpha\beta}(q_{\mathrm{L}}^{\beta\gamma})^2-\frac{p-2}{\sqrt{p-1}}q_{\mathrm{L}}^{\alpha\beta}(q_{\mathrm{T}}^{\beta\gamma})^2-\frac{p-2}{\sqrt{p-1}}q_{\mathrm{T}}^{\alpha\beta}q_{\mathrm{T}}^{\beta\gamma}q_{\mathrm{T}}^{\gamma\alpha}\right]$$
$$+\frac{\mu^4}{48}\sum_{\alpha\neq\beta}\left[\frac{p^4-6p^3+12p^2-12p+6}{(p-1)^2}(q_{\mathrm{L}}^{\alpha\beta})^4-6\frac{p(p-2)^2}{(p-1)^2}(q_{\mathrm{L}}^{\alpha\beta})^2(q_{\mathrm{T}}^{\alpha\beta})^2+4\frac{p(p-2)(p-3)}{(p-1)^2}q_{\mathrm{L}}^{\alpha\beta}(q_{\mathrm{T}}^{\alpha\beta})^3\right.$$
$$\left.+\frac{(p-2)(p^4-8p^3+19p^2-16p+6)}{(p-1)^2}(q_{\mathrm{T}}^{\alpha\beta})^4\right]+\frac{\mu^4}{8}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}\left[2\frac{(p-2)^2}{p-1}q_{\mathrm{L}}^{\alpha\beta}(q_{\mathrm{L}}^{\beta\gamma})^2q_{\mathrm{L}}^{\gamma\alpha}+\frac{(p-2)^2}{p-1}(q_{\mathrm{L}}^{\alpha\beta})^2(q_{\mathrm{L}}^{\beta\gamma})^2\right.$$
$$+2\frac{p-2}{p-1}q_{\mathrm{L}}^{\alpha\beta}(q_{\mathrm{T}}^{\beta\gamma})^2q_{\mathrm{L}}^{\gamma\alpha}-\frac{(p-2)^2}{p-1}(q_{\mathrm{L}}^{\alpha\beta})^2(q_{\mathrm{T}}^{\beta\gamma})^2+4\frac{p-2}{p-1}q_{\mathrm{T}}^{\alpha\beta}q_{\mathrm{L}}^{\beta\gamma}q_{\mathrm{T}}^{\beta\gamma}q_{\mathrm{T}}^{\gamma\alpha}+2\frac{p(p-2)(p-3)}{p-1}q_{\mathrm{T}}^{\alpha\beta}(q_{\mathrm{T}}^{\beta\gamma})^2q_{\mathrm{T}}^{\gamma\alpha}$$
$$\left.+\frac{(p-2)^2}{p-1}(q_{\mathrm{T}}^{\alpha\beta})^2(q_{\mathrm{T}}^{\beta\gamma})^2\right]+\frac{\mu^4}{8}\sum_{\substack{\alpha\beta\gamma\delta\\ \text{all different}}}[q_{\mathrm{L}}^{\alpha\beta}q_{\mathrm{L}}^{\beta\gamma}q_{\mathrm{L}}^{\gamma\delta}q_{\mathrm{L}}^{\delta\alpha}+(p-2)q_{T}^{\alpha\beta}q_{\mathrm{T}}^{\beta\gamma}q_{\mathrm{T}}^{\gamma\delta}q_{\mathrm{T}}^{\delta\alpha}]. \tag{17}$$

The expansion may be valid when q_{L}, q_{T}, and z are small. This condition is satisfied for a small field near the continuous transition from the paramagnetic to an ordered phase.

Without a uniform field, A is reduced to

$$A \simeq \frac{\mu}{4}(p-1)n + \frac{\mu(\mu-1)}{4}\sum_{\alpha\neq\beta}(p-1)(q^{\alpha\beta})^2 + \frac{\mu^3}{12}\sum_{\alpha\neq\beta}(p-1)(p-2)(q^{\alpha\beta})^3 + \frac{\mu^3}{6}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}(p-1)q^{\alpha\beta}q^{\beta\gamma}q^{\gamma\alpha}$$
$$+ \frac{\mu^4}{48}\sum_{\alpha\neq\beta}(p-1)(p^2-6p+6)(q^{\alpha\beta})^4 + \frac{\mu^4}{4}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}(p-1)(p-2)q^{\alpha\beta}(q^{\beta\gamma})^2q^{\gamma\alpha} + \frac{\mu^4}{8}\sum_{\substack{\alpha\beta\gamma\delta \\ \text{all different}}}(p-1)q^{\alpha\beta}q^{\beta\gamma}q^{\gamma\delta}q^{\delta\alpha}, \quad (18)$$

that has been obtained in Ref. 4.

III. REPLICA SYMMETRIC SOLUTION AND ITS STABILITY

In this section, we look for the replica symmetric solution and study its stability. Assuming $q_{\rm L}^{\alpha\beta}=q_{\rm L}$ and $q_{\rm T}^{\alpha\beta}=q_{\rm T}$ for all $\alpha\neq\beta$, (17) becomes

$$\frac{A}{n} \simeq_{(n\to 0)} \frac{\mu}{4}(p-1) + \frac{\beta}{2}[\beta(\hat{J}z)^2 - \hat{J}m^2] - \frac{\mu(\mu-1)}{4}[q_{\rm L}^2 + (p-2)q_{\rm T}^2] + \frac{\beta^3}{6}(\hat{J}z)^3\frac{p-2}{\sqrt{p-1}} - \frac{\beta^2\mu}{2}(\hat{J}z)^2 q_{\rm L} - \frac{\beta\mu^2}{2}\hat{J}z(q_{\rm L}^2 - q_{\rm T}^2)\frac{p-2}{\sqrt{p-1}}$$
$$+ \frac{\mu^3}{12}\left[-\frac{p^2-8p+8}{p-1}q_{\rm L}^3 - 3\frac{p-2}{p-1}q_{\rm L}q_{\rm T}^2 - \frac{(p-2)(p^2-7p+4)}{p-1}q_{\rm T}^3\right] + \frac{\beta^4}{24}\frac{p^2-6p+6}{p-1}(\hat{J}z)^4 - \frac{\beta^3\mu}{4}(\hat{J}z)^3 q_{\rm L}\frac{p-2}{\sqrt{p-1}}$$
$$- \frac{\beta^2\mu^2}{4}(\hat{J}z)^2\left[2\frac{p^2-6p+6}{p-1}q_{\rm L}^2 - \frac{(p-2)(p-3)}{p-1}q_{\rm T}^2\right] + \frac{\beta\mu^3}{6}\hat{J}z\left[\frac{(p-2)(p^2-15p+15)}{(p-1)^{3/2}}q_{\rm L}^3 + 3\frac{(p-2)(2p-3)}{(p-1)^{3/2}}q_{\rm L}q_{\rm T}^2\right.$$
$$\left. - \frac{(p-2)(p^2-9p+6)}{(p-1)^{3/2}}q_{\rm T}^3\right] - \frac{\mu^4}{48}\left[\frac{p^4-42p^3+228p^2-372p+186}{(p-1)^2}q_{\rm L}^4 + 6\frac{(p-2)(3p^2-14p+12)}{(p-1)^2}q_{\rm L}^2q_{\rm T}^2\right.$$
$$\left. + 4\frac{(p-2)(p^2-15p+12)}{(p-1)^2}q_{\rm L}q_{\rm T}^3 + \frac{(p-2)(p^4-32p^3+139p^2-123p+18)}{(p-1)^2}q_{\rm T}^4\right]. \quad (19)$$

We seek a saddle point of the free energy. The saddle point equation with respect to m is

$$0 = \beta\hat{J}\left\{(\beta\hat{J}z - m) + \frac{\beta^2}{2}\frac{p-2}{\sqrt{p-1}}(\hat{J}z)^2 - \beta\mu\hat{J}zq_L - \frac{\mu^2}{2}(q_L^2 - q_T^2)\right.$$
$$\times\frac{p-2}{\sqrt{p-1}} + \frac{\beta^2}{6}\frac{p^2-6p+6}{p-1}(\hat{J}z)^3 - \frac{3\beta^2\mu}{4}(\hat{J}z)^2 q_{\rm L}\frac{p-2}{\sqrt{p-1}}$$
$$- \frac{\beta\mu^2}{2}\hat{J}z\left(2\frac{p^2-6p+6}{p-1}q_{\rm L}^2 - \frac{p^2-5p+6}{p-1}q_{\rm T}^2\right)$$
$$+ \frac{\mu^3}{6}\frac{p-2}{(p-1)^{3/2}}[(p^2-15p+15)q_{\rm L}^3 + 3(2p-3)q_{\rm L}q_{\rm T}^2$$
$$\left. - (p^2-9p+6)q_{\rm T}^3]\right\}. \quad (20)$$

