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Phases intermediate between a two-dimensional electron liquid and Wigner crystal

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We show that there can be no direct first-order transition between a Fermi liquid and an insulating electronic (Wigner) crystalline phase in a clean two-dimensional electron gas in a metal-oxide-semiconductor field-effect transistor (MOSFET); rather, there must always exist intermediate "microemulsion" phases, and an accompanying sequence of continuous phase transitions. Among the intermediate phases which we find are a variety of electronic liquid crystalline phases, including stripe-related analogues of classical smectics and nematics. The existence of these phases can be established in the neighborhood of the phase boundaries on the basis of an *asymptotically exact* analysis, and reasonable estimates can be made concerning the ranges of electron densities and device geometries in which they exist. They likely occur in clean Si MOSFETs in the range of densities in which an "apparent metal to insulator transition" has been observed in existing experiments. We also point out that, in analogy with the Pomaranchuk effect in ^{3}He, the Wigner crystalline phase has higher spin entropy than the liquid phase, leading to an increasing tendency to crystallization with increasing temperature.

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I. INTRODUCTION

In discussions of the theory of the two-dimensional electron gas (2DEG), it is generally accepted that, as a function of electron density n, there is a first-order quantum $(T=0)$ phase transition from a high-density liquid[1] to a low-density Wigner crystalline phase.[2] This assumption is reasonable in the case of a triangular Wigner crystal due to the presence of cubic invariants in the Landau free energy,[3] and for other lattices due to the general expectation[4] that fluctuations will always render a freezing transition first order. The transition is thought to occur when the dimensionless ratio $r_s \equiv [\pi n (a_B)^2]^{-1/2}$ exceeds a critical value[5] $r_s = r_c \sim 38$, where a_B is the effective Bohr radius in the semiconductor. However, this generally accepted picture is manifestly incorrect for the 2DEG in a metal-oxide-semiconductor field-effect transistor (MOSFET), and possibly more generally!

Each electron in the 2DEG in a clean MOSFET drags along with it an image charge in the ground plane above. Consequently, at small separations, the interaction between the electrons is the $V(r) \sim e^2/\epsilon r$ Coulomb interaction, while for separations larger than the distance to the gate d, it is the repulsive dipole-dipole interaction, $V(r) \sim 4e^2 d^2/\epsilon r^3$. (Here ϵ is the dielectric constant of the host semiconductor.) In two-dimensional (2D) systems with dipolar interactions, the following simple argument leads to the concussion that first-order phase transitions are forbidden: In systems with interactions that fall more rapidly than $1/r^2$, there exists a "forbidden" range of densities in the neighborhood of a first-order phase transition where macroscopic phase separation reduces the free energy of the system. However, when we come to compute the surface tension between two macroscopic phases, we find that $1/r^3$ interactions are marginal: for shorter range interactions, there is a well-defined scale independent surface tension, σ, while for longer range interactions, σ is scale dependent. Specifically, for dipolar interactions, the interfacial contribution to the free energy of an arbitrary macroscopic mixture of two phases is (see, e.g., Refs. 6 and 7)

$$F_\sigma = \int ds \sigma_0(\hat{\theta}) - \frac{\sigma_1}{2} \int \frac{d\mathbf{l} \cdot d\mathbf{l}'}{\sqrt{|\mathbf{l} - \mathbf{l}'|^2 + d^2}}. \tag{1}$$

Here, the arclength integral ds, runs along[8] the interfaces between the two phases, $\hat{\theta}(s)$ is the local orientation of the interface, $\sigma_0(\hat{\theta})$ is the (in general orientation dependent and by assumption positive) short-range piece of the surface tension, $d\mathbf{l}$ runs along the interfaces,[8] $\sigma_1 = 2e^2(\Delta n)^2 d^2/\epsilon$, and Δn is the density difference between the coexisting phases. The second (nonlocal) term in Eq. (1) comes from the long-range parts of the dipolar interaction. One can also view it as the leading finite-size correction to the capacitance of parallel-plate capacitors due to the fringing fields.[9]

It is important to note that the second term in Eq. (1) gives a negative contribution to the effective surface tension which diverges logarithmically with length; for example, an isolated straight segment of interface of length L has $F_\sigma = L\{\sigma_0 - \sigma_1 \ln[L/2d]\}$. This implies that there is an absolute instability of the macroscopically phase separated state - in the regime of the phase diagram where a classical Maxwell construction would lead to two-phase coexistence, a state formed from a "microemulsion" of the two phases (with a character and length scale to be determined), has lower free energy. Thus, instead of a first-order transition between two phases, there must always be an intermediate regime in which one or more microemulsion phase occurs, bounded by one or more line of continuous phase transitions.

