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Electronic thermal conductivity of disordered metals

Roberto Raimondi
Dipartimento di Fisica "E. Amaldi," Università di Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy and INFN, Laboratori Nazionali di Frascati, P.O. Box 13, 00044 Frascati, Italy

Giorgio Savona
Dipartimento di Fisica "E. Amaldi," Università di Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy

Peter Schwab and Thomas Lück
Institut für Physik, Universität Augsburg, 86135 Augsburg, Germany

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We calculate the thermal conductivity of interacting electrons in disordered metals. In our analysis we point out that the interaction affects thermal transport through two distinct mechanisms, associated with quantum interference corrections and energy exchange of the quasiparticles with the electromagnetic environment, respectively. The latter is seen to lead to a violation of the Wiedemann-Franz law. Our theory, valid to all orders in perturbation theory, predicts a strong enhancement of the Lorenz ratio $\kappa/\sigma T$ over the value predicted by the Wiedemann-Franz law, when the electrons encounter a large environmental impedance.

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The Wiedemann-Franz law relates the electronic thermal conductivity κ and the electrical conductivity σ and states that the Lorenz ratio $L=\kappa/\sigma T$ is a universal constant given by $L=\pi^2 k_B^2/3e^2$. In this equation k_B is the Boltzmann constant, $-e$ the electron charge, and T the temperature. The validity of the Wiedemann-Franz law relies mainly on a single-particle description of the transport properties, on the Fermi statistics of the charge carriers, and on the assumption of purely elastic scattering.[1] In a Fermi liquid, one expects that this law still holds at low enough temperature, when the quasiparticles cannot exchange energy during collisions.[2] Deviations from the Wiedemann-Franz law as recently observed in the normal state of a copper oxide superconductor have thus been interpreted as an evidence for the breakdown of Fermi-liquid theory.[3]

The effects of Coulomb interaction on the electrical transport at low temperature can be broadly grouped in two main types. From one side, transport implies adding charges to a conductor. This has an energy cost that depends on the size and shape of the conductor itself. For example, in tunnel junctions the energy transfer between quasiparticles and the electrodynamical environment causes the Coulomb blockade phenomenon. On the other hand, Coulomb interaction leads in a disordered conductor to an additional source of random scattering that interferes with the scattering from the impurities. This is a quantum effect and depends on the details of charge diffusion.

In the 1980s these quantum interferences were shown to lead to corrections to the electrical conductivity beyond the standard Fermi-liquid results. It turned out that these corrections may, in fact, be incorporated into a scale-dependent renormalization of the Landau Fermi-liquid parameters.[4–9] Within this framework, Castellani and co-workers[10] demonstrated that the scale-dependent corrections to the thermal conductivity are the same as the corrections to the electrical conductivity. This led them to conclude that the Wiedemann-Franz law is valid up to the metal-insulator transition.

This conclusion was challenged by Livanov *et al.*[11] and Arfi[12] who found for a two-dimensional system with long-range Coulomb interaction additional contributions to the thermal conductivity. Recently, the issue has been reexamined by Niven and Smith,[13] who also concluded that the Wiedemann-Franz law is violated.

In this article we study the problem by means of the quasiclassical Green's function approach, which has proved to be a powerful tool in describing the dynamical properties of superconductors[14] and the transport in hybrid mesoscopic structures.[15] Recently it was further demonstrated that both Coulomb blockade phenomenon and quantum interference corrections to the charge transport can be conveniently described within this theoretical framework.[16] Advantages of the method are that it is not restricted to the linear response regime, and often provides more compact derivations than the standard diagrammatic techniques.