The saddle point equation for $q_{\rm L}$ is

$$0 = -\frac{\mu(\mu-1)}{2}q_{\rm L} - \frac{\beta^2\mu}{2}(\hat{J}z)^2 - \beta\mu^2\hat{J}zq_{\rm L}\frac{p-2}{\sqrt{p-1}}$$
$$- \frac{\mu^3}{4}\frac{1}{p-1}[(p^2-8p+8)q_{\rm L}^2 + (p-2)q_{\rm T}^2]$$
$$- \frac{\beta^3\mu}{4}(\hat{J}z)^3\frac{p-2}{\sqrt{p-1}} - \beta^2\mu^2(\hat{J}z)^2\frac{p^2-6p+6}{p-1}q_{\rm L}$$
$$- \frac{\beta\mu^3}{2}\hat{J}z\frac{p-2}{(p-1)^{3/2}}[(p^2-15p+15)q_{\rm L}^2 + (2p-3)q_{\rm T}^2]$$
$$- \frac{\mu^4}{12}\frac{1}{(p-1)^2}[(p^4-42p^3+228p^2-372p+186)q_{\rm L}^3$$
$$+ 3(p-2)(3p^2-14p+12)q_{\rm L}q_{\rm T}^2$$
$$+ (p-2)(p^2-15p+12)q_{\rm T}^3]. \quad (21)$$

The saddle point equation for $q_{\rm T}$ becomes

$$0 = \mu(p-2)q_{\rm T}\left\{-\frac{\mu-1}{2} + \beta\mu\hat{J}z\frac{1}{\sqrt{p-1}} - \frac{\mu^2}{4}\frac{1}{p-1}\right.$$
$$\times[2q_{\rm L} + (p^2-7p+4)q_{\rm T}] + \frac{\beta^2\mu}{2}(\hat{J}z)^2\frac{p-3}{p-1}$$
$$- \frac{\beta\mu^2}{2}\hat{J}z\frac{1}{(p-1)^{3/2}}[2(2p-3)q_{\rm L} - (p^2-9p+6)q_{\rm T}]$$
$$- \frac{\mu^3}{12}\frac{1}{(p-1)^2}[3(3p^2-14p+12)q_{\rm L}^2 + 3(p^2-15p+12)$$
$$\left.\times q_{\rm L}q_{\rm T} + (p^4-32p^3+139p^2-123p+18)q_{\rm T}^2]\right\}. \quad (22)$$

Because we are interested in glassy states, the effective ferromagnetic interaction $\hat{J}$ that varies with temperature will be set small near the paramagnetic to glassy phase transition. The choice for the ferromagnetic interaction J_0 adopted here is

$$\frac{J_0}{J} + \frac{1}{2}(p-2) = 0. \tag{23}$$

From this choice and the saddle point equation for the magnetization (20), magnetization and $\hat{J}z$ become

$$m \simeq \beta h \tag{24}$$

and

$$\hat{J}z \simeq h, \tag{25}$$

respectively. The remaining two saddle point equations, (21) and (22), are used to obtain the transition line for the freezing of transverse degrees of freedom. From (21), the longitudinal order parameter becomes

$$q_{\rm L} \simeq -\frac{2(p-1)}{p^2-8p+8}\Bigg[-\frac{\delta T}{J} + \frac{p-2}{\sqrt{p-1}}\frac{h}{J} + \sqrt{\left(\frac{\delta T}{J}\right)^2 - 2\frac{p-2}{\sqrt{p-1}}\frac{\delta T}{J}\frac{h}{J} + \frac{p^2}{2(p-1)}\left(\frac{h}{J}\right)^2}\Bigg], \tag{26}$$

where

$$\frac{\delta T}{J} \equiv \frac{T}{J} - 1. \tag{27}$$

This quantity is small near the continuous transition for a small field. Using (22), the continuous transverse freezing transition line is obtained from

$$\frac{\delta T}{J} + \frac{1}{\sqrt{p-1}}\frac{h}{J} - \frac{1}{2(p-1)}q_{\rm L} \simeq 0. \tag{28}$$

From this equation and (26) the transition line is

$$\frac{\delta T_{\rm c}}{J} \simeq \frac{-1}{p^2-8p+6}\left[\frac{2(p^2-7p+5) \pm \sqrt{2p^2+8p-8}}{2\sqrt{p-1}}\right]\frac{h_{\rm c}}{J}. \tag{29}$$

The $\pm$ correspond to $h_{\rm c}/J>0$ and $h_{\rm c}/J<0$, respectively. The relation between $\delta T_{\rm c}/J$ and $h_{\rm c}/J$ is linear. This contrasts with the case of the vector spin glass where the Gabay-Toulouse (GT) line has the form $\delta T_{\rm c}/J \propto (h_{\rm c}/J)^2$ (Ref. 10). Just below the transition line,

$$q_{\rm T} \simeq \frac{4(p-1)}{p^2-7p+4}\alpha\frac{\Delta T}{J}, \tag{30}$$

where

$$\frac{\Delta T}{J} \equiv \frac{\delta T}{J} - \frac{\delta T_{\rm c}}{J} < 0 \tag{31}$$

and

$$\alpha = \frac{1}{p^2-8p+8}\left[p^2-8p+7 + \frac{\dfrac{\delta T_{\rm c}}{J} - \dfrac{p-2}{\sqrt{p-1}}\dfrac{h_{\rm c}}{J}}{\sqrt{\left(\dfrac{\delta T_{\rm c}}{J}\right)^2 - 2\dfrac{p-2}{\sqrt{p-1}}\dfrac{\delta T_{\rm c}}{J}\dfrac{h_{\rm c}}{J} + \dfrac{p^2}{2(p-1)}\left(\dfrac{h_{\rm c}}{J}\right)^2}}\right]. \tag{32}$$

The relation between $\delta T_{\rm c}/J$ and $h_{\rm c}/J$ is given by (29). A first-order transition with a jump in the order parameter would appear for $p>(7+\sqrt{33})/2 \simeq 6.37$. It is noted that the first-order transition appears for $p>6$ in the zero field.[1]

To investigate the stability of the replica symmetric solutions, the order parameters are written as

$$q_{ab}^{\alpha\beta} = (q_{\rm L}-q_{\rm T})\delta_{a1}\delta_{b1} + q_{\rm T}\delta_{ab} + \eta_{ab}^{\alpha\beta} \tag{33}$$

and

$$m_a^\alpha = m\delta_{a1} + \varepsilon_a^\alpha, \tag{34}$$

where $q_{\rm L}$, $q_{\rm T}$, and m represent the replica symmetric solution. From (16), we have

$$\hat{J}z_a^\alpha = \hat{J}z\delta_{a1} + \hat{J}\varepsilon_a^\alpha. \tag{35}$$

Expanding (13) to second order in the fluctuations η and ε, A becomes

$$A \simeq A_{\rm MF}^{\rm RS} - \frac{1}{4}M_{abcd}^{\alpha\beta\gamma\nu}\eta_{ab}^{\alpha\beta}\eta_{cd}^{\gamma\nu} - \frac{1}{4}L_{abc}^{\alpha\beta\gamma}\varepsilon_a^\alpha\eta_{bc}^{\beta\gamma} - \frac{1}{4}K_{ab}^{\alpha\beta}\varepsilon_a^\alpha\varepsilon_b^\beta, \tag{36}$$

where $A_{\rm MF}^{\rm RS}$ is the replica symmetric quantity. The stability matrices **M**, **L**, and **K** are given by

$$M_{abcd}^{\alpha\beta\gamma\nu} = \delta_{ac}\delta_{bd}\Bigg(\Bigg\{-\mu(\mu-1) + 2\beta\mu^2\hat{J}z\frac{1}{\sqrt{p-1}} - \mu^3(q_{\rm L}-q_{\rm T})\frac{1}{p-1} + \mu^3 q_{\rm T} + \beta^2\mu^2(\hat{J}z)^2\frac{p-3}{p-1} + 2\beta\mu^3\hat{J}z(q_{\rm L}-q_{\rm T}) \times\frac{1}{\sqrt{p-1}}\left(\frac{1}{p-1}+n-2\right) + 2\beta\mu^3\hat{J}zq_{\rm T}\frac{1}{\sqrt{p-1}}(n-3) + \mu^4(q_{\rm L}-q_{\rm T})^2\left[\frac{p(p-2)}{2(p-1)^2} + (n-2)\frac{p-3}{p-1}\right]$$