At this point we would like to compare this situation with the Coulomb case (no ground plane) where macroscopic phase separation is forbidden. The nature of the phases that result from the "Coulomb frustrated phase separation[10,11]" in what would otherwise have been the forbidden range of densities is an issue of potentially relevance in many highly correlated materials. However, the inhomogeneities that occur in this situation are typically microscopic in scale, and so difficult to distinguish from more familiar charge density wave structures.[12] Moreover, the relevant microscopic details are difficult to treat with any degree of rigor. (It is an interesting[13] question, which we would like to reopen, whether there are intermediate phases between the Fermi liquid and Wigner crystal phases in the 2DEG with pure Coulomb interactions.)

The character of the microemulsion of the two coexisting phases is determined by minimizing F_σ in Eq. (1); the result depends on how anisotropic the function $\sigma_0(\hat{\theta})$ is. The case where $\sigma_0(\hat{\theta})$ is independent of $\hat{\theta}$ has been considered in different contexts, including lipid films (e.g., Ref. 14), two-dimensional uniaxial ferromagnets (e.g., Ref. 15), and the 2DEG in MOSFET's.[16,17] The resulting phase diagram includes both stripe and bubble phases, with stripes preferred in the center of the phase separated region and bubbles generally thought to be slightly lower in energy when one phase is in extreme minority. Current estimates[7] place the difference between the dilute stripe and bubble energies at about 6%. In the earlier literature, it was assumed[14,15] that there is a direct first-order transition between uniform stripe and bubble phases. This is incorrect, even at mean-field level, since, as we have shown, first-order phase transitions are forbidden. Thus, a sequence of continuous phase transitions (which we discuss below) must replace the putative first-order transition.[16]

In the present case, where at least one of the two coexisting phases is crystalline, the angular dependence of $\sigma_0(\hat{\theta})$ is not negligible, reflecting the tendency of crystals to facet. Clearly, a strong angle dependence of $\sigma_0(\hat{\theta})$ tends to favor stripe phases (where all interfaces lie along the direction in which $\sigma_0(\hat{\theta})$ is minimal) relative to any form of bubble phase.

In the present paper, we characterize the phase diagram, and in particular, the universal aspects of the intermediate phases and phase transitions that are expected at low or zero temperature in an ideal MOSFET (i.e., in the absence any disorder). We will consider explicitly the case in which d is large compared to the spacing between electrons, $nd^2 \gg 1$, as in this limit (as we shall see) fluctuation effects are parametrically small and an appropriate mean-field theory provides a valid zeroth-order description of the phases. In Sec. I, we first discuss the mean-field phase diagram, then in Secs. II and III we discuss the effects of weak thermal and quantum fluctuations, respectively. In Sec. IV, we discuss some of the implications of the present results for the properties of real devices (which, alas, have non-negligible disorder), and in Sec. V we discuss some incompletely developed ideas concerning further implications of the present line of analysis.

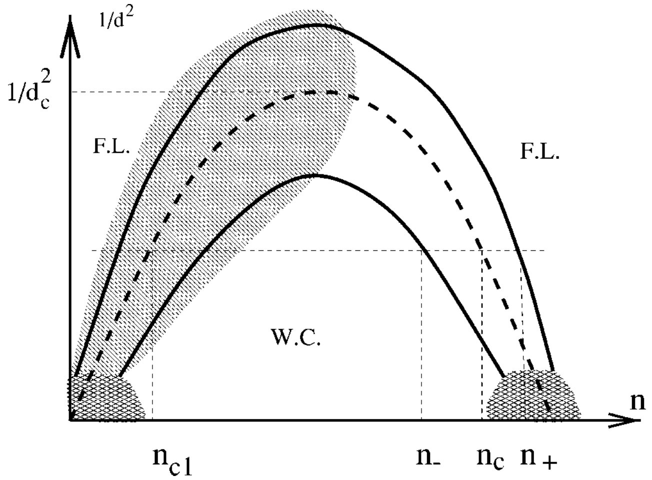

FIG. 1. The $T=0$ phase diagram of the 2DEG in an MOSFET. The dashed line indicates the mean-field critical density $n_c(d)$, where the free energies of the uniform Wigner crystal (W.C.) and Fermi liquid (F.L.) phases cross. The solid lines mark the boundaries of the regime of the intermediate microemulsion (stripe or bubble) phases. At mean-field level, these solid lines are Lifshitz transitions. They approximately coincide with the regime of macroscopic two-phase coexistence ($n_- < n < n_+$) derived from a Maxwell construction. The hatched area represents the regime in which the regions of the two coexisting phases have sizes of order the electron spacing, so quantum fluctuations are order 1, and hence may substantially alter the mean-field character of the phases and phase transitions. The cross-hatched areas denote the regimes of Coulomb frustrated phase separation where even the mean-field character of the phase diagram is not known.

II. MEAN-FIELD PHASE DIAGRAM

Two dimensionless parameters determine the physics of the 2DEG in a MOSFET, r_s (defined above) and a_B/d. Let us start with a discussion of the zero-temperature mean-field phase diagram of this system, assuming only uniform states. If $nd^2 \gg 1$, the free energy per unit area can be represented by the sum $f(n)=f^{(C)}+f^{(el)}$ of the energy density of a capacitor $f^{(C)}=(en)^2/2C$ and the internal free-energy density of the electron liquid $f^{(el)}$. Here $C=(\epsilon d)^{-1}$ is the capacitance per unit area. At high electron densities, $r_s \ll 1$, the kinetic energy of the electrons is much larger than their potential energy, so the system forms a Fermi liquid. At small densities $r_s \gg 1$ (but still $nd^2 \gg 1$) the Coulomb energy of the electrons is much larger than the kinetic energy, so the ground state is crystalline.

