Berry phases for composite fermions: Effective magnetic field and fractional statistics

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The quantum Hall superfluid is presently the only viable candidate for a realization of quasiparticles with fractional Berry phase statistics. For a simple vortex excitation, relevant for a subset of fractional Hall states considered by Laughlin, nontrivial Berry phase statistics were demonstrated many years ago by Arovas, Schrieffer, and Wilczek. The quasiparticles are in general more complicated, described accurately in terms of excited composite fermions. We use the method developed by Kjønsberg, Myrheim, and Leinaas to compute the Berry phase for a single composite-fermion quasiparticle and find that it agrees with the effective magnetic field concept for composite fermions. We then evaluate the "fractional statistics," related to the change in the Berry phase for a closed loop caused by the insertion of another composite-fermion quasiparticle in the interior. Our results support the general validity of fractional statistics in the quantum Hall superfluid, while also giving a quantitative account of corrections to it when the quasiparticle wave functions overlap. Many caveats, both practical and conceptual, are mentioned that will be relevant to an experimental measurement of the fractional statistics. A short report on some parts of this article has appeared previously.

I. INTRODUCTION

When hard-core particles are confined in two dimensions, the configuration space is multiply connected, and paths with different winding numbers are topologically distinct because they cannot be continuously deformed into one another. The particles are said to have statistics θ^* if a path-independent phase $2\pi\theta^*$ results when one particle traverses around another in a complete loop. A half loop is equivalent to an exchange of particles, assuming translational invariance, which produces a phase factor $e^{i\pi\theta^*}=(-1)^{\theta^*}$. As pointed out by Leinaas and Myrheim,[1] nonintegral values of θ^* are allowed, which are referred to as fractional statistics. Clearly, fractional statistics can be consistently defined only in two dimensions, because in higher dimensions the notion of a particle going around another is topologically ill defined.

Given the experimental fact that all fundamental particles in nature are either bosons or fermions, any discussion of fractional statistics may appear academic. That would perhaps be true from an elementary particle physicist's perspective. However, there is no fundamental principle that precludes the possibility that certain *emergent* quasiparticles of a condensed matter system might possess fractional statistics; indeed, an appearance of such statistics would be an interesting example of how entirely new concepts can emerge[2] in a many-body system. Obviously, it would take a highly nontrivial state of matter in order for fractional statistics particles to emerge, and nature has graciously provided a possible candidate—namely, the quantum Hall superfluid (QHS).[3] The QHS is formed when interacting electrons confined to two dimensions are exposed to a strong magnetic field. It is characterized by quantized Hall plateaus and a vanishing resistance at zero temperature (in spite of the presence of disorder). The investigation of the QHS has given rise to much interesting physics since its discovery in the early 1980s.[3]

The possibility of fractional statistics in the QHS was first recognized by Halperin,[4] demonstrated in a microscopic theory by Arovas, Schrieffer, and Wilczek[5] for a "vortex" excitation in Laughlin's wave function[6] at filling factors of the form $\nu=1/m$, m odd, and argued to be a general consequence of incompressibility at a fractional filling of the lowest Landau level.[7]

It is by now clear that the physics of the QHS can be understood, both qualitatively and quantitatively, without any mention of fractional statistics. Jain showed[8] that the nonperturbative physics of the QHS lies in the formation of particles that are *fermions*, called composite fermions, which are bound states of electrons and an even number of quantized vortices. Many essential properties of the QHS can be explained by neglecting the interaction between composite fermions altogether, as properties of almost free fermions. Extensive experimental and theoretical work has established that the composite-fermion (CF) theory gives a *complete* description of the low-energy Hilbert space of the system,[9–11] in that it correctly predicts the quantum numbers of the low-energy states and also gives an accurate microscopic wave function for them. Thus, neither the explanation of the phenomenology nor a calculation of the experimentally measurable quantities requires, in principle, any consideration of fractional statistics.