$$
\begin{aligned}
&+\mu^4(q_{\rm L}-q_{\rm T})q_{\rm T}\left[\frac{p}{p-1}+2(n-2)\frac{p-3}{p-1}\right]+\mu^4q_{\rm T}^2\left(\frac{p-2}{2}+n-2\right)\Bigg\}\mathbf{A}_{\alpha\beta\gamma\nu}\\
&-\left[\mu^3q_{\rm T}-3\beta\mu^3\hat{J}zq_{\rm T}\frac{1}{\sqrt{p-1}}+3\mu^4(q_{\rm L}-q_{\rm T})q_{\rm T}\frac{1}{p-1}+\mu^4q_{\rm T}^2\left(n-5+\frac{p-2}{2}\right)\right]\mathbf{B}_{\alpha\beta\gamma\nu}-\mu^4q_{\rm T}^2\mathbf{C}_{\alpha\beta\gamma\nu}\Bigg)\\
&-\frac{v_{abcd}}{p}\Bigg\{\left[\mu^3q_{\rm T}-2\beta\mu^3\hat{J}zq_{\rm T}\frac{1}{\sqrt{p-1}}+\mu^4(q_{\rm L}-q_{\rm T})q_{\rm T}\frac{1}{p-1}+\mu^4q_{\rm T}^2\left(\frac{p-2}{2}+n-2\right)\right]\mathbf{A}_{\alpha\beta\gamma\nu}+3\mu^4q_{\rm T}^2\mathbf{B}_{\alpha\beta\gamma\nu}\Bigg\}\\
&+\delta_{ab}\delta_{cd}\mu^4q_{\rm T}^2(\mathbf{A}_{\alpha\beta\gamma\nu}+\mathbf{B}_{\alpha\beta\gamma\nu})+\frac{1}{4}\left(\frac{v_{abc}}{p}\delta_{d1}+\frac{v_{abd}}{p}\delta_{c1}+\frac{v_{acd}}{p}\delta_{b1}+\frac{v_{bcd}}{p}\delta_{a1}\right)\Bigg[-2\beta\mu^3\hat{J}zq_{\rm T}\\
&+6\mu^4(q_{\rm L}-q_{\rm T})q_{\rm T}\frac{1}{\sqrt{p-1}}\Bigg]\mathbf{B}_{\alpha\beta\gamma\nu}+\frac{1}{4}(\delta_{ac}\delta_{b1}\delta_{d1}+\delta_{ad}\delta_{b1}\delta_{c1}+\delta_{bc}\delta_{a1}\delta_{d1}+\delta_{bd}\delta_{a1}\delta_{c1})\Bigg(\Bigg\{\mu^4(q_{\rm L}-q_{\rm T})^2\Bigg[-\frac{p-2}{p-1}\\
&-(p-4)(n-2)\Bigg]+\mu^4(q_{\rm L}-q_{\rm T})q_{\rm T}[2-2(p-4)(n-2)]\Bigg\}\mathbf{A}_{\alpha\beta\gamma\nu}+\Bigg\{\beta\mu^3\hat{J}z(q_{\rm L}-q_{\rm T})\frac{5-p}{\sqrt{p-1}}-2\beta\mu^3\hat{J}zq_{\rm T}\sqrt{p-1}\\
&+\mu^4(q_{\rm L}-q_{\rm T})^2\left[\frac{-p^2+8p-14}{2(p-1)}-(n-3)\right]+\mu^4(q_{\rm L}-q_{\rm T})q_{\rm T}[8-p-2(n-3)]\Bigg\}\mathbf{B}_{\alpha\beta\gamma\nu}-\mu^4(q_{\rm L}-q_{\rm T})q_{\rm T}\mathbf{C}_{\alpha\beta\gamma\nu}\Bigg)\\
&+(\delta_{ab}\delta_{c1}\delta_{d1}+\delta_{cd}\delta_{a1}\delta_{b1})\mu^4(q_{\rm L}-q_{\rm T})q_{\rm T}(\mathbf{A}_{\alpha\beta\gamma\nu}+\mathbf{B}_{\alpha\beta\gamma\nu})+\delta_{a1}\delta_{b1}\delta_{c1}\delta_{d1}\Bigg(\Bigg\{-\mu^3(q_{\rm L}-q_{\rm T})(p-1)-\beta^2u^2(\hat{J}z)^2(p-1)\\
&-2\beta\mu^3\hat{J}z(q_{\rm L}-q_{\rm T})\sqrt{p-1}(p-2)-2\beta\mu^3\hat{J}zq_{\rm T}(p-1)^{3/2}+\mu^4(q_{\rm L}-q_{\rm T})^2\left[-\frac{p^2-4p+2}{2}-(p-1)(n-2)\right]\\
&+\mu^4(q_{\rm L}-q_{\rm T})q_{\rm T}(p-1)[-(p-3)-2(n-2)]\Bigg\}\mathbf{A}_{\alpha\beta\gamma\nu}+[-3\beta\mu^3\hat{J}z(q_{\rm L}-q_{\rm T})\sqrt{p-1}+\mu^4(q_{\rm L}-q_{\rm T})^2(7-2p)\\
&-6\mu^4(q_{\rm L}-q_{\rm T})q_{\rm T}(p-1)]\mathbf{B}_{\alpha\beta\gamma\nu}-\mu^4(q_{\rm L}-q_{\rm T})^2\mathbf{C}_{\alpha\beta\gamma\nu}\Bigg),
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
L_{abc}^{\alpha\beta\gamma}=&\frac{v_{abc}}{p}\Bigg\{\Bigg[-\beta\mu^2\hat{J}q_{\rm T}+2\beta^2\mu^2\hat{J}\hat{J}zq_{\rm T}\frac{1}{\sqrt{p-1}}-\beta\mu^3\hat{J}(q_{\rm L}-q_{\rm T})q_{\rm T}\frac{1}{p-1}-\frac{\beta\mu^3}{2}\hat{J}q_{\rm T}^2(2n-3)\Bigg](\delta_{\alpha\beta}+\delta_{\alpha\gamma})-3\beta\mu^3\hat{J}q_{\rm T}^2(1-\delta_{\alpha\beta})\\
&\times(1-\delta_{\alpha\gamma})\Bigg\}+(\delta_{ab}\delta_{c1}+\delta_{ac}\delta_{b1})\Bigg(\Bigg\{-\beta^2\mu\hat{J}\hat{J}z+\beta\mu^2\hat{J}(q_{\rm L}-q_{\rm T})\frac{1}{\sqrt{p-1}}-\frac{\beta^3\mu}{4}\hat{J}(\hat{J}z)^2\frac{p-4}{\sqrt{p-1}}+\beta^2\mu^2\hat{J}\hat{J}z(q_{\rm L}-q_{\rm T})\Bigg[\frac{p-3}{p-1}-(n\\
&-2)\Bigg]-\beta^2\mu^2\hat{J}\hat{J}zq_{\rm T}(n-2)-\frac{\beta\mu^3}{2}\hat{J}(q_{\rm L}-q_{\rm T})^2\left[\frac{-p}{(p-1)^{3/2}}+\frac{p-2}{\sqrt{p-1}}(n-2)\right]-\beta\mu^3\hat{J}(q_{\rm L}-q_{\rm T})q_{\rm T}\left[\frac{1}{\sqrt{p-1}}+\frac{p-2}{\sqrt{p-1}}(n-2)\right]\Bigg\}\\
&\times(\delta_{\alpha\beta}+\delta_{\alpha\gamma})+\left[-2\beta^2\mu^2\hat{J}\hat{J}zq_{\rm T}+4\beta\mu^3\hat{J}(q_{\rm L}-q_{\rm T})q_{\rm T}\frac{1}{\sqrt{p-1}}\right](1-\delta_{\alpha\beta})(1-\delta_{\alpha\gamma})+\delta_{a1}\delta_{bc}\Big[2\beta^2\mu^2\hat{J}\hat{J}zq_{\rm T}(\delta_{\alpha\beta}+\delta\alpha\gamma)\\
&+2\beta\mu^3\hat{J}(q_{\rm L}-q_{\rm T})q_{\rm T}(1-\delta_{\alpha\beta})(1-\delta_{\alpha\gamma})+\delta_{a1}\delta_{b1}\delta_{c1}\Bigg(\Bigg\{-2\beta\mu^2\hat{J}(q_{\rm L}-q_{\rm T})\sqrt{p-1}-\beta^3\mu\hat{J}(\hat{J}z)^2\sqrt{p-1}-\beta^2\mu^2\hat{J}\hat{J}z(q_{\rm L}-q_{\rm T})\\
&\times(3p-8)-4\beta^2\mu^2\hat{J}\hat{J}zq_{\rm T}(p-1)-\beta\mu^3\hat{J}(q_{\rm L}-q_{\rm T})^2\frac{1}{\sqrt{p-1}}[(p-2)^2+2(p-1)(n-2)]-2\beta\mu^3\hat{J}(q_{\rm L}-q_{\rm T})q_{\rm T}\sqrt{p-1}\\
&\times[p-3+2(n-2)]\Bigg\}(\delta_{\alpha\beta}+\delta_{\alpha\gamma})+\left[-4\beta^2\mu^2\hat{J}\hat{J}z(q_{\rm L}-q_{\rm T})-2\beta\mu^3\hat{J}(q_{\rm L}-q_{\rm T})^2\frac{p-2}{\sqrt{p-1}}-4\beta\mu^3\hat{J}(q_{\rm L}-q_{\rm T})q_{\rm T}\sqrt{p-1}\right]\\
&\times(1-\delta_{\alpha\beta})(1-\delta_{\alpha\gamma})\Bigg),
\end{aligned}
\tag{38}
$$