However, at even smaller densities when $nd^2 \ll 1$, the electrons interact only via dipole interactions, so the kinetic energy is larger than the potential, and the system again has a Fermi liquid groundstate. [See discussion surrounding Eq. (12).] For $d/a_B \gg 1$, this implies that the phase diagram of the system has reentrant transitions as a function of n (along the dashed-dotted trajectory in Fig. 1) from a Fermi liquid phase for $n > n_c \approx r_c^{-2}(\pi a_B^2)^{-1}$ to a Wigner crystal phase for $n_c > n > n_{c1} \sim (\pi d^2)^{-1}$, to a Fermi liquid phase for $n_{c1} > n$. With decreasing d/a_B, n_{c1} and n_c move toward each other, until for $d < d_c \sim r_c a_B$, the Wigner crystal phase disappears entirely. This is represented by the dashed line in Fig. 1.

As a next step, we improve this phase diagram by allowing for the possibility of inhomogeneous states. There is a range of forbidden densities about the critical density in which macroscopic phase separation into regions of high-and low-density phase has lower free energy than the uniform state.

Let us briefly review the salient features of the Maxwell construction for phase coexistence, as applied in the present context. For given average density, n, we consider a state in which a fraction, x, of the system is at a higher than average density, $n_+ > n$, and a fraction $(1-x)$ is at a lower than average density, $n_- < n$, such that $xn_+ + (1-x)n_- = n$. We then minimize the total free energy with respect to n_+ and n_-. The result of this minimization is an implicit expression for the densities of the two coexisting phases,

$$\mu_+ + \frac{n_+}{C} = \mu_- + \frac{n_-}{C} = \frac{[f(n_-) - f(n_+)]}{\Delta n}, \tag{2}$$

where $\mu_\pm = -\partial f^{(el)}(n_\pm)/\partial n_\pm$ are the chemical potentials in the two phases, and $\Delta n \equiv [n_+ - n_-]$. Phase coexistence occurs for $n_- < n < n_+$, where the fraction of the two phases is determined by the lever rule,

$$x = (n - n_-)/\Delta n. \tag{3}$$

Equation (2) is somewhat complicated, but it can be greatly simplified when the forbidden region is relatively small ($\Delta n \ll n_c$); in this case, we can linearize the density dependence of the free energy about the critical density,

$$f^{(el)}(n_\pm) = f^{(el)}(n_c) - \mu_\pm(n_\pm - n_c) + \cdots, \tag{4}$$

where $\cdots$ represents higher-order terms in powers of $(n_\pm - n_c)$. To this level of approximation,

$$n_\pm = n_c \pm \frac{\Delta n}{2}; \quad \Delta n = \frac{(\mu_- - \mu_+)\epsilon}{e^2 d}. \tag{5}$$

The discontinuity of the chemical potential $(\mu_- - \mu_+) > 0$ is determined by microscopic physics, and is only small to the extent that the putative transition is weakly first order. Whether or not the transition is strongly first order, for d large, Δn is self-consistently small.

The validity of the Maxwell construction rests on the implicit assumption that the interface energy between the coexisting phases is positive, so the amount of interface is minimized. As we have seen, in the dipolar case this assumption is invalid. We can construct a state with lower free energy by making an inhomogeneous mixture of the two coexisting phases to increase the amount of interface. To complete the mean-field analysis, one should minimize Eq. (1) with respect to the shape of the minority phase regions at given area of the phase, the area being given, to first approximation, by the Maxwell rule.

A. Stripe phases

To begin with, let us consider only striped phases. This is fully justified in the case of strong anisotropy of the surface energy. (As we will see later even in the opposite case when $\sigma(\hat{\theta})$ is isotropic, there are regions in the phase diagram where this assumption is relevant.) The interfacial free-energy density for a striped phase is easily computed from Eq. (1) to be

$$f_\sigma = L^{-1}\{2\sigma_0 - 4\sigma_1 \ln[L_+ L_-/dL]\}, \tag{6}$$

where $L_\pm$ are the widths of the high- and low-density regions, respectively, and $L = L_+ + L_-$ is the period of the stripe structure. Minimizing Eq. (6) at fixed areal fraction of the high-density phase, $x \equiv L_+/L$, we get

$$L_- = \frac{d}{x}e^{1+\gamma}, \quad L_+ = \frac{d}{(1-x)}e^{1+\gamma} \tag{7}$$

with $\gamma = \sigma_0/2\sigma_1$. It is important to note that as $x \to 0$, the stripes of the high density phase approach a finite limiting width, $L_+ \to L_0 = de^{1+\gamma}$, although the spacing between stripes, L_-, diverges in proportion to $1/x$. Also, because the minimized value of $f_\sigma = -4\sigma_1/L$ is negative, the region of stability of the striped phase in fact extends somewhat beyond the edges (n_- and n_+) of the two-phase region derived from the Maxwell construction.