With the exception of the "quasihole" at $\nu=1/m$, the excitations of fractional quantum Hall effect (FQHE) are not described by a simple vortex, but have a much more complicated structure, just as the FQHE ground states in general have much more complex wave functions than those at $\nu=1/m$. Analytical calculations for the Berry phase statistics have not been possible for the nonvortex excitations, but numerical calculations have been carried out and showed surprising results. For the *quasiparticles* (as opposed to quasiholes) at $\nu=1/m$, a calculation by Kjønsberg and Myrheim[12] with a trial wave function suggested by Laughlin[6] showed that they do *not* possess a well-defined fractional statistics, in the sense that the calculated statistics parameter shows rapid variations with the separation between the two quasiparticles and fails to reach an asymptotic limit. That

appears to cast doubt on the generality of the concept of fractional statistics and also on the validity of the earlier heuristic derivations that were based on nothing more than incompressibility at a fractional filling.[7]

The microscopic understanding in terms of composite fermions has enabled further progress. Because the CF theory gives a good account of the low-energy physics, it must also contain information about fractional Berry phase statistics, provided it really exists. One might naively think that the fractional statistics is incompatible with the CF theory, but that is not the case. As discussed in several articles[13,14] the CF theory, in fact, provides a straightforward heuristic "derivation" for fractional statistics. The fractional statistics simply keeps track of how the effective magnetic field experienced by composite fermions is affected by local deformations in the density, as obtained, for example, by creation of a localized excitation. The CF theory allows one to go beyond the simple vortex at $\nu = 1/m$ through the general understanding of quasiparticles as excited composite fermions. The wave function for the CF quasiparticle at $\nu = 1/m$ written by Jain[8] is known to be more accurate than the one suggested by Laughlin.[15–18] An important step in the clarification of the issue of fractional statistics was taken by Kjønsberg and Leinaas, who showed that when the former wave function is used for a calculation of the statistics, a definite value is obtained.[19] The present study confirms that the result is robust to projection into the lowest Landau level, sorts out certain subtle corrections left out in the earlier study, and extends the calculation to more complex excitations of other incompressible states. A brief account of some of the results below has appeared previously in a short article by the authors.[20]

The logical route to fractional statistics is displayed in Fig. 1, which serves to clarify the cause-and-effect relationship between various concepts. The fractional statistics is a consequence of incompressibility at fractional filling factors;[21] the incompressibility itself results from the formation of composite fermions. Two notable facts consistent with the directions of arrows in Fig. 1 are that (a) the fractional statistics can be derived from composite fermions, but the reverse is not possible, and (b) the fractional statistics is tied to incompressibility, whereas composite fermions are more general and apply to compressible states as well. The CF theory is the microscopic theory of the QHS, whereas a description in terms of fractional statistics quasiparticles is an effective theory obtained when all but a few degrees of freedom are integrated out.

It should be stressed that fractional statistics is a rather delicate concept. The effective magnetic field of composite fermions is a robust $O(N)$ quantity, which has been directly measured in experiments and gives an immediate explanation for the FQHE and many other phenomena. The fractional statistics, on the other hand, provides a natural interpretation for a subtle, but perhaps measurable, property of composite fermions, which specifies how the effective magnetic field changes upon an $O(1)$ variation in the particle density. That is the reason why it has not been possible to confirm it so far, although several proposals have recently been advanced.[22]

This paper is organized as follows: Section II is devoted to the introduction of the composite-fermion concept and the
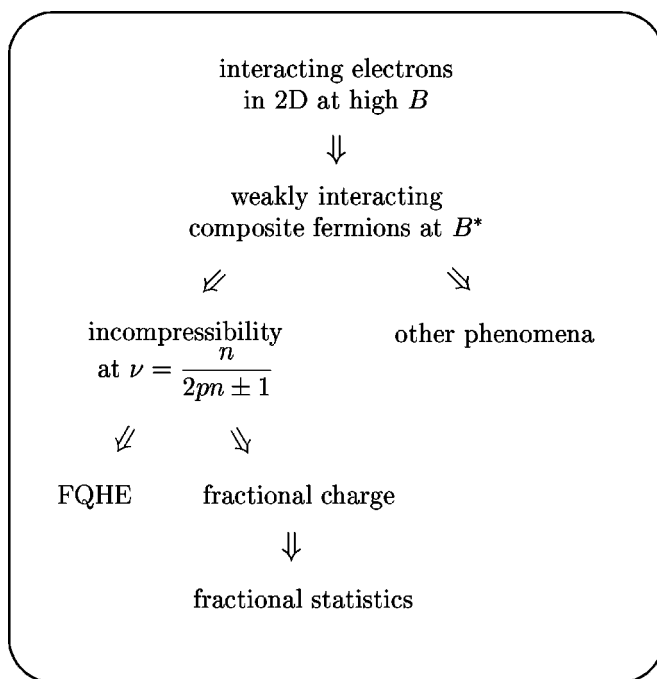