and

$$K_{ab}^{\alpha\beta} = \delta_{ab}\Bigg(\Bigg\{-2\beta^2\hat{J}^2 + 2\beta^2\hat{J} + 2\beta^3\hat{J}^2\hat{J}z + \beta^4\hat{J}^2\frac{3p-4}{3(p-1)} + 2\beta^3\mu\hat{J}^2\hat{J}zq_{\mathrm{T}}\frac{1}{\sqrt{p-1}}(n-1) + \beta^3\mu\hat{J}^2\hat{J}zq_{\mathrm{L}}\frac{1}{\sqrt{p-1}}(n-1) + \beta^2\mu^2 \times\hat{J}^2(q_{\mathrm{L}}-q_{\mathrm{T}})^2(n-1)\frac{p-2}{p-1} + \beta^2\mu^2\hat{J}^2(q_{\mathrm{L}}-q_{\mathrm{T}})q_{\mathrm{T}}\left[2(n-1)-\frac{2}{p-1}\right]\Bigg\}\Bigg)\delta_{\alpha\beta} + \Bigg\{-2\beta^2\mu\hat{J}^2q_{\mathrm{T}} - \beta^2\mu^2\hat{J}^2(q_{\mathrm{L}}-q_{\mathrm{T}})q_{\mathrm{T}}\frac{1}{p-1} - \beta^2\mu^2\hat{J}^2q_{\mathrm{T}}^2[p-2+2(n-2)]\Bigg\}(1-\delta_{\alpha\beta}) + \delta_{a1}\delta_{b1}\Bigg(\Bigg[-2\beta^3\hat{J}^2\hat{J}z\sqrt{p-1} - \frac{\beta^4}{3}\hat{J}^2(\hat{J}z)^2(p-2) - 2\beta^3\mu\hat{J}^2\hat{J}zq_{\mathrm{T}}\sqrt{p-1} - \beta^3\mu\hat{J}^2\hat{J}zq_{\mathrm{L}}\sqrt{p-1} - \beta^2\mu^2\hat{J}^2(q_{\mathrm{L}}-q_{\mathrm{T}})^2(p-2)(n-1) - 2\beta^2\mu^2\hat{J}^2(q_{\mathrm{L}}-q_{\mathrm{T}})q_{\mathrm{T}}(p-2)(n-1)\Bigg]\delta_{\alpha\beta} + \Bigg\{-2\beta^2\mu\hat{J}^2(q_{\mathrm{L}}-q_{\mathrm{T}}) - \beta^2\mu^2\hat{J}^2(q_{\mathrm{L}}-q_{\mathrm{T}})^2\left[\frac{(p-2)^2}{p+1}+2(n-2)\right] - 2\beta^2\mu^2\hat{J}^2(q_{\mathrm{L}}-q_{\mathrm{T}})q_{\mathrm{T}}[p-2+2(n-2)]\Bigg\}(1-\delta_{\alpha\beta})\Bigg), \tag{39}$$

where the replica matrices are given by

$$\mathbf{A}_{\alpha\beta\gamma\nu} = \delta_{\alpha\gamma}\delta_{\beta\nu}, \tag{40}$$

$$\mathbf{B}_{\alpha\beta\gamma\nu} = \delta_{\alpha\gamma}(\mathbf{1}-\delta_{\beta\nu}) + \delta_{\beta\nu}(\mathbf{1}-\delta_{\alpha\gamma}), \tag{41}$$

and

$$\mathbf{C}_{\alpha\beta\gamma\nu} = (\mathbf{1}-\delta_{\alpha\gamma})(\mathbf{1}-\delta_{\alpha\nu})(\mathbf{1}-\delta_{\beta\gamma})(\mathbf{1}-\delta_{\beta\nu}). \tag{42}$$

As the stability of the glassy phase without ferromagnetic order is expected to be related to the stability matrix **M**, the fluctuation in (34) is ignored and only **M** is considered. In addition, fluctuations are assumed to be diagonal in the Potts index as

$$q_{ab}^{\alpha\beta} = [(q_{\mathrm{L}}-q_{\mathrm{T}})\delta_{a1} + q_{\mathrm{T}} + \eta_a^{\alpha\beta}]\delta_{ab}. \tag{43}$$

The stability matrix **M** reduces to

$$M_{abcd}^{\alpha\beta\gamma\nu} = M_{ac}^{\alpha\beta\gamma\nu} = \delta_{ac}\Bigg(\Bigg\{\Bigg[-\mu(\mu-1) + 2\beta\mu^2\hat{J}z\frac{1}{\sqrt{p-1}} - \mu^3(q_{\mathrm{L}}-q_{\mathrm{T}})\frac{1}{p-1} + \mu^3q_{\mathrm{T}} + \beta^2\mu^2(\hat{J}z)^2\frac{p-3}{p-1} + 2\beta\mu^3\hat{J}z(q_{\mathrm{L}}-q_{\mathrm{T}})\frac{1}{\sqrt{p-1}}\left(\frac{1}{p-1}+n-2\right) + 2\beta\mu^3\hat{J}zq_{\mathrm{T}}\frac{1}{\sqrt{p-1}}(n-3) + \mu^4(q_{\mathrm{L}}-q_{\mathrm{T}})^2\left[\frac{p(p-2)}{2(p-1)^2} + (n-2)\frac{p-3}{p-1}\right] + \mu^4(q_{\mathrm{L}}-q_{\mathrm{T}})q_{\mathrm{T}}\left[\frac{p}{p-1} + 2(n-2)\frac{p-3}{p-1}\right] + \mu^4q_{\mathrm{T}}^2\left(\frac{p-2}{2}+n-2\right)\Bigg\}\mathbf{A}_{\alpha\beta\gamma\nu} - \Bigg[\mu^3q_{\mathrm{T}} - 3\beta\mu^3\hat{J}zq_{\mathrm{T}}\frac{1}{\sqrt{p-1}} + 3\mu^4(q_{\mathrm{L}}-q_{\mathrm{T}}) \times q_{\mathrm{T}}\frac{1}{p-1} + \mu^4q_{\mathrm{T}}^2\left(n-5+\frac{p-2}{2}\right)\Bigg]\mathbf{B}_{\alpha\beta\gamma\nu} - \mu^4q_{\mathrm{T}}^2\mathbf{C}_{\alpha\beta\gamma\nu}\Bigg) - \frac{v_{aacc}}{p}\Bigg\{\Bigg[\mu^3q_{\mathrm{T}} - 2\beta\mu^3\hat{J}zq_{\mathrm{T}}\frac{1}{\sqrt{p-1}} + \mu^4(q_{\mathrm{L}}-q_{\mathrm{T}})q_{\mathrm{T}}\frac{1}{p-1} + \mu^4q_{\mathrm{T}}^2\left(\frac{p-2}{2}+n-2\right)\Bigg]\mathbf{A}_{\alpha\beta\gamma\nu} + 3\mu^4q_{\mathrm{T}}^2\mathbf{B}_{\alpha\beta\gamma\nu}\Bigg\} + \mu^4q_{\mathrm{T}}^2(\mathbf{A}_{\alpha\beta\gamma\nu}+\mathbf{B}_{\alpha\beta\gamma\nu}) + \delta_{c1}\Bigg\{2\mu^4(q_{\mathrm{L}}-q_{\mathrm{T}})q_{\mathrm{T}}\mathbf{A}_{\alpha\beta\gamma\nu} + \left[2\beta\mu^3\hat{J}zq_{\mathrm{T}}\frac{1}{\sqrt{p-1}} + 2\mu^4(q_{\mathrm{L}}-q_{\mathrm{T}})q_{\mathrm{T}}\frac{p-3}{p-1}\right]\mathbf{B}_{\alpha\beta\gamma\nu}\Bigg\} + \delta_{a1}\delta_{c1}\Bigg(\Bigg\{-2\beta\mu^2\hat{J}z\sqrt{p-1} - \mu^3(q_{\mathrm{L}}-q_{\mathrm{T}})(p-3) - \beta^2\mu^2(\hat{J}z)^2(2p-5) - 2\beta\mu^3\hat{J}z(p_{\mathrm{L}}-q_{\mathrm{T}})\frac{p^2-4p+5}{\sqrt{p-1}} - 2\beta\mu^3\hat{J}zq_{\mathrm{T}}\sqrt{p-1}(p-2) - 2\beta\mu^3\hat{J}zq_{\mathrm{L}}\sqrt{p-1}(n-2) + \mu^4(q_{\mathrm{L}}-q_{\mathrm{T}})^2\left[1 - \frac{(p-2)(p^2-3p+4)}{2(p-1)} - (2p-5)(n-2)\right]$$

$$- \mu^4(q_L - q_T)q_T[p^4 - 4p + 5 + 2(2p-5)(n-2)]\Big\}\mathbf{A}_{\alpha\beta\gamma\nu} + \Big\{ -\mu^3(q_L - q_T) - \beta^2\mu^2(\hat{J}z)^2 - 5\beta\mu^3$$
$$\times \hat{J}z(q_L - q_T)\frac{p-2}{\sqrt{p-1}} - 5\beta\mu^3\hat{J}zq_T\sqrt{p-1} + \mu^4(q_L-q_T)^2\left[-\frac{7p^2}{2(p-1)} + 17 - n\right] + \mu^4(q_L-q_T)q_T(-7p+25-2n)\Big\}\mathbf{B}_{\alpha\beta\gamma\nu}$$
$$+ [-\mu^4(q_L-q_T)^2 - 2\mu^4 - 2\mu^4(q_L-q_T)q_T]\mathbf{C}_{\alpha\beta\gamma\nu}\Big). \tag{44}$$

In the case of no field, $\hat{J}z=0$ and the glass order parameters are isotropic as $q_L = q_T \equiv q$. The stability matrix becomes

$$M_{ac}^{\alpha\beta\gamma\nu} = \delta_{ac}\Bigg\{\left[-\mu(\mu-1) + \mu^3 q + \mu^4 q^2\left(\frac{p}{2} - 3 + n\right)\right]\mathbf{A}_{\alpha\beta\gamma\nu} - \left[\mu^3 q + \mu^4 q^2\left(\frac{p}{2} - 6 + n\right)\right]\mathbf{B}_{\alpha\beta\gamma\nu} - \mu^4 q^2 \mathbf{C}_{\alpha\beta\gamma\nu}\Bigg\} - \frac{v_{aacc}}{p}\Bigg\{\left[\mu^3 q + \mu^4 q^2\left(\frac{p}{2} - 3 + n\right)\right]\mathbf{A}_{\alpha\beta\gamma\nu} + 3\mu^4 q^2 \mathbf{B}_{\alpha\beta\gamma\nu}\Bigg\} + \mu^4 q^2(\mathbf{A}_{\alpha\beta\gamma\nu} + \mathbf{B}_{\alpha\beta\gamma\nu}). \tag{45}$$

This form is consistent with the one in Ref. 4. It can be shown that this form includes the most unstable diagonal replicon mode.