Finally, it is necessary to estimate the magnitude of γ; if it is of order 1, then $L_0 \sim d$, but if $\gamma \gg 1$, then L_0 is exponentially larger than atomic lengths. So long as the stripe phase occurs in a relatively narrow range of n, we can use Eq. (5) to estimate σ_1, with the result that $\gamma \sim \sigma_0 e^2/\epsilon[\mu_+ - \mu_-]^2$, which is a ratio of microscopic electronic energies. Thus, except under special circumstances, we expect that $\gamma \sim 1$, and hence that $L_0 \sim d$. However, so long as $nd^2 \gg 1$, the stripe widths are still large compared to the spacing between electrons, which validates the macroscopic approach taken here.

In short, at mean-field level, as a function of decreasing density the system evolves from the Fermi liquid phase, through intermediate stripe phases, to the Wigner crystal, as summarized in Fig. 2(a).

(1) Starting in the uniform Fermi liquid phase, as the density is varied across n_+, the system undergoes a transition to a stripe phase, consisting of a periodic array of far separated stripes of Wigner crystal, with characteristic width L_0. This transition is analogous to a Lifshitz transition, in that the period of the ordered phase diverges at the transition.[19] Thus, the arguments[4] that fluctuations will generally drive an otherwise continuous freezing transition first order do not apply; the continuous character of this transition is robust.

(2) There is, of course, some coupling between the translational motion of the crystalline order in neighboring stripes, so at mean-field level the crystalline order will be locked from stripe to stripe. Consequently, the stripe ground state breaks translation symmetry not only in the direction perpendicular to the stripes, but along the stripe direction as well. However, near the transition, where $x \ll 1$, the spacing between stripes is large compared to L_0, so this coupling is exponentially small; consequently, this locking can be neglected for all practical purposes. Therefore, this phase should operationally be classified as an electron smectic,[20] in which translation symmetry is unbroken along the stripe direction. (There remains the interesting academic question of principle whether or not quantum fluctuations are able to truly stabilize this smectic phase at $T=0$, this is closely related to the issue of whether "floating phases" are stable in quasi-1D electronic systems.[17,21])

(3) Near $x = 1/2$, the stripes of Wigner crystal and the intervening Fermi liquid are comparable in width. As $x \to 1$,

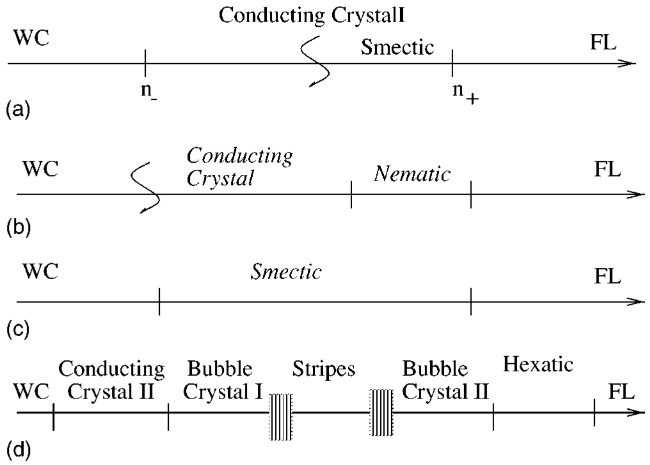

FIG. 2. Schematic representation of the sequence of intermediate states as the 2DEG evolves from the Fermi liquid (FL) to the Wigner crystal (W.C.). In a, b, and c, we assume that $\sigma(\theta)$ is sufficiently anisotropic that bubble phases are suppressed. (a) The mean-field phase diagram. Under appropriate circumstances, this also represents the true sequence of phase transitions at $T=0$. (b) The phase diagram at nonzero temperature with a rotationally invariant Hamiltonian. (c) The phase diagram at nonzero temperature with a preferred axis, for instance, due to an in-field magnetic field. Vertical lines represent phase transitions and wavy lines crossovers. Phases with power-law order are named in italics. The smectic phase in (a) is weakly unstable to crystallization at mean-field level, but may be stabilized by quantum fluctuations. (d) The phase diagram including bubble phases at $T=0$ in the presence of quantum fluctuations. The hatched areas correspond to the sequence of transitions involving megastripes of bubble and stripe phases discussed in the text.

the system is better thought of as stripes of Fermi liquid separated by broad regions of Wigner crystal. At some point, the crystalline order becomes so rigid that the coupling across the liquid stripes is no longer negligible. At this point, the striped state is fully crystalline, in the sense that translation symmetry is broken in both directions, and the structure factor contains Bragg peaks. However, this phase is still qualitatively distinct from the Wigner crystal. Since generally speaking the Fermi wave vector is unrelated to the Bragg vectors of the Wigner crystal, the liquid in the stripe "rivers" can still conduct current in the stripe direction. For want of a better name, we christen this state a striped "conducting crystalI." (See Fig. 2.)