A perturbative calculation of the thermal conductivity, besides confirming Ref. 13, allows us to clarify the origin of the apparent discrepancies in the literature. To do so we separate the different physical mechanisms by their different range of exchanged energies and relevant length scales. For instance, the quantum interference effects occur over distances from the mean free path up to the thermal diffusion length $\sqrt{\hbar D/k_B T}$, and imply energy exchanges larger than the temperature T. These yield corrections are logarithmically divergent and can be readily related to the scale-dependent renormalization of the electrical conductivity. Here the temperature acts as an infrared cutoff. The interaction effects responsible for the deviations from the Wiedemann-Franz law are associated with the long-range part of the Coulomb interaction and their singular behavior has the temperature as the upper cutoff. We successively concentrate on the long-range part of the Coulomb interaction. In particular, to all orders in perturbation theory, we predict a sizable enhancement of the Lorenz ratio when the sheet resistance is of the order h/e^2 in the case of a two-dimensional electron system, and a strong enhancement of the Lorenz ratio for thin metallic wires when the total resistance of the wire is larger than h/e^2.

From now on we use units where $\hbar = k_B = 1$, but we put back the constants in the final results. We start with a brief introduction of the quasiclassical formalism. For a more detailed description we defer the reader to Ref. 17. Within this formalism the short-distance behavior of the electron Green's function is taken into account in an averaged way from the outset by introducing the quasiclassical Green's function, which solves the Eilenberger equation[18]

$$[\partial_{t_1} + \partial_{t_2} + v_F \hat{\mathbf{p}} \cdot \partial_{\mathbf{x}}]\check{g}_{t_1 t_2}(\mathbf{x},\hat{\mathbf{p}}) = -\mathrm{i}[\check{\Sigma}(\mathbf{x},\hat{\mathbf{p}}),\check{g}(\mathbf{x},\hat{\mathbf{p}})]. \quad (1)$$

In contrast to the Dyson equation for the ordinary Green's function, the Eilenberger equation for $\check{g}$ is homogeneous and requires a normalization condition, which can be chosen of the form $\check{g}\check{g} = \check{1}$. The Green's function has a two-by-two matrix structure in Keldysh space,

$$\check{g} = \begin{pmatrix} g^R & g^K \\ 0 & g^A \end{pmatrix}. \quad (2)$$

Matrix products imply both summation and integration over Keldysh indices and time variables, respectively. We recall that, whereas the diagonal components of $\check{g}$ describe the spectral properties of the system, the off-diagonal Keldysh component carries information about the distribution function. In this respect, the Keldysh component of Eq. (1) is the quantum analog of the Boltzmann equation.

Impurity scattering is introduced by means of the standard white-noise random potential and is described by the self-energy in the self-consistent Born approximation as

$$\check{\Sigma}_{t_1 t_2}(\mathbf{x}) = -\frac{\mathrm{i}}{2\tau}\int \frac{d\hat{\mathbf{p}}}{\Omega_d}\check{g}_{t_1 t_2}(\mathbf{x},\hat{\mathbf{p}}), \quad (3)$$

where τ is the elastic scattering time and Ω_d is the d-dimensional solid angle. The charge current density has the form

$$\mathbf{j}_e(\mathbf{x},t) = e\pi N_0 v_F \int \frac{d\hat{\mathbf{p}}}{\Omega_d}\hat{\mathbf{p}} g^K_{t_1 t_2}(\mathbf{x},\hat{\mathbf{p}})\big|_{t_1=t_2}. \quad (4)$$

The expression for the heat current density is not obvious. In the absence of interactions, but in the presence of disorder, the heat current is found from the thermal average of the operator[2]

$$\mathbf{j}_Q(\mathbf{x},t) = -\frac{1}{m}[\dot{\Psi}^+(\mathbf{x},t)\nabla\Psi(\mathbf{x},t) + \nabla\Psi^+(\mathbf{x},t)\dot{\Psi}(\mathbf{x},t)], \quad (5)$$

where the dots denote the time derivatives of the Heisenberg operators. The quasiclassical implementation of this expression reads

$$\mathbf{j}_Q(\mathbf{x},t) = \frac{\mathrm{i}\pi N_0 v_F}{2}(\partial_{t_2} - \partial_{t_1})\int \frac{d\hat{\mathbf{p}}}{\Omega_d}\hat{\mathbf{p}} g^K_{t_1 t_2}(\mathbf{x},\hat{\mathbf{p}})\big|_{t_1=t_2}. \quad (6)$$

Generally, in an interacting system Eqs. (5) and (6) have to be modified.[19–21] On the other hand, in the case of short-range interactions or in the case of Coulomb interaction, when the screening length is short compared to the other relevant length scales, Eqs. (5) and (6) remain valid. This is demonstrated in the Appendix.