FIG. 1. The logical path to fractional statistics. First, the interacting electrons transform into weakly interacting composite fermions at an effective magnetic field. Composite fermions form incompressible states at certain fractional fillings of the lowest Landau level. Incompressibility implies fractional charge, which, finally, leads to fractional statistics.

interpretation in terms of effective magnetic fields. In Sec. III, we will first calculate the Berry phase for a single CF quasiparticle and show that it is consistent with the effective magnetic field principle. The effective magnetic field principle is also shown to explain the situation when the CF quasiparticle lies outside the quantum Hall droplet. The fractional statistics of CF quasiparticles is calculated microscopically in Sec. IV. We will see that very small deviations in the trajectory make corrections that are on the same order as the statistics itself. The CF quasiparticles in different CF–quasi-Landau levels are also found to exhibit the same fractional statistics. Finally, constraints on experimental observations of the fractional statistics are discussed.

II. COMPOSITE FERMIONS AND EFFECTIVE MAGNETIC FIELD

The physics of the QHS is governed by the Hamiltonian

$$H = \sum_j \frac{1}{2m_b}\left[\frac{\hbar}{i}\boldsymbol{\nabla}_j + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_j)\right]^2 + \sum_{j<k}\frac{e^2}{|\boldsymbol{r}_j - \boldsymbol{r}_k|}. \quad (1)$$

In the limit of very large magnetic fields, all electrons can be taken to reside in the lowest Landau level (LL), and the Hamiltonian reduces, in appropriate units, to

$$H = \mathcal{P}_{LLL}\sum_{j<k}\frac{1}{|\boldsymbol{r}_j - \boldsymbol{r}_k|}\mathcal{P}_{LLL}, \quad (2)$$

where $\mathcal{P}_{LLL}$ denotes projection into the lowest LL.

The problem is highly nontrivial because of the lack of a small parameter, but we have a good understanding of its physics, both qualitatively and quantitatively, based on the formation of a new kind of fermions called composite fermions, which are bound states of electrons and an even number of quantized vortices. Through composite fermions, many essential features can be understood through an analogy to the well-understood integral quantum Hall effect (IQHE). The wave function of the QHS has the following structure:

$$\Psi_\nu = \mathcal{P}_{LLL} \Phi_{\nu^*} \prod_{j<k} (z_j - z_k)^{2p}. \tag{3}$$

Here Ψ_ν is the wave function of interacting electrons at filling factor ν, defined as $\nu = \rho \phi_0 / B$ (ρ is the two-dimensional density of electrons, B is the external magnetic field, and $\phi_0 = hc/e$ is called the magnetic flux quantum). Φ_{ν^*} is the wave function for *weakly interacting* electrons at filling factor ν^*, related to ν by

$$\nu = \frac{\nu^*}{2p\nu^* + 1}. \tag{4}$$

The complex number $z_j = x_j - iy_j$ denotes the position of the jth electron in the x-y plane. Ψ_ν are known to be accurate representations of the actual eigenfunctions of the lowest LL projected Coulomb Hamiltonian,[9–11] and it will be assumed below that they provide a correct account of topological properties like the fractional statistics.

The filling factor ν is typically <1, whereas ν^* is typically >1. The mapping in Eq. (3) leads to a simplification of the problem because the space of candidate wave functions at ν^* is much smaller than that at ν. In particular, when $\nu^* = n$ (an integer), the ground-state wave function Φ_n is unique, giving a unique wave function Ψ_ν at $\nu = n/(2pn+1)$. That explains the FQHE at these filling factors, which are the prominently observed sequences of fractions.