(4) At $x=1$, the transition from the conducting to Wigner crystal mirrors the smectic to Fermi liquid transition; it is also a Lifshitz transition at which the period of the stripe order diverges.

B. Bubble phases

So far, this analysis ignores the possibility of bubble phases. Whether or not there is a regime in which the lowest energy mean-field state is a bubble phase depends, as we mentioned before, on the degree of anisotropy of the microscopic surface tension $\sigma_0(\hat{\theta})$. It may happen, due to the anisotropy of the Wigner crystal, that $\sigma_0(\hat{\theta})$ is sufficiently anisotropic that bubble phases never intrude upon the phase diagram. It is also possible to force the issue by artificially enhancing the anisotropy of $\sigma_0(\hat{\theta})$. This can be done by explicitly breaking the rotational symmetry of the 2DEG, for instance, by applying an in-plane magnetic field or by using a sufficiently anisotropic surface in the construction of the MOSFET. In this case, no more need be said.

However, the Wigner crystal is generally thought to be triangular. In this case the surface energy is sufficiently isotropic that for x near 0 or 1, there will be a range of x in which bubble phases have lower energy than the stripe phase; for x near 0, such a phase consists of far separated crystallites in a metallic sea, while for x near 1, it is far separated bubbles of fluid in a Wigner crystalline host. We will call these phases bubble crystals I and II, respectively. [See Fig. 2(d).] As $x\to 0$ or $x\to 1$, the period of the bubble crystals diverge, leading at mean-field level to another Lifshitz transition, much as in the stripe case. (However, fluctuation effects are much different near these transition in the bubble and stripe cases, as we will discuss in the next sections.)

However, this is not the end of the story. The stripe phase is always the lower energy one near $x=1/2$, so if a bubble phase occurs for small x, there must be a critical value of $x=x_c$ at which the energy of the bubble and stripe phases cross, seemingly implying a first-order transition. Since we have proven in general that first-order transitions are forbidden, this first-order transition, too, must be replaced by a regime of intermediate phases consisting of a mixture of bubble and stripe phases.[16] Now, however, because of the large anisotropy of the stripe phase, the surface tension between these two phases must be highly anisotropic. Thus, this intermediate phase will most probably be of the form of alternating mega-stripes of bubble and stripe phase regions. These regions are shown in Fig. 2(d) by hatched boxes.

III. THERMAL EFFECTS

A. The Pomaranchuk effect

The most dramatic effect of finite T is its effect on the balance between the liquid and Wigner crystalline phases, the fraction of the Wigner crystal phase grows as the temperature increases.[16] This phenomenon is similar to the Pomeranchuk effect in He^3 and has the same origin: the spin entropy of the crystal phase is substantially larger than that of the liquid state. The same considerations lead, as well, to the conclusion that the crystalline phase is preferred relative to the liquid in the presence of an in-plane magnetic field, $H_{\|}$.

Due to the Pauli exclusion principle, an effective exchange energy of order the Fermi energy, $E_F^* \sim \hbar^2 n/2m^*$ quenches the spin entropy in the liquid phase. In contrast, the exchange[22] energy in the Wigner crystal is exponentially small, $J\propto \exp[-\alpha\sqrt{r_s}]$ where α is a number of order 1. For example an estimate made in Ref. 23 yields $J\sim 10^{-7}\mathrm{Ry}^*$ where $\mathrm{Ry}^*=e^4m^*/2\hbar^2 e$ is the effective Rydberg. Thus a combination of the quantum character of the liquid and the smallness of exchange processes in the solid imply that the solid phase is stabilized by nonzero T relative to the liquid phase, for n near n_c, the 2DEG freezes upon heating. In the present context, this means that for fixed n, the relative fraction of Wigner crystalline regions increases with increasing T or $H_\parallel$. A simple estimate of the magnitude of this effect can be made for the range of temperatures $J \ll T \ll E_F^* = \mathrm{Ry}^*/\pi(r_s)^2$ and $\hbar\mu_B H_\parallel \ll E_F^*$, where the entropy of the liquid is negligible, as are the subtleties of the ground-state magnetic structure of the Wigner crystal. In this case,

$$f(n_-,T,H_\parallel) \approx f(n_-,0,0) - k_B T n_- \ln[2\cosh(\hbar\mu_B H_\parallel/k_B T)] \tag{8}$$

and $f(n_+,T,H_\parallel) \approx f(n_+,0,0)$. The fact that temperature and magnetic field stabilize the Wigner crystal in qualitatively similar fashion is one of the striking aspects of this relation: For $T \gg \hbar\mu_B H_\parallel$, $f(n_-,T,H_\parallel) - f(n_-,0,0) \approx -k_B T n_- \ln[2]$ while for $T \ll \hbar\mu_B H_\parallel$, $f(n_-,T,B) - f(n_-,0,0) \approx -\hbar\mu_B H_\parallel n_-$.