In the dirty limit, the variation of the Green's function is on space and time scales larger than the elastic mean free path $l = v_F\tau$ and scattering time τ, respectively. In this limit one may expand the Green's function $\check{g}$ in spherical harmonics and keep the s- and p-wave components only: $\check{g}(\hat{\mathbf{p}}) = \check{g}_s + \hat{\mathbf{p}}\check{g}_p + \ldots$. The Eilenberger equation is then replaced by ($D = v_F^2\tau/d$) the Usadel equation,[22] as

$$\partial_t \check{g}_s(\mathbf{x}) - D\partial_{\mathbf{x}}[\check{g}_s\partial_{\mathbf{x}}\check{g}_s] = 0, \quad (7)$$

which is the analog of the saddle-point condition in the nonlinear σ-model matrix field theory.[6] As a result, the currents are expressed in terms of the s-wave component of the Green's function, as

$$\mathbf{j}_e(\mathbf{x},t) = -\frac{eN_0D}{2}\int d\epsilon(\check{g}_s\partial_{\mathbf{x}}\check{g}_s)^K, \quad (8)$$

$$\mathbf{j}_Q(\mathbf{x},t) = \frac{N_0D}{2}\int d\epsilon\ \epsilon(\check{g}_s\partial_{\mathbf{x}}\check{g}_s)^K, \quad (9)$$

with ϵ corresponding to the Fourier transform of the relative time $t_1 - t_2$. As a simple application of the formalism we derive the Drude formula for the electrical and thermal conductivity. In the absence of interactions the Green's function $\check{g}$ reads

$$\check{g}_s(\mathbf{x},t;\epsilon) = \begin{pmatrix} 1 & 2F \\ 0 & -1 \end{pmatrix}. \quad (10)$$

In thermal equilibrium the function F is given by $F = \tanh(\epsilon/2T)$. Near local equilibrium with a local temperature $T(\mathbf{x})$ and a local chemical potential $\mu(\mathbf{x})$ the Drude expressions for both electrical and heat currents are found, so that

$$\mathbf{j}_e = 2e^2DN_0(\nabla\mu/e), \quad (11)$$

$$\mathbf{j}_Q = \frac{\pi^2}{3}k_B^2 2N_0DT(-\nabla T), \quad (12)$$

and in particular the Wiedemann-Franz law holds.

To include the effects of the Coulomb interaction, we introduce[16,23,24] a Hubbard-Stratonovich matrix field

$$\check{\phi} = \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_1 \end{pmatrix} \quad (13)$$

whose fluctuations describe the retarded, advanced, and Keldysh components of the screened Coulomb interaction

$$-\mathrm{i}e^2\begin{pmatrix} \langle\phi_1\phi_1\rangle & \langle\phi_1\phi_2\rangle \\ \langle\phi_2\phi_1\rangle & \langle\phi_2\phi_2\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} V^K & V^R \\ V^A & 0 \end{pmatrix}. \quad (14)$$

In the presence of the field $\check{\phi}$, one first adds a term $\mathrm{i}e[\check{\phi},\check{g}]$ to the right-hand side of the Eilenberger (1) or the Usadel equation (7). Secondly, the resulting solution $\check{g}[\phi]$ is averaged over the fluctuations of $\check{\phi}$ according to Eq. (14). Analogously to the noninteracting case [cf. Eq. (10)], one can define the distribution function in the presence of interactions via the relation between the Keldysh and the retarded, advanced components of the Green's function $\langle g^K\rangle = \langle g^R\rangle F - F\langle g^A\rangle$. We further assume a system which is—with the exception of a weak temperature gradient—translational invariant. It is then convenient to expand the distribution function as