The stability in the high temperature phase with $q_T=0$ can be studied by

$$M_{ac}^{\alpha\beta\gamma\nu} \simeq \delta_{ac}\left(2\frac{\delta T}{J} + 2\frac{h}{J}\frac{1}{\sqrt{p-1}} - q_L\frac{1}{p-1}\right)\mathbf{A}_{\alpha\beta\gamma\nu} + \delta_{a1}\delta_{c1} \times \left\{\left[-2\frac{h}{J}\sqrt{p-1} - q_L(p-3)\right]\mathbf{A}_{\alpha\beta\gamma\nu} - q_L\mathbf{B}_{\alpha\beta\gamma\nu}\right\}, \tag{46}$$

to first order in q_L and $\hat{J}z$. This matrix is diagonal in the Potts index. For the most unstable replicon mode, the eigenvalues of $\mathbf{A}_{\alpha\beta\gamma\nu}$ is 1 and that of $\mathbf{B}_{\alpha\beta\gamma\nu}$ is −2, respectively.[4] Using these eigenvalues, the most unstable eigenvalue of the stability matrix (46) is

$$\delta_{ac}\left(2\frac{\delta T}{J} + 2\frac{h}{J}\frac{1}{\sqrt{p-1}} - q_L\frac{1}{p-1}\right) + \delta_{a1}\delta_{c1}\left[-2\frac{h}{J}\sqrt{p-1} - q_L(p-5)\right]. \tag{47}$$

What is the most unstable mode depends on the sign of the second term on the transverse freezing line (28) where the first term is zero. The replicon mode for $a=c=1$ has the eigenvalue

$$2\frac{\delta T}{J} - 2\frac{h}{J}\frac{p-2}{\sqrt{p-1}} - q_L\frac{p^2-6p+6}{p-1}. \tag{48}$$

Using (26), the stability line of this mode becomes

$$\frac{\delta T_{sl}}{J} = \left[\frac{p-2}{\sqrt{p-1}} - \frac{-p^2+6p-6}{\sqrt{2(p-1)}(p-2)}\right]\frac{h_{sl}}{J}. \tag{49}$$

This is compared with the transverse freezing line (29) that is the stability line of the replicon mode for $a=c\neq 1$. For $p > p_{Lc} \simeq 3.2$, the stability line of the replicon mode for $a=c=1$ is located at higher temperature than that of the replicon mode for $a=c\neq 1$ and $h_c/J > 0$. This indicates a replica symmetry breaking phase without transverse freezing for $p > p_{Lc}$. This symmetry breaking will be shown rather weak later.

For $p < p_{Lc}$ where the stability line and the transverse freezing line coincide, the stability analysis is made near the transverse freezing temperature in the transverse freezing phase. To the first order in q_T, q_L, and $\hat{J}z$, (44) becomes

$$M_{ac}^{\alpha\beta\gamma\nu} \simeq \delta_{ac}\Bigg\{\left[-\mu(\mu-1) + 2\beta\mu^2\hat{J}z\frac{1}{\sqrt{p-1}} - \mu^3 q_L\frac{1}{p-1} + \mu^3 q_T\frac{p}{p-1}\right]\mathbf{A}_{\alpha\beta\gamma\nu} - \mu^3 q_T\mathbf{B}_{\alpha\beta\gamma\nu}\Bigg\} - \frac{v_{aacc}}{p}\mu^3 q_T\mathbf{A}_{\alpha\beta\gamma\nu}. \tag{50}$$

The Potts matrix $v_{ac} \equiv v_{aacc}$ in (44) has an eigenvalue $p(p-1)$ with eigenvector $P_a=1$ for all a. This eigenvalue is simply assumed to be maximum here. The most unstable replicon mode has the eigenvalue

$$-\mu(\mu-1)+2\beta\mu^2\hat{J}z\frac{1}{\sqrt{p-1}}-\mu^3 q_{\mathrm{L}}\frac{1}{p-1}+q_{\mathrm{T}}\frac{-p^2+5p-3}{p-1}. \tag{51}$$

With (26) and (61), this eigenvalue at $T/J=T_c/J+\Delta T/J$ for $\Delta T/J<0$ is

$$2\frac{(p-1)(2-p)}{p^2-7p+4}\alpha\frac{\Delta T}{J}, \tag{52}$$

where α is given by (32). This quantity is minus for $p>2$, which indicates that the instability and replica symmetry is broken in the transverse freezing phase.

IV. ORDER PARAMETER FUNCTION IN HIGH TEMPERATURE LONGITUDINAL GLASSY PHASE

As described in the previous section, replica symmetry of the longitudinal glass order parameter is broken in the high temperature phase for $p>p_{\mathrm{Lc}}\simeq 3.2$. In this section, replica symmetry breaking solution in the high temperature glassy phase will be reviewed.[11]

In the high temperature phase with $q_{\mathrm{T}}^{\alpha\beta}=0$, (17) reduces to

$$\begin{aligned}A \simeq{}& \frac{\mu}{4}(p-1)n+\frac{\mu(\mu-1)}{4}\sum_{\alpha\neq\beta}(q_{\mathrm{L}}^{\alpha\beta})^2+\frac{n\beta}{2}[\beta(\hat{J}z)^2-\hat{J}m^2]\\&+\frac{\beta^2\mu}{2}(\hat{J}z)^2\sum_{\alpha\neq\beta}q_{\mathrm{L}}^{\alpha\beta}+\frac{\beta\mu^2}{2}\hat{J}z\sum_{\alpha\neq\beta}(q_{\mathrm{L}}^{\alpha\beta})^2\frac{p-2}{\sqrt{p-1}}\\&+\frac{\mu^3}{12}\sum_{\alpha\neq\beta}\frac{(p-2)^2}{p-1}(q_{\mathrm{L}}^{\alpha\beta})^3+\frac{\mu^3}{6}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}q_{\mathrm{L}}^{\alpha\beta}q_{\mathrm{L}}^{\beta\gamma}q_{\mathrm{L}}^{\gamma\alpha}\\&+\frac{\mu^4}{48}\sum_{\alpha\neq\beta}\frac{p^4-6p^3+12p^2-12p+6}{(p-1)^2}(q_{\mathrm{L}}^{\alpha\beta})^4\\&+\frac{\mu^4}{8}\sum_{\alpha\neq\beta\neq\gamma\neq\alpha}\left[2\frac{(p-2)^2}{p-1}q_{\mathrm{L}}^{\alpha\beta}(q_{\mathrm{L}}^{\beta\gamma})^2q_{\mathrm{L}}^{\gamma\alpha}\right.\\&\left.+\frac{(p-2)^2}{p-1}(q_{\mathrm{L}}^{\alpha\beta})^2(q_{\mathrm{L}}^{\beta\gamma})^2\right]+\frac{\mu^4}{8}\sum_{\substack{\alpha\beta\gamma\delta\\ \text{all different}}}q_{\mathrm{L}}^{\alpha\beta}q_{\mathrm{L}}^{\beta\gamma}q_{\mathrm{L}}^{\gamma\delta}q_{\mathrm{L}}^{\delta\alpha},\end{aligned} \tag{53}$$

where higher order terms in $\hat{J}z$ and $q_{\mathrm{L}}^{\alpha\beta}$ are neglected. Using order parameter function[14] $q_{\mathrm{L}}(x)$ $(0\leqslant x\leqslant 1)$, (53) is written as

$$\begin{aligned}\frac{A}{n} \simeq{}& \frac{\mu}{4}(p-1)-\frac{\mu(\mu-1)}{4}\int_0^1 dx\, q_{\mathrm{L}}(x)^2+\frac{\beta}{2}[\beta(\hat{J}z)^2-\hat{J}m^2]-\frac{\beta^2\mu}{2}(\hat{J}z)^2\int_0^1 dx\, q_{\mathrm{L}}(x)-\frac{\beta\mu^2}{2}\hat{J}z\frac{p-2}{\sqrt{p-1}}\int_0^1 dx\, q_{\mathrm{L}}(x)^2\\&-\frac{\mu^3}{12}\frac{(p-2)^2}{p-1}\int_0^1 dx\, q_{\mathrm{L}}(x)^3+\frac{\mu^3}{6}\left(3\int_0^1 dx\, q_{\mathrm{L}}(x)\int_0^x dy\, q_{\mathrm{L}}(y)^2+\int_0^1 dx\, xq_{\mathrm{L}}(x)^3\right)\\&-\frac{\mu^4}{48}\frac{p^4-6p^3+12p^2-12p+6}{(p-1)^2}\int_0^1 dx\, q_{\mathrm{L}}(x)^4+\frac{\mu^4}{8}\frac{(p-2)^2}{p-1}\left[4\int_0^1 dx\, q_{\mathrm{L}}(x)\int_0^x dy\, q_{\mathrm{L}}(y)^3+4\int_0^1 dx\, q_{\mathrm{L}}(x)^2\int_0^x dy\, q_{\mathrm{L}}(y)^2\right.\\&\left.+\int_0^1 dx(2x+1)q_{\mathrm{L}}(x)^4\right]-\frac{\mu^4}{8}\left[\int_0^1 dx(x^2+1)q_{\mathrm{L}}(x)^4+4\int_0^1 dxq_{\mathrm{L}}(x)\int_0^x dyyq_{\mathrm{L}}(y)^3+12\int_0^1 dx\, q_{\mathrm{L}}(x)\int_0^x dy\, q_{\mathrm{L}}(y)\int_0^y dz\, q_{\mathrm{L}}(z)^2\right.\\&\left.+6\int_0^1 dx\, q_{\mathrm{L}}(x)^2\int_0^x dy\, q_{\mathrm{L}}(y)^2\right].\end{aligned} \tag{54}$$