Of course, at high enough temperatures, all tendencies to ordered states are suppressed. This occurs above the characteristic temperature at which the Wigner crystal melts. In the limit of very large r_s, this occurs at the classical melting temperature of the Wigner crystal, which has been estimated in accurate numerical experiments[24] to be

$$T_{melt} = A(e^2/\epsilon)\sqrt{\pi n} = 2A\ \mathrm{Ry}^*/r_s \tag{9}$$

where $A = 1/125[1 \pm 0.04]$. However, at smaller r_s, where E_F of the competing fluid phase is larger than the putative classical melting temperature, the implied reduction of the entropy of the fluid state means that the melting temperature is set, by $T_{melt} \propto E_F$. Far from the Lifshitz points, the melting temperatures of the various microemulsion phases are determined by these same considerations, and are of similar magnitude. Here, the fraction of the system that is crystalline is a nonmonotonic function of T, first increasing and then dropping to zero at T_{melt}. Near the Lifshitz points, more delicate considerations determine the melting point.

B. Thermal fluctuations in the stripe phases

Let us now consider the role of thermal fluctuations on the stripe phases. We distinguish two cases:

(1) If the Hamiltonian is rotationally invariant, then the smectic phase is unstable at any nonzero temperature to the proliferation of dislocations. Thus, the mean-field smectic phase is replaced by a nematic phase, which, in keeping with the Mermin-Wagner theorem, does not actually break rotational symmetry, but rather has power-law orientational order. A free dislocation has a logarithmically divergent energy in both the Wigner and conducting-crystal phases, so they are robust against thermal fluctuations at low temperatures, although with power law rather than long-range crystalline order. The resulting phase diagram is shown schematically in Fig. 2(b).

(2) If, however, the Hamiltonian has a preferred axis, for instance, if we consider the 2DEG in the presence of an in-plane magnetic field, the effects of low-temperature thermal fluctuations are much less severe. Here, the smectic and both crystalline phases remain well defined at nonzero T, although again with power-law spatial correlations rather than with true long-range order, as shown schematically in Fig. 2(c).

Because first-order transitions are forbidden, the transition between the isotropic fluid and the nematic phase must be of the Beresinskii-Kosterlitz-Thouless (BKT) type. Near the mean field Lifshitz point we can estimate this transition temperature as follows: The distance between stripes is large so the stripes of minority phase evaporate when the energy to break off a piece, $\sim \sigma_0 L_0$, is less than the configurational entropy of a state where rare droplets of the minority phase are distributed randomly. Equating these two free energies leads to the estimate

$$T_c|\ln[x(1-x)]| \sim \sigma_0 L_0. \tag{10}$$

In the presence of an in-plane magnetic field, there is no sharply defined nematic phase, since rotational symmetry is explicitly broken. However, by the same token, free dislocations in the smectic state have a logarithmically divergent energy, so a power-law smectic phase exists at nonzero T. With increasing temperature, the smectic to liquid phase transition is also of the BKT type. Indeed, so long as the symmetry breaking term in the Hamiltonian is small, the transition temperature is roughly the same as in Eq. (10), above.

C. Thermal fluctuations in the bubble phases

We now consider the effect of thermal fluctuations on bubble phases. As can be seen from Eq. (1), the interaction energy between far separated bubbles decreases at large r_0 as

$$V_{bubble} \sim \sigma_1 L_0^4/r_0^3, \tag{11}$$

where L_0 and r_0 are the radius of and distance between bubbles. Thus, where the bubbles are far separated, because of the screening by the ground plane the BKT melting temperature will tend to rapidly to zero, $T_{BKT} \propto [(1-x)x]^{3/2}$, as the spacing between bubbles increases. The result is that, near the mean-field Lifshitz point the bubble phase is always melted by the thermal fluctuations. On the other hand, at smaller r_0 the bubble phase survives thermal fluctuations in the usual sense that the correlations of bubble positions exhibit power-law decay.

The nature of the transition between the bubble phase and the uniform phase is not, presently, settled. Of course, a direct first-order transition is forbidden. One possibility is that there is a sequence of two BKT transitions, as in the Halperin-Nelson theory[25] of melting, with an intermediate hexatic phase. Alternatively, there may be a further set of hierarchical microemulsion phases.

IV. QUANTUM FLUCTUATIONS

A. Stripe phases

So long as $nL_0^2 \gg 1$ ($nd^2 \gg 1$), the stripes are many electrons wide, so quantum fluctuations of their positions are intrinsically small; $1/nL_0^2$ is a small parameter in the problem, which permits an asymptotically exact treatment of quantum fluctuation effects. At zero temperature, the conducting-crystal phase is clearly stable in the presence of small quantum fluctuations, although, as mentioned previously, the jury is still out on whether the smectic phase is unstable to crystallization.[20,21] Only where the stripe width is of order of the interelectron distance (i.e., when $nL_0^2 \sim 1$), quantum fluctuations become very significant. This applies to the hatched region in Fig. 1, where the quantum properties of the system are still uncertain.