$$F_{\epsilon-\omega}(\mathbf{x}_1) \approx F_{\epsilon-\omega}(\mathbf{x}) - \partial_T F_{\epsilon-\omega}(\mathbf{x}) \nabla T \cdot (\mathbf{x}_1 - \mathbf{x}) \qquad (15)$$

and to Fourier transform from real to momentum space. The correction to the thermal current is finally obtained as $\delta\mathbf{j}_Q = \delta\mathbf{j}_Q^a + \delta\mathbf{j}_Q^b$ with

$$\delta\mathbf{j}_Q^a = DN_0 \nabla T \int d\epsilon\, \epsilon \int \frac{d\omega}{2\pi} \partial_T [F_{\epsilon-\omega}(\mathbf{x}) F_\epsilon(\mathbf{x})] \times \mathrm{Im} \sum_q \frac{1}{(-\mathrm{i}\omega + Dq^2)^2} V_\omega^R(\mathbf{q}) \qquad (16)$$

and

$$\delta\mathbf{j}_Q^b = DN_0 \nabla T \int d\epsilon\, \epsilon \int \frac{d\omega}{2\pi} F_\epsilon(\mathbf{x}) \partial_T F_{\epsilon-\omega}(\mathbf{x}) \times \frac{4}{d}\, \mathrm{Im} \sum_q \frac{Dq^2}{(-\mathrm{i}\omega + Dq^2)^3} V_\omega^R(\mathbf{q}), \qquad (17)$$

where d is the dimension of the system under consideration. Our result [Eqs. (16) and (17)] for the thermal current is equivalent to the thermal conductivity found[13] by using the diagrammatic method and the Matsubara technique. We notice that the diffusive pole appearing in Eqs. (16) and (17) originates from the Usadel equation (7).

Using the relation $F_\epsilon F_{\epsilon-\omega} = 1 - (F_\epsilon - F_{\epsilon-\omega}) B(\omega/2T)$ with $B(x) = \coth(x)$ allows us to evaluate the ϵ-integrations in Eqs. (16) and (17) with the result

$$\int d\epsilon\, \epsilon \partial_T (F_\epsilon F_{\epsilon-\omega}) = -\omega^2 \partial_T B\left(\frac{\omega}{2T}\right), \qquad (18)$$

$$\int d\epsilon\, \epsilon F_\epsilon \partial_T F_{\epsilon-\omega} = -\frac{2\pi^2 T}{3} \partial_\omega \left[\omega B\left(\frac{\omega}{2T}\right)\right] + \frac{\omega^3}{3T} \partial_\omega B\left(\frac{\omega}{2T}\right). \qquad (19)$$

From Eqs. (18) and (19) we observe that $\delta\mathbf{j}_Q^a$ is dominated by diffusive modes of frequency $|\omega| < T$, whereas modes with frequencies $|\omega| > T$ give the dominant contribution to $\delta\mathbf{j}_Q^b$. To appreciate the role played by the different frequency ranges, we begin by evaluating the current in two dimensions. The retarded component of the dynamically screened Coulomb interaction reads

$$V^R(\mathbf{q}, \omega) \approx \frac{1}{2N_0} \frac{\kappa_{2d}}{q} \frac{-\mathrm{i}\omega + Dq^2}{-\mathrm{i}\omega + D\kappa_{2d} q}, \qquad (20)$$

where $\kappa_{2d} = 4\pi e^2 N_0$ is the screening vector in two dimensions. By considering first $\delta\mathbf{j}_Q^b$, one notices that the momentum integration to be performed is identical to the momentum integral in the correction to the electrical conductivity,[4–7] i.e., the integration is logarithmically divergent in the ultraviolet and must be cut off with the diffusive condition $Dq^2\tau < 1$. In the ω-integration there is a minor difference at low frequencies $|\omega| < T$ due to the second term on the right-hand side of Eq. (19). In two dimensions, with logarithmic accuracy, this difference is negligible and one has

$$\delta\mathbf{j}_Q^b \approx \frac{\pi^2}{3} \frac{T}{e^2} \delta\sigma(-\nabla T), \qquad (21)$$