The order parameter function $q_{\mathrm{L}}(x)$ may be obtained following the method in Ref. 15. Variation with respect to $q_{\mathrm{L}}(x)$ and further differentiation with respect to x lead to $q'_{\mathrm{L}}(x)=0$ or a differential equation for $q_{\mathrm{L}}(x)$. Making a linear approximation for $q_{\mathrm{L}}(x)$ with respect to x, the order parameter function $q_{\mathrm{L}}(x)$ is obtained as

$$q_{\rm L}(x)=\begin{cases} q_0, & 0\leqslant x\leqslant x_0,\\ c\left[x-\dfrac{(p-2)^2}{2(p-1)}\right], & x_0\leqslant x\leqslant x_1,\\ q_1, & x_1\leqslant x\leqslant 1,\end{cases} \qquad (55)$$

where

$$q_0=\frac{\sqrt{2(p-1)}}{p-2}\frac{h}{J}, \qquad (56)$$

$$q_1=-\frac{2(p-1)}{-p^2+6p-6}\frac{\delta T}{J}+\frac{2\sqrt{p-1}(p-2)}{-p^6+6p-6}\frac{h}{J}, \qquad (57)$$

and

$$c=\frac{2(p-1)^2}{6p^6-60p^5+235p^4-444p^3+384p^2-72p-60}. \qquad (58)$$

Because the value of c for $p=4$ is $9/338$ and rather small, the replica symmetry breaking may be considered rather weak. The line of replica symmetry breaking may be obtained from $q_0=q_1$ that gives the previously obtained stability line (49). This shows an adequateness of the obtained solution for $q_{\rm L}(x)$. The continuous replica symmetry breaking scheme presented above is valid only for $p<3+\sqrt{3}$ as can be seen in (57).

Another replica symmetry breaking pattern is the one-step breaking.[3] We have obtained a one-step breaking solution and studied its stability following the method in Ref. 4. Although q_0 and q_1 for the one-step breaking solution is the same as that of (56) and (57), the solution is unstable.

From the above results, the continuous replica symmetry breaking scheme of the high temperature longitudinal glassy phase is valid for $3.2\simeq p_{Lc}<p<3+\sqrt{3}\simeq 4.7$. It is not clear by now what happens for $p>3+\sqrt{3}$.

V. REPLICA SYMMETRY BREAKING OF THE TRANSVERSE FREEZING

The replica symmetry breaking pattern for the transverse glass order parameter will be considered in this section. As stated in the previous section, the replica symmetry breaking for the longitudinal glass order parameter is weak. To avoid complexity, the longitudinal glass order parameter will be assumed to be replica symmetric in this section and we concentrate on the replica symmetry breaking of the transverse glass order parameter.

In the presence of the transverse glass order parameter function $q_{\rm T}(x)$, (54) is extended to

$$\begin{aligned}\frac{A}{n}\simeq{}&\frac{\mu}{4}(p-1)-\frac{\mu(\mu-1)}{4}\left[\int_0^1 dx\, q_{\rm L}(x)^2+(p-2)\int_0^1 dx\, q_{\rm T}(x)^2\right]+\frac{\beta}{2}[\beta(\hat{J}z)^2-\hat{J}m^2]-\frac{\beta^2\mu}{2}(\hat{J}z)^2\int_0^1 dx\, q_{\rm L}(x)\\
&-\frac{\beta\mu^2}{2}\hat{J}z\frac{p-2}{\sqrt{p-1}}\left[\int_0^1 dx\, q_{\rm L}(x)^2-\int_0^1 dx\, q_{\rm T}(x)^2\right]-\frac{\mu^3}{12}\left[\frac{(p-2)^2}{p-1}\int_0^1 dx\, q_{\rm L}(x)^3+3\frac{p-2}{p-1}\int_0^1 dx\, q_{\rm L}(x)q_{\rm T}(x)^2\right.\\
&\left.+\frac{p(p-2)(p-3)}{p-1}\int_0^1 dx\, q_{\rm T}(x)^3\right]+\frac{\mu^3}{6}\left(3\int_0^1 dx\, q_{\rm L}(x)\int_0^x dy\, q_{\rm L}(y)^2+\int_0^1 dx\, xq_{\rm L}(x)^3\right)\\
&+\frac{\mu^3}{6}(p-2)\left(3\int_0^1 dx\, q_{\rm T}(x)\int_0^x dy\, q_{\rm T}(y)^2+\int_0^1 dx\, xq_{\rm T}(x)^3\right)-\frac{\mu^4}{48}\left[\frac{p^4-6p^3+12p^2-12p+6}{(p-1)^2}\int_0^1 dx\, q_{\rm L}(x)^4\right.\\
&-6\frac{p(p-2)}{(p-1)^2}\int_0^1 dx\, q_{\rm L}(x)^2q_{\rm T}(x)^2+4\frac{p(p-2)(p-3)}{(p-1)^2}\int_0^1 dx\, q_{\rm L}(x)q_{\rm T}(x)^3\\
&\left.+\frac{(p-2)(p^4-8p^3+19p^2-15p+6)}{(p-1)^2}\int_0^1 dx\, q_{\rm T}(x)^4\right]+\frac{\mu^4}{8}\frac{(p-2)^2}{p-1}\left[4\int_0^1 dx\, q_L(x)\int_0^x dy\, q_L(y)^3\right.\\
&\left.+4\int_0^1 dx\, q_L(x)^2\int_0^x dy\, q_L(y)^2+\int_0^1 dx(2x+1)q_L(x)^4\right]+\frac{\mu^4}{4}\frac{p-2}{p-1}\left[2\int_0^1 dx\, q_{\rm L}(x)\int_0^x dy\, q_{\rm T}(y)^2q_{\rm L}(y)\right.\\
&\left.+\int_0^1 dx\, q_{\rm T}(x)^2\int_0^x dyq_{\rm L}(y)^2+\int_0^1 dx\, xq_{\rm T}(x)^2q_{\rm L}(x)^2\right]-\frac{\mu^4}{4}\frac{(p-2)^2}{p-1}\left[\int_0^1 dx\, q_{\rm L}(x)^2\int_0^x dy\, q_{\rm T}(y)^2\right.\\
&\left.+\int_0^1 dx\, q_{\rm T}(x)^2\int_0^x dy\, q_{\rm L}(y)^2+\int_0^1 dx\, q_{\rm L}(x)^2q_{\rm T}(x)^2\right]+\frac{\mu^4}{2}\frac{p-2}{p-1}\left[2\int_0^1 dx\, q_{\rm T}(x)\int_0^x dy\, q_{\rm T}(y)^2q_{\rm L}(y)\right.\\
&\left.+\int_0^1 dx\, q_{\rm L}(x)q_{\rm T}(x)\int_0^x dy\, q_{\rm T}(y)^2+\int_0^1 dx\, xq_{\rm L}(x)q_{\rm T}(x)^3\right]+\frac{\mu^4}{4}\frac{p(p-2)(p-3)}{p-1}\left[2\int_0^1 dx\, q_{\rm T}(x)\int_0^x dy\, q_{\rm T}(y)^3\right.\end{aligned}$$

$$+\int_0^1 dx\, q_{\mathrm{T}}(x)^2\int_0^x dy q_{\mathrm{T}}(y)^2+\int_0^1 dx\, x q_{\mathrm{T}}(x)^4\Bigg]+\frac{\mu^4}{8}\frac{(p-2)^2}{p-1}\Bigg[2\int_0^1 dx\, q_{\mathrm{T}}(x)^2\int_0^x dy\, q_{\mathrm{T}}(y)^2+\int_0^1 dx\, q_{\mathrm{T}}(x)^4\Bigg]$$
$$-\frac{\mu^4}{8}\Bigg[\int_0^1 dx(x^2+1)q_{\mathrm{L}}(x)^4+4\int_0^1 dx\, q_{\mathrm{L}}(x)\int_0^x dy\, y q_{\mathrm{L}}(y)^3+12\int_0^1 dx\, q_{\mathrm{L}}(x)\int_0^x dy\, q_{\mathrm{L}}(y)\int_0^y dz\, q_{\mathrm{L}}(z)^2$$
$$+6\int_0^1 dx\, q_{\mathrm{L}}(x)^2\int_0^x dy\, q_{\mathrm{L}}(y)^2\Bigg]-\frac{\mu^4}{8}(p-2)\Bigg[\int_0^1 dx(x^2+1)q_{\mathrm{T}}(x)^4+4\int_0^1 dx\, q_{\mathrm{T}}(x)\int_0^x dy\, y q_{\mathrm{T}}(y)^3$$
$$+12\int_0^1 dx\, q_{\mathrm{T}}(x)\int_0^x dy\, q_{\mathrm{T}}(y)\int_0^y dz\, q_{\mathrm{T}}(z)^2+6\int_0^1 dx\, q_{\mathrm{T}}(x)^2\int_0^x dy\, q_{\mathrm{T}}(y)^2\Bigg]. \tag{59}$$