The quantum nature of the system near the Lifshitz points is determined by the quantum nature of the interface between the crystal and the liquid, a problem which itself is still unsolved. This interface may be quantum rough or quantum smooth. If it is smooth, the Lifshitz transition from the uniform fluid to smectic phase is not fundamentally affected by quantum fluctuations, provided the width of the stripes is large enough. However, if an isolated interface is rough, the stripe order in the vicinity of the mean-field Lifshitz point is quantum melted; in this case, for the rotationally invariant system, the proscription against first order transitions implies that there must be an intermediate zero temperature nematic phase between the isotropic and the stripe ordered phases. It was recently shown[13] that a nematic Fermi fluid is necessarily a non-Fermi liquid in the sense that quasiparticles are not well defined elementary excitations. We believe that, depending on microscopic details and on the value of nd^2, both scenarios are possible.

It is worth mentioning why the quantum nature of the crystal-liquid interface is so subtle. Consider the motion of a step in the interface. Quantum-mechanically, an isolated step might be expected to propagate along the interface.[26] Because the steps interact by a short-range dipolar interaction, the steps should then form a delocalized 1D quantum liquid along the interface. However, because of the density mismatch between the solid and liquid, the situation is more complicated. Motion of the step requires a flux of mass into the liquid of a magnitude proportional to the density difference between the solid and liquid and to the step's velocity. In a Fermi liquid this flux of mass is carried by quasiparticles, making the step motion highly dissipative. Thus, characterizing the interface involves interesting, but as far as we know unsolved issues in dissipative quantum mechanics.

B. Bubble phases

In contrast to stripe phases, quantum fluctuations always melt the bubble phases when the bubbles are sufficiently dilute. To see this, we can estimate the characteristic potential energy of a bubble crystal as in Eq. (11), and can make a corresponding dimensional estimate of the bubble kinetic energy $K_{bubble} \sim \hbar^2/r_0^2 m^*$, where m^* is the bubble effective mass. Therefore the ratio of these energies is

$$\frac{V_{bubble}}{K_{bubble}} \sim \left(\frac{\sigma_1 L_0^4 m^*}{\hbar^2}\right)\frac{1}{r_0} \qquad (12)$$

vanishes as $r_0 \to \infty$. This analysis fleshes out the same argument mentioned in the introduction that leads to the conclusion that there is no stable Wigner crystal phase at small d. However, whereas in that case, the proportionality constant is a_B^*, in the present case the same constant is parametrically large, both due to the explicit factors of L_0 and due to the fact that m^* increases with increasing L_0 (in a way that depends on whether the interface is quantum rough or smooth). The result is that, for large nd^2, the regime in which the bubble crystal is quantum melted is extremely small. However, if $nd^2 \sim 1$ quantum melting is a significant phenomenon.

The character of the bubble liquid phase is different depending on the character of the minority phase. When the majority phase is Wigner crystalline with dilute inclusions of liquid, the melting of the bubble crystal results in a type of "conducting crystal".[16] In this state, crystalline long-range order coexists with fluid-like conductivity, but in this case the conductivity is associated with the motion of the bubbles themselves. Phenomenologically this state is similar to the "supersolid" phase which has been discussed[27] in the context of He^4. In both cases the number of electrons (or aroms) is not equal to the number of the crystalline sites.[28] The difference is that unlike the case of He^4 where vacancies are bosons, in our case the statistics of the droplets is not known, and hence the liquid state may not be a superfluid. Therefore, we refer to this state as a "conducting crystal II" in Fig. 2(d) to distinguish it from the highly anisotropic conducting crystal [see Figs. 2(a) and 2(d)] which originates from the existence of stripes.

When the Fermi liquid is the majority phase, with a fluid of "icebergs" floating in it, no spatial symmetries need be broken. However, elementary excitation spectrum is likely to be different from that of a conventional Fermi liquid.

Since the majority phase already brakes rotational symmetry, the bubble crystallization transition which transforms the system from the conducting crystal to the insulating bubble crystal phase can be a simple continuous transition. However, the freezing of the icebergs into a triangular crystal of Wigner crystalline bubbles is more problematic. As with the thermal transition, there may be a two-step freezing transition, with an intermediate quantum hexatic phase,[13] or another hierarchy of microemulsion phases. The sequence of the phases at $T=0$ is shown in Fig. 2(d).

V. EXPERIMENTAL CONSEQUENCES

Obviously, there are many experimental consequences of the existence of intermediate phases, of which we here list only a few. It should be kept in mind that macroscopic spatial symmetry breaking, the sort which precisely characterizes the various phases we have discussed, does not truly occur in 2D in the presence of quenched disorder. This complicates the actual observation of various phenomena.