where $\delta\sigma = -e^2/(2\pi^2) \ln(1/T\tau)$ is the interaction correction to the electrical conductivity and $T\tau < 1$. The other contribution to the thermal current, $\delta\mathbf{j}_Q^a$, does not depend on the ultraviolet cutoff $1/\tau$, so that

$$\delta\mathbf{j}_Q^a \approx -D \nabla T \int_0^T \frac{d\omega}{2\pi} \omega \int \frac{d^2q}{(2\pi)^2} \times \mathrm{Im}\left(\frac{1}{-\mathrm{i}\omega + Dq^2} \frac{\kappa_{2d}}{q} \frac{1}{-\mathrm{i}\omega + Dq\kappa_{2d}}\right) \qquad (22)$$

since the temperature acts as an upper cutoff in the frequency integration. In contrast, in the limit of good metallic screening when $\kappa_{2d} \to \infty$, the integration becomes infared divergent. By combining the two contributions, we finally write the expression for the thermal conductivity in a form which shows that, although the Wiedemann-Franz law is violated,

$$\kappa = \frac{\pi^2}{3} \frac{k_B^2 T}{e^2} \left[\sigma + \delta\sigma + \frac{1}{2} \frac{e^2}{\pi h} \ln(\hbar D \kappa_{2d}^2 / k_B T)\right], \qquad (23)$$

the integration of diffusive modes in the region $T < Dq^2, \omega < \tau^{-1}$ yields the same scaling equations for σ and κ,

$$\frac{d \ln \sigma}{d \ln l} = \frac{d \ln \kappa}{d \ln l}, \qquad (24)$$

so that the apparent discrepancies in the literature are not contradictory.

We observe that in the last term of Eq. (23), responsible for the violation of the Wiedemann-Franz law, only the extreme long wavelength modes of the dynamically screened Coulomb interaction with $Dq^2 < |\omega| < T$ are relevant [cf. Eqs. (16) and (18)]. It has been shown in Ref. 25 that these modes can be conveniently handled by making the following gauge transformation

$$\check{g}_{t_1 t_2}(\mathbf{x}; \hat{\mathbf{p}}) = e^{\mathrm{i}\check{\varphi}(\mathbf{x}, t_1)} \check{\tilde{g}}_{t_1 t_2}(\mathbf{x}; \hat{\mathbf{p}}) e^{-\mathrm{i}\check{\varphi}(\mathbf{x}, t_2)}, \qquad (25)$$

where $\partial_t \check{\varphi}(\mathbf{x}, t) = e\check{\phi}(\mathbf{x}, t)$. In the case of only homogeneous field fluctuations, Eq. (25) gives the Green's function to all orders in the interaction when inserting for $\check{\tilde{g}}_{t_1 t_2}$ the Green's function in the absence of interaction. In general, $\check{\tilde{g}}_{t_1 t_2}$ has a perturbative expansion in terms of gradients of φ. Whereas the gauge factors drop in the expression for the electrical current, i.e., the long wavelength modes of the Coulomb interaction do not modify the electrical conductivity, they survive in the heat current to yield

$$\mathbf{j}_Q = -\frac{N_0}{2}\int \frac{d\hat{\mathbf{p}}}{\Omega_d} v_F \hat{\mathbf{p}} \int d\epsilon \Bigg\{ \epsilon \tilde{g}^K(\mathbf{x},t;\hat{\mathbf{p}},\epsilon) - \frac{1}{2} e[\langle \check{\phi}(\mathbf{x},t)\check{\tilde{g}}(\mathbf{x},t;\hat{\mathbf{p}},\epsilon) + \check{\tilde{g}}(\mathbf{x},t;\hat{\mathbf{p}},\epsilon)\check{\phi}(\mathbf{x},t)\rangle]^K \Bigg\}. \tag{26}$$