Making the replica symmetric approximation of $q_{\mathrm{L}}(x)=q_{\mathrm{L}}$, variation with respect to $q_{\mathrm{T}}(x)$ and successive differentiations with respect to x lead to $q'_{\mathrm{T}}(x)=0$ or

$$\left\{\left[x-\frac{p(p-3)}{p-1}\right]^3+\left[x-\frac{p(p-3)}{p-1}\right]\frac{p(-5p^3+28p^2-35p-9)}{6(p-1)^2}\right\}q'_T(x)+\frac{p(5p^3-28p^2+35p+9)}{6(p-1)^2}q_{\mathrm{T}}(x)+\frac{p(p-3)}{6(p-1)}\left(1+\frac{5q_L}{p-1}\right)$$
$$\simeq 0. \tag{60}$$

The solution is obtained as

$$q_{\mathrm{T}}(x)=\begin{cases}0, & 0\leqslant x\leqslant x_0,\\ \left[x-\dfrac{p(p-3)}{2(p-1)}\right]\left(1+\dfrac{5q_{\mathrm{L}}}{p-1}\right)\dfrac{4(p-1)^2}{p(-7p^3+38p^2-43p-18)}, & x_0\leqslant x\leqslant x_1,\\ \dfrac{2(p-1)}{p^2-5p+2}\alpha\dfrac{\Delta T}{J}, & x_1\leqslant x\leqslant 1,\end{cases} \tag{61}$$

where $x_0=p(p-3)/2(p-1)$, $\Delta T/J$ is defined in (31), and α is given by (32). This continuous order parameter function is valid for $p<p_{\mathrm{T}1}$ where $p_{\mathrm{T}1}$ is a solution of

$$-7p^3+38p^2-43p-18=0. \tag{62}$$

At $p=p_{T1}\simeq 3.4$, the coefficient of x in $q_{\mathrm{T}}(x)$ diverges, which indicates a one-step replica symmetry breaking.

For $p>p_{T1}$, a one-step replica symmetry breaking solution is considered. The transverse glass order parameter with one-step replica symmetry breaking is expressed by

$$q_{\mathrm{T}}^{\alpha\beta}=\begin{cases}q_{\mathrm{T}}, & \text{if } (\alpha,\beta) \text{ belong to the same group;}\\ q_{0\mathrm{T}}, & \text{otherwise.}\end{cases} \tag{63}$$

The n replicas are divided in groups of m_{q} that are determined by the saddle point equations. The saddle point equation for $q_{0\mathrm{T}}$ has the solution $q_{0\mathrm{T}}=0$. Adopting this solution, (17) becomes

$$\frac{A}{n}\simeq\frac{\mu}{4}(p-1)+\frac{\mu(\mu-1)}{4}[(n-1)q_{\mathrm{L}}^2+(p-2)(m_{\mathrm{q}}-1)q_{\mathrm{T}}^2]+\frac{\beta}{2}[\beta(\hat{J}z)^2-\hat{J}m^2]+\frac{\beta^3}{6}(\hat{J}z)^3\frac{p-2}{\sqrt{p-1}}+\frac{\beta^2\mu}{2}(\hat{J}z)^2(n-1)q_{\mathrm{L}}$$
$$+\frac{\beta\mu^2}{2}\hat{J}z\frac{p-2}{\sqrt{p-1}}[(n-1)q_{\mathrm{L}}^2-(m_{\mathrm{q}}-1)q_{\mathrm{T}}^2]+\frac{\mu^3}{12}\left[\frac{(p-2)^2}{p-1}(n-1)q_{\mathrm{L}}^3+3\frac{p-2}{p-1}(m_{\mathrm{q}}-1)q_{\mathrm{L}}q_{\mathrm{T}}^2+\frac{p(p-2)(p-3)}{p-1}(m_{\mathrm{q}}-1)q_{\mathrm{T}}^3\right]$$
$$+\frac{\mu^3}{6}[(n-1)(n-2)q_{\mathrm{L}}^3+(m_{\mathrm{q}}-1)(m_{\mathrm{q}}-2)(p-2)q_{\mathrm{T}}^3]+\frac{\beta^4}{24}\frac{p^2-6p+6}{p-1}(\hat{J}z)^4$$
$$+\frac{\beta^3\mu}{4}(\hat{J}z)^3\frac{p-2}{\sqrt{p-1}}(n-1)q_{\mathrm{L}}+\frac{\beta^2\mu^2}{4}(\hat{J}z)^2\left[2(n-1)\left[\frac{(p-2)^2}{p-1}+n-2\right]q_{\mathrm{L}}^2-(m_{\mathrm{q}}-1)\frac{(p-2)(p-3)}{p-1}q_{\mathrm{T}}^2\right]$$
$$+\frac{\beta\mu^3}{6}\hat{J}z\frac{p-2}{(p-1)^{3/2}}\{(n-1)(p^3-3p+3)q_{\mathrm{L}}^3+(m_{\mathrm{q}}-1)[-3q_{\mathrm{L}}q_{\mathrm{T}}^2-p(p-3)q_{\mathrm{T}}^3]\}$$

$$+\frac{\beta\mu^3}{2}\hat{J}z\frac{p-2}{\sqrt{p-1}}[2(n-1)(n-2)q_{\rm L}^3-(n-2)(m_{\rm q}-1)q_{\rm L}q_{\rm T}^2-(m_{\rm q}-1)(m_{\rm q}-2)q_{\rm T}^3]-\frac{\mu^4}{48}\frac{1}{(p-1)^2}[(p^4-6p^3$$
$$+12p^2-12p+6)(n-1)q_{\rm L}^4-6p(p-2)^2(m_{\rm q}-1)q_{\rm L}^2q_{\rm T}^2+4p(p-2)(p-3)(m_{\rm q}-1)q_{\rm L}q_{\rm T}^3+(p-2)(p^4-8p^3+19p^2-15p+6)$$
$$\times(m_{\rm q}-1)q_{\rm T}^4]+\frac{\mu^4}{4}\frac{p-2}{p-1}\{(n-1)(n-2)(p-2)q_{\rm L}^4+(n-2)(m_{\rm q}-1)q_{\rm L}^2q_{\rm T}^2+(m_{\rm q}-1)(m_{\rm q}-2)[2q_{\rm L}q_{\rm T}^3+p(p-3)q_{\rm T}^4]\}$$
$$+\frac{\mu^4}{8}\frac{(p-2)^2}{p-1}[(n-1)(n-2)q_{\rm L}^4-2(n-2)(m_{\rm q}-1)q_{\rm L}^2q_{\rm T}^2+(m_{\rm q}-1)(m_{\rm q}-2)q_{\rm T}^4]+\frac{\mu^4}{8}[(n-1)(n-2)(n-3)q_{\rm L}^4$$
$$+(m_{\rm q}-1)(m_{\rm q}-2)(m_{\rm q}-3)(p-2)q_{\rm T}^4]. \quad (64)$$

The saddle point equations with respect to $q_{\rm L}$, $q_{\rm T}$, and $m_{\rm q}$ become

$$0=-\frac{\mu(\mu-1)}{2}q_{\rm L}-\frac{\beta^2\mu}{2}(\hat{J}z)^2-\beta\mu^2\hat{J}z\frac{p-2}{\sqrt{p-1}}q_{\rm L}-\frac{\mu^3}{4}\frac{p^2-8p+8}{p-1}q_{\rm L}^2+\frac{\mu^3}{4}\frac{p-2}{p-1}(m_{\rm q}-1)q_{\rm T}^2-\frac{\beta^3\mu}{4}(\hat{J}z)^3\frac{p-2}{\sqrt{p-1}}$$
$$-\beta^2\mu^2(\hat{J}z)^2\frac{p^2-6p+6}{p-1}q_{\rm L}-\frac{\beta\mu^3}{2}\hat{J}z\frac{p-2}{(p-1)^{3/2}}(p^2-15p+15)q_{\rm L}^2-\frac{\beta\mu^3}{2}\hat{J}z\frac{p-2}{(p-1)^{3/2}}(2p-3)(1-m_{\rm q})q_{\rm T}^2$$
$$-\frac{\mu^4}{12}\frac{1}{(p-1)^2}(p^4-42p^3+228p^2-372p+186)q_{\rm L}^3+\frac{\mu^4}{4}\frac{p-2}{(p-1)^2}(-3p^2+14p-12)(1-m_{\rm q})q_{\rm L}q_{\rm T}^2-\frac{\mu^4}{12}\frac{p-2}{(p-1)^2}(1-m_{\rm q})$$
$$\times[6(p-1)m_{\rm q}+p^2-15p+12]q_{\rm T}^3, \quad (65)$$

$$0=\mu(p-2)(1-m_{\rm q})q_{\rm T}\Bigg(-\frac{\mu-1}{2}-\beta\mu\hat{J}z\frac{1}{\sqrt{p-1}}-\frac{\mu^2}{2}\frac{1}{p-1}q_{\rm L}-\frac{\mu^2}{4}\frac{1}{p-1}[p^2-7p+4+2m_{\rm q}(p-1)]q_{\rm T}+\frac{\beta^2\mu}{2}(\hat{J}z)^2\frac{p-3}{p-1}$$
$$+\frac{\beta\mu^2}{2}\hat{J}z\frac{1}{(p-1)^{3/2}}\{q_{\rm L}[2-4(p-1)]+q_{\rm T}[p^2-9p+6+3m_{\rm q}(p-1)]\}+\frac{\mu^3}{12}\frac{1}{(p-1)^2}\{3q_{\rm L}^2(-3p^2+14p-12)+3q_{\rm L}q_{\rm T}$$
$$\times[-p^2+15p-12-6m_{\rm q}(p-1)]-q_{\rm T}^2[p^4-32p^3+139p^2-123p+18+6m_{\rm q}(2p^3-12p^2+13p-3)+6m_{\rm q}^2(p-1)^2]\}\Bigg), \quad (66)$$