The majority of industrially produced Si MOSFET's have gates relatively close to the 2DEG, $d \ll d_c$, so the electron liquid is weakly interacting at all n. However, a small number of high mobility Si MOSFET's (For a review, see Ref. 29.) and p-type of GaAs double layers[30] with large $d \sim 1000$ Å have been studied in the past few years, and found to exhibit transport anomalies that have been interpreted as evidence for an unexpected metal-insulator transition. While these devices certainly are not ideal, in the sense that they have non-zero quenched disorder, we would like to propose that a natural explanation of those phenomena is that they reflect the existence in the zero disorder limit of the electronic microemulsion phases identified in the present paper.

One robust consequence of two-phase coexistence is that the conductivity is a decreasing function of the volume fraction of Wigner crystal. This volume fraction, in turn, is strongly temperature and magnetic field dependent due to the Pomeranchuk effect, as explained above. As a result, the fraction of crystal grows with temperature and magnetic field, leading to a corresponding increase of the resistivity. As has been pointed out previously,[16] this basic physics may underly the transport anomalies observed in large d Si MOSFETs. In particular, it offers a candidate explanation of the anomalous metallic $(d\rho/dT>0)$ temperature dependence and large positive magneto resistance observed in these systems despite the fact that the resistivity, itself, exceeds the Ioffe-Regel limit. [Ideally, one might want to explore the scaling relation between the temperature and magnetic field dependence of the resistivity implied by the thermodynamic relation in Eq. (9).]

Of course, each new phase has different patterns of spatial symmetry breaking, and hence has new collective modes and modified hydrodynamics. Even when the effects of quenched disorder or thermal fluctuations restore the symmetry at macroscopic distances, the existence of these collective modes can have readily detectable consequences for the dynamical responses of the system. Small explicit symmetry breaking fields can be used to overcome the destructive effects of quenched disorder and reveal the true tendency to symmetry breaking. For instance, an in-plane magnetic field explicitly breaks rotational symmetry; where some form of stripe or nematic phase exists in the absence of quenched disorder, the small symmetry breaking produced by such a field can give rise to a large resistivity anisotropy, as has been seen for the analogous states in quantum Hall devices.[31]

VI. EXTENSIONS

We end with some speculative observations concerning intermediate phases of the 2DEG.

A. Spin physics

Other than the Pomeranchuk effect, we have largely ignored the physics of the electron spins. The exchange interactions in the Wigner crystal phase are generally found to be very small,[22] and so are only important at very low temperatures. At $T=0$, however, the fact that the magnetic Hamiltonian is highly frustrated and may have important multispin ring exchange interactions, can lead to a variety of possible magnetic phases, and this complexity could be inherited, to some degree, by the intermediate phases discussed here. Moreover, at a liquid-crystalline interface, the quantum dynamics of the interface itself (mentioned above) can produce effective exchange interactions, likely with much larger energy scales than in the bulk Wigner crystal. There is thus the very real possibility that the magnetic structure of the interfaces is very rich, and characterized by substantial energy scales.

B. Superconductivity

The parallels between the 2DEG in a MOSFET and Coulomb frustrated phase separation in a doped Mott insulator naturally lead one to *speculate* concerning the possibility of superconductivity in the present system. In the bubble related conducting-crystal phase, each bubble has a fixed number of electrons; when that number is even, the bubbles are likely bosonic and a supersolid phase with low superfluid density is possible.[16,27] In the hatched region of the phase diagram, where quantum effects are most severe, a more robust mechanism is possible, based on the "spin-gap proximity effect:"[32] Small clusters of Wigner crystal (be they stripe or bubble like) will often have a spin gap. (Near the cluster edge, this gap may be larger than in the bulk.) Where this gap is large enough, it suppresses single-particle exchange between the crystal clusters and the surrounding Fermi fluid, but pair exchange is still permitted. When this dominates, it induces global superconductivity by a process analogous to the conventional proximity effect.

C. Double layers

In a double layer system, with two nearby 2DEG's, the two layers screen each other in much the same way as the metal layer screens the 2DEG in a MOSFET. However, here the types of phases, and the available experiments are still richer. One particularly interesting point is that the conductivity measured in drag can explore the nature of the interlayer screening. The presence of a crystalline component of an electron fluid has the potential to greatly increase the drag conductivity relative to a Fermi liquid; in particular, whereas the drag conductivity vanishes as $T\rightarrow 0$ in a Fermi liquid, we believe it can approach a nonzero constant in some of the intermediate phases we have explored.

D. Other applications

The present ideas are pretty clearly applicable in a host of additional physical contexts. What is needed is short-range tendency to phase separation, i.e., a concave local free energy, opposed by dipolar forces. Under appropriate circumstances, this situation may pertain in the 2DEG at higher densities $r_s<r_c$ and it certainly applies in various regimes to the physics of lipid films and planar ferromagnets.

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[1] We will refer to the high-density liquid phase as a Fermi liquid, although it may be that it is unstable at very low temperature to the formation of a high angular momentum superconducting phase. Nearer to the critical r_s there is also the possibility that various other types of non-Fermi liquid states, distinct from those discussed here, could occur. None of these subtleties are relevant for the present discussion.

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