To consider the effect of the long wavelength modes it is sufficient to expand $\check{\tilde{g}}$ to first order in the gradient of φ, so that only the second term of Eq. (26) is modified since φ has Gaussian correlations. Equation (26) makes then clear the physical origin of the violation of the Wiedemann-Franz law. While the first term on the right-hand side reproduces the noninteracting contribution to the thermal current, the second may be interpreted as the effect of the time dependent fluctuations of the quasiparticle energy in the presence of an electromagnetic environment. Indeed, the extra heat current is proportional to the correlation of voltage and current fluctuations in the system, $\delta \mathbf{j}_Q = \langle \phi_1(\mathbf{x},t)\delta \mathbf{j}_e(\mathbf{x},t)\rangle$, which then leads to the strikingly simple result

$$\mathbf{j}_Q = \frac{\pi^2}{3}\frac{k_B^2 T}{e^2}\sigma(-\nabla T) - \frac{1}{2}\sigma \nabla \langle \phi_1(\mathbf{x},t)\phi_1(\mathbf{x},t)\rangle. \tag{27}$$

Notice that due to the linear current-voltage characteristics of the system under consideration only the first order in the Coulomb interaction contributes to the heat current. By using the fluctuation dissipation theorem [or equivalently Eq. (14)]

$$\langle \phi_1(\mathbf{x},t)\phi_1(\mathbf{x},t)\rangle = -\frac{1}{e^2}\int \frac{d\omega}{2\pi} B\left(\frac{\omega}{2T(\mathbf{x})}\right)\sum_q \operatorname{Im} V^R(\mathbf{q},\omega) = \int \frac{d\omega}{2\pi} B\left(\frac{\omega}{2T(\mathbf{x})}\right)\omega \operatorname{Re} Z(\omega), \tag{28}$$

direct contact can be made with the conventional perturbation theory, i.e., with $\delta \mathbf{j}_Q^a$ in Eq. (16).

Instead of parametrizing the local voltage fluctuations in terms of an interaction $V^R(\mathbf{q},\omega)$, we will in the following parametrize them in terms of the impedance of the local electromagnetic environment, $Z(\omega)$. By doing so, the thermal conductivity reads

$$\kappa = \frac{\pi^2}{3}\frac{k_B^2 T}{e^2}\sigma + \frac{\sigma k_B}{e^2}\int dE \left(\frac{E/2k_B T}{\sinh(E/2k_B T)}\right)^2 \frac{\operatorname{Re} Z(E/\hbar)}{h/e^2}, \tag{29}$$

where for clarity we put back the $\hbar$ and k_B. We will now discuss three different examples for the impedance Z. The simplest situation consists of a purely ohmic environment where $Z(E/\hbar)=R$. The thermal conductivity is found linear in the temperature, strong deviations from the Wiedemann-Franz law are found when the environmental resistance is of the order of the resistance quantum h/e^2 or larger. The explicit result is

$$\kappa = \frac{\pi^2}{3}\frac{k_B^2 T}{e^2}\sigma[1 + 2R/(h/e^2)]. \tag{30}$$

From the retarded Coulomb interaction as given in Eq. (20) we determine the impedance of a thin film as $\operatorname{Re} Z(E/\hbar) = (1/4\pi\sigma)\ln(\hbar D \kappa_{2d}^2 k_B T/E^2)$. Due to the weak logarithmic energy dependence of the impedance, the thermal conductivity is to good accuracy obtained from Eq. (30) with $R = \operatorname{Re} Z(k_B T/\hbar)$. As a third example we consider a RC-transmission line, as a model of a gated wire. The impedance is $\operatorname{Re} Z = \frac{1}{2}\sqrt{R_0/2|\omega|C_0}$, where R_0 and C_0 are respectively, the resistance and capacitance per unit length. We find a contribution to the thermal conductivity that is proportional to the square root of the temperature, as

$$\kappa = \frac{\pi^2}{3}\frac{k_B^2 T}{e^2}\sigma + 2.456\frac{k_B}{e^2}\sigma \frac{\sqrt{\hbar k_B T R_0/C_0}}{h/e^2}, \tag{31}$$

with 2.456 the approximate numerical value for $3\zeta(3/2)\Gamma(3/2)/2^{3/2}$.