and

$$0=\mu(p-2)q_{\rm T}^2\Bigg(\frac{\mu-1}{4}-\frac{\beta\mu}{2}\hat{J}z\frac{1}{\sqrt{p-1}}+\frac{\mu^2}{12}\frac{1}{p-1}\{3q_{\rm L}+[p^2-9p+6+4m_{\rm q}(p-1)]q_{\rm T}\}-\frac{\beta^2\mu}{4}(\hat{J}z)^2\frac{p-3}{p-1}+\frac{\beta\mu^2}{6}\hat{J}z\frac{1}{(p-1)^{3/2}}$$
$$\times\{(6p-9)q_{\rm L}-[p^2-12p+9+6m_{\rm q}(p-1)]q_{\rm T}\}+\frac{\mu^3}{48}\frac{1}{(p-1)^2}\{(18p^2-84p+72)q_{\rm L}^2+[4p^2-84p+72+48m_{\rm q}(p-1)]q_{\rm L}q_{\rm T}$$
$$+[p^4-44p^3+211p^2-201p+36+12m_{\rm q}(2p^3-13p^2+15p-4)+18m_{\rm q}^2(p-1)^2]q_{\rm T}^2\}\Bigg), \quad (67)$$

respectively.

From (65), $q_{\rm L}$ is given by the replica symmetric solution (26). From (66) and (67), we have

$$\frac{\mu^2}{4}\frac{1}{p-1}[p^2-7p+4+2m_{\rm q}(p-1)]q_{\rm T}\simeq-\frac{\mu-1}{2}+\beta\mu\hat{J}z\frac{1}{\sqrt{p-1}}-\frac{\mu^2}{2}\frac{1}{p-1}q_{\rm L} \quad (68)$$

and

$$\frac{\mu^2}{6}\frac{1}{p-1}[p^2-9p+6+4m_{\rm q}(p-1)]q_{\rm T}\simeq-\frac{\mu-1}{2}+\beta\mu\hat{J}z\frac{1}{\sqrt{p-1}}-\frac{\mu^2}{2}\frac{1}{p-1}q_{\rm L}. \quad (69)$$

The right hand sides of these equations become zero at the transverse glass transition line (28). Using (31), this quantity just below the transition line is $\alpha(\Delta T/J)$ where α is given in (32). From (68) and (69), $m_{\rm q}$ and $q_{\rm T}$ are given by

$$m_q = \frac{p(p-3)}{2(p-1)} \tag{70}$$

and

$$q_T \simeq \frac{2(p-1)}{p^2-5p+2}\alpha\frac{\Delta T}{J}. \tag{71}$$

These values are consistent with x_0 and $q_T(x)$ for the continuous replica symmetry breaking solution. At $p=(5+\sqrt{17})/2 \simeq 4.56$, $m_q=1$ and q_T diverges. For $p>(5+\sqrt{17})/2$, a first order transition with a jump in the order parameter q_T is expected.[4]

To locate the transition point and obtain the jump in q_T, a condition for the free energy $F_0=F_G$ is added, where F_0 is the free energy for $q_T=0$ and F_G is the free energy of the transverse glassy phase. Using (64), this condition is obtained as

$$\begin{aligned}0 = \mu(p-2)(m_q-1)q_T^2\Bigg(&\frac{\mu-1}{4} - \frac{\beta\mu}{2}\hat{J}z\frac{1}{\sqrt{p-1}} \\ &+ \frac{\mu^3}{12}\frac{1}{p-1}\{3q_L + [p^2-7p+4+2m_q(p-1)q_T\} \\ &+ \frac{\mu^4}{48}\frac{1}{(p-1)^2}\{(18p^2-84p+72)q_L^2 + [4p^2-60p+48 \\ &+ 24m_q(p-1)]q_Lq_T + [p^4-32p^3+139p^2-123p+18 \\ &+ 6m_q(2p^3-12p^2+13p-3) + 6m_q^2(p-1)^2]q_T^2\}\Bigg).\end{aligned} \tag{72}$$

From this equation, (66), and (67) we have three equations for $\Delta T/J$, q_T, and m_q. They are given by

$$\begin{aligned}-\frac{1}{2}\alpha\frac{\Delta T}{J} &+ \frac{1}{12}\frac{1}{p-1}[p^2-7p+4+2m_q(p-1)]q_T \\ &+ \frac{1}{48}\frac{1}{(p-1)^2}[p^4-32p^3+139p^2-123p+18 \\ &+ 6m_q(2p^3-12p^2+13p-3)+6m_q^2(p-1)^2]q_T^2 \simeq 0,\end{aligned} \tag{73}$$

$$\begin{aligned}\alpha\frac{\Delta T}{J} &- \frac{1}{4}\frac{1}{p-1}[p^2-7p+4+2m_q(p-1)]q_T \\ &- \frac{1}{12}\frac{1}{(p-1)^2}[p^4-32p^3+139p^2-123p+18 \\ &+ 6m_q(2p^3-12p^2+13p-3)+6m_q^2(p-1)^2]q_T^2 \simeq 0,\end{aligned} \tag{74}$$

and

$$\begin{aligned}-\frac{1}{2}\alpha\frac{\Delta T}{J} &+ \frac{1}{12}\frac{1}{p-1}[p^2-9p+6+4m_q(p-1)]q_T \\ &+ \frac{1}{48}\frac{1}{(p-1)^2}[p^4-44p^3+211p^2-201p+36 \\ &+ 12m_q(2p^3-13p^2+15p-4)+18m_q^2(p-1)^2]q_T^2 \simeq 0,\end{aligned} \tag{75}$$

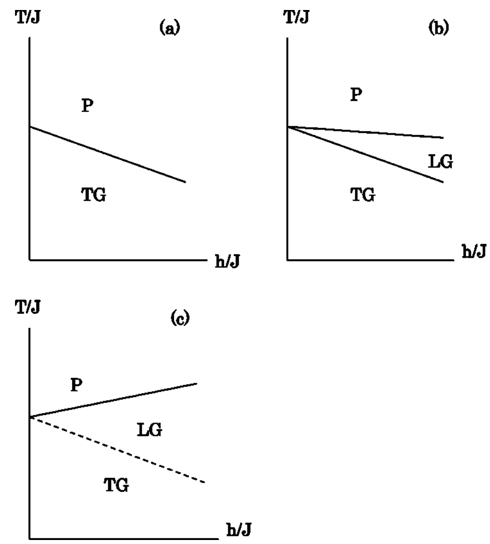

FIG. 1. Schematic phase diagrams of the mean-field p-state Potts glass model in a field for (a) $p \lesssim 3.2$, (b) $3.2 \lesssim p \lesssim 4.6$, and (c) $p \gtrsim 4.6$. P, TG, and LG represent the paramagnetic phase, glass phase with a nonzero transverse glass order parameter, and a weakly replica symmetry breaking longitudinal glassy phase, respectively. Real and broken lines are second and first order transition lines, respectively.

respectively. The solution for $p=(5+\sqrt{17})/2+\varepsilon$ ($\varepsilon \ll 1$) is

$$\frac{\Delta T_g}{J} \equiv \frac{\delta T_g}{J} - \frac{\delta T_c}{J} \simeq 0.0110\varepsilon^2, \tag{76}$$

$$q_{Tg} \simeq 0.141\varepsilon, \tag{77}$$

and

$$m_{q_g} = 1. \tag{78}$$

Close to the transition, the free energy is given by

$$\frac{\delta F_G}{J} \equiv \frac{F_G - F_0}{J} \simeq 1.54 q_{Tg}\left(\frac{T-T_g}{J}\right)^2. \tag{79}$$

As in the case without a field,[3,4] there is no latent heat.

VI. SUMMARY

The results are summarized as schematic phase diagrams in Fig. 1. P, TG, and LG represent the paramagnetic phase, the glass phase with a nonzero transverse glass order parameter, and the weakly replica symmetry breaking longitudinal glassy phase, respectively. Real and broken lines are second and first order transition lines, respectively. The longitudinal glassy phase appears for $p \gtrsim 3.2$. The replica symmetry breaking of the longitudinal glassy phase and the transition from the paramagnetic phase to the longitudinal glassy phase for $p \gtrsim 4.7$ has not yet been well understood.

The transition lines are straight in the phase diagrams for small fields. This is in contrast with the Gabay-Toulouse line for the vector spin glass.

The intermediate phase with the longitudinal freezing is characteristic of the Potts glass model although the effect of the replica symmetry breaking is rather weak.

The qualitative feature of the transition to the transverse glassy phase is similar to that without a field although the values of the Potts index where the changes in the nature of the transition occur are different. Within the replica symmetric approximation of the longitudinal order parameter, the transverse glass order parameter function is continuous for $p \lesssim 3.4$ and becomes one-step replica symmetry breaking for $3.4 \lesssim p \lesssim 4.6$. For $p \gtrsim 4.6$, the transition is discontinuous but without latent heat.

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