In summary, we calculated the thermal conductivity of disordered metals. In the two-dimensional electron system, the scaling equations for the thermal and the electrical conductivity are the same, nevertheless, the Wiedemann-Franz law does not hold. The deviations from the Wiedemann-Franz law are comparable in size to the localization effects. It is interesting to note that this is in qualitative agreement with observations made in the cuprates:[3,26] The resistivity of PCCO in Ref. 3 shows a well pronounced low temperature anomaly that has been attributed to localization effects, and at the same time the low temperature heat conductivity is larger than what would be expected from the Wiedemann-Franz law. In the low temperature resistivity of Tl-2201 in Ref. 26, no indications of localization effects are seen and the Wiedemann-Franz law is perfectly obeyed within the experimental accuracy. Quantitatively, on the other hand, the agreement of our theory with Ref. 3 remains poor, since the sheet resistance was estimated as $R_\square \approx h/(60e^2)$, from which we expect a much smaller enhancement of the heat conductivity than observed experimentally. By measuring the Lorenz ratio in a gated film or wire as function of the gate capacitance, it should be possible to test our predictions experimentally.

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APPENDIX: HEAT CURRENT FOR INTERACTING ELECTRONS

We obtain the conservation law for the energy by considering the s-wave part of the Keldysh component of the Eilenberger equation [Eq. (1)]. The impurity self-energy does not contribute to the s-wave part of the equation. We recall that by setting the relative time $\eta = t_1 - t_2$ to zero, the Eilenberger equation becomes the continuity equation for charge. The continuity equation for energy is obtained by first taking the derivative with respect to η and then taking the $\eta=0$ limit, as

$$\frac{\partial}{\partial t}\left[-\mathrm{i}\pi N_0\int\frac{d\hat{\mathbf{p}}}{\Omega_d}\frac{\partial}{\partial\eta}g^K(\hat{\mathbf{p}},\eta;\mathbf{x},t)\right] + \partial_{\mathbf{x}}\left[-\mathrm{i}\pi N_0\int\frac{d\hat{\mathbf{p}}}{\Omega_d}\frac{\partial}{\partial\eta}v_F\hat{\mathbf{p}}g^K(\hat{\mathbf{p}},\eta;\mathbf{x},t)\right] = e\pi N_0\dot{\phi}_1(\mathbf{x},t)\int\frac{d\hat{\mathbf{p}}}{\Omega_d}g^K(\hat{\mathbf{p}},\eta;\mathbf{x},t). \tag{A1}$$

On the left-hand side of the equation appear the energy density and energy current in the absence of the interaction. To appreciate this we recall that the time representation of the distribution function reads in thermal equilibrium

$$\int\frac{d\hat{\mathbf{p}}}{\Omega_d}g^K(\hat{\mathbf{p}},\eta;\mathbf{x},t) = -\frac{2\mathrm{i}}{\pi\eta} + \frac{2\mathrm{i}\pi T^2\eta}{6} + \ldots \tag{A2}$$

in the small η limit. By using the above expansion in the energy density defined by the expression under the time derivative in Eq. (A1), one obtains the standard expression for the specific heat in a noninteracting electron gas, $c_v=(2\pi^2/3)N_0T$.

Clearly due to the term on the right-hand side of Eq. (A2), both the expressions for energy and current density will be modified. By exploiting the functional dependence of $g^K(\hat{\mathbf{p}},\eta;\mathbf{x},t)$ on the internal field $\check{\phi}$, we rewrite the right-hand side of Eq. (A1) as

$$\frac{\mathrm{i}}{2}\int d\mathbf{x}_1 V(\mathbf{x}-\mathbf{x}_1)\frac{\partial}{\partial t_1}\chi^K(\mathbf{x},t;\mathbf{x}_1,t_1), \tag{A3}$$

where $V(\mathbf{x}-\mathbf{x}_1)$ is the statically screened Coulomb interaction, and

$$\chi^K(\mathbf{x},t;\mathbf{x}_1,t_1) = \frac{\pi N_0}{e}\int\frac{\mathrm{d}\hat{\mathbf{p}}}{\Omega_d}\left\langle\frac{\delta g^K(\hat{\mathbf{p}},\eta=0;\mathbf{x},t)}{\delta\phi_2(\mathbf{x}_1,t_1)}\right\rangle, \tag{A4}$$

is the dynamical part of the symmetric density-density correlation function. When the statically sceened interaction is short ranged, $V(\mathbf{x}-\mathbf{x}_1)=V\delta(\mathbf{x}-\mathbf{x}_1)$, by making use of the symmetry of χ^K with respect to the interchange of its pairs of arguments, $(\mathbf{x},t)$ and $(\mathbf{x}_1,t_1)$, it is possible to write the expression as a total time derivative so that the correction to the energy density reads

$$\delta\rho_\epsilon = -\frac{\mathrm{i}}{4}V\chi^K(\mathbf{x},t;\mathbf{x},t), \tag{A5}$$

while the energy current density remains unchanged. This justifies the use of Eq. (6) even in the presence of the interaction.[27]

In the case in which the interaction has a finite range, we exploit the fact that $\chi^K(\mathbf{x},t;\mathbf{x}_1,t_1)$ depends only weakly on the center-of-mass coordinate $(\mathbf{x}+\mathbf{x}_1)/2$ as compared to the relative coordinate $\mathbf{r}=\mathbf{x}-\mathbf{x}_1$. By expanding with respect to the center-of-mass coordinate one then finds

$$\delta\rho_\epsilon = -\frac{\mathrm{i}}{4}\int d\mathbf{r}V(\mathbf{r})\chi\left(\mathbf{x}+\frac{\mathbf{r}}{2},t;\mathbf{x}-\frac{\mathbf{r}}{2},t\right), \tag{A6}$$

$$\delta\mathbf{j}_Q = \frac{\mathrm{i}}{4}\int d\mathbf{r}V(\mathbf{r})\mathbf{r}\partial_{t_1}\chi\left.\left(\mathbf{x}+\frac{\mathbf{r}}{2},t;\mathbf{x}-\frac{\mathbf{r}}{2},t_1\right)\right|_{t_1=t}. \tag{A7}$$

To conclude this Appendix, we refer to the results of Moreno and Coleman[20] and Paul and Kotliar,[21] who derived the heat current operator in an interacting electron system. Consider electrons with kinetic energy $\epsilon_{\mathbf{k}}$ and interaction $V(\mathbf{k})$, such that the Hamiltonian reads

$$H = \sum_{\mathbf{k},s}\epsilon_{\mathbf{k}}\Psi^+_{\mathbf{k},s}\Psi_{\mathbf{k},s} + \frac{1}{2}V(\mathbf{k}):n_{-\mathbf{k}}n_{\mathbf{k}}:, \tag{A8}$$

where s is the spin, $n(\mathbf{k})$ denotes the density, and $:n_{-\mathbf{k}}n_{\mathbf{k}}:$ the normal ordering of the Fermionic operators. The heat current operator becomes

$$\mathbf{j}_Q = \frac{\mathrm{i}}{2}\sum_{\mathbf{k},s}\nabla_{\mathbf{k}}\epsilon(\Psi^+_{\mathbf{k},s}\dot{\Psi}_{\mathbf{k},s} - \dot{\Psi}^+_{\mathbf{k},s}\Psi_{\mathbf{k},s}) + \frac{\mathrm{i}}{4}\sum_{\mathbf{k}}\nabla_{\mathbf{k}}V(\mathbf{k}):n_{-\mathbf{k}}\dot{n}_{\mathbf{k}} - \dot{n}_{-\mathbf{k}}n_{\mathbf{k}}:. \tag{A9}$$

When using the free particle dispersion $\epsilon_{\mathbf{k}}=\mathbf{k}^2/2m$, the first line of Eq. (A9) agrees with Eq. (5). The term in the second line, on the other hand, is consistent with Eq. (A7) and disappears for the **k**-independent interaction.

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[27] We notice that—consistently—exact agreement between the heat current as given in Eqs. (16) and (17) with Niven and Smith (Ref. 13) is obtained by neglecting the momentum dependence of the statically screened interaction.