The physics of the wave function Ψ_ν is best understood prior to the lowest LL projection—that is, with

$$\Psi_\nu^{\text{up}} = \Phi_{\nu^*} \prod_{j<k} (z_j - z_k)^{2p}. \tag{5}$$

The Jastrow factor $\prod_{j<k}(z_j - z_k)^{2p}$ binds $2p$ vortices to each electron. The bound state is interpreted as a particle—namely, the composite fermion. Now imagine a composite fermion—i.e., an electron along with $2p$ vortices—traversing a closed loop enclosing an area A (in the counterclockwise direction). The phase associated with this process contains two terms:

$$\Phi^* = -2\pi \frac{BA}{\phi_0} + 2\pi 2p N_{\text{enc}}, \tag{6}$$

where N_{enc} is the number of composite fermions inside the loop. The first term is the usual Aharonov-Bohm (AB) phase

$$\Phi = -2\pi \frac{BA}{\phi_0}, \tag{7}$$

produced when a particle of charge $-e$ executes a counterclockwise loop, with the magnetic field pointing in the $+z$ direction. The second term is the phase due to $2p$ vortices

going around N_{enc} particles inside the loop, with each particle contributing a phase of 2π. Equation (6) will play a fundamental role in what follows. As we shall see, this equation implies incompressibility at certain fractional fillings and also explains the origin of fractional statistics.

We interpret the net phase as the AB phase due to an *effective* magnetic field B^*:

$$\Phi^* \equiv -2\pi \frac{B^* A}{\phi_0}. \tag{8}$$

In a mean-field approximation, we replace N_{enc} by its average value ρA, which gives

$$B^* = B - 2p\rho\phi_0. \tag{9}$$

Thus, the composite fermions behave as though they were in a much smaller effective magnetic field.

To understand why the Berry phases coming from the vortices in the Jastrow factor effectively *cancel* (as opposed to augment) the magnetic field, it is instructive to understand the effective magnetic field by eliminating the phases of the Jastrow factor in favor of a vector potential following the standard approach.[23,24] Consider the Schrödinger equation

$$\left[\frac{1}{2m_b} \sum_i \left(\boldsymbol{p}_i + \frac{e}{c} \boldsymbol{A}(\boldsymbol{r}_i) \right)^2 + V \right] \prod_{j<k} (z_j - z_k)^{2p} \Phi_{\nu^*}$$
$$= E \prod_{j<k} (z_j - z_k)^{2p} \Phi_{\nu^*}, \tag{10}$$

where V is the interaction. The kinetic energy term will be the important one in what follows. (We note that the unprojected wave function is not an exact eigenfunction of the Hamiltonian. For the sake of the present argument, one may think of Φ_{ν^*} as an arbitrary wave function rather than the solution of the noninteracting problem at ν^*; then, the exact eigenstate in question can always be written in the above form. While performing the actual calculations of the Berry phase, we will of course use the projected wave functions which have a close to 100% overlap with the exact eigenstates.) Display the phases due to the Jastrow factor explicitly:

$$\prod_{j<k} (z_j - z_k)^{2p} = e^{-i2p\Sigma_{j<k}\phi_{jk}} \prod_{j<k} |z_j - z_k|^{2p}. \tag{11}$$

Here

$$\phi_{jk} = i \ln \frac{z_j - z_k}{|z_j - z_k|}. \tag{12}$$

We have been careful above to keep track of the fact that $z = re^{-i\phi}$, as appropriate for external magnetic field in the $+z$ direction. The Schrödinger equation can be rewritten as

$$\left[\frac{1}{2m_b} \sum_i \left(\boldsymbol{p}_i + \frac{e}{c} \boldsymbol{A}(\boldsymbol{r}_i) + \frac{e}{c} \boldsymbol{a}(\boldsymbol{r}_i) \right)^2 + V \right] \prod_{j<k} |z_j - z_k|^{2p} \Phi_{\nu^*}$$
$$= E \prod_{j<k} |z_j - z_k|^{2p} \Phi_{\nu^*}, \tag{13}$$

where the additional vector potential, which simulates the effect of the phases of the Jastrow factor, is given by

$$a(r_i) = - \frac{2p}{2\pi} \phi_0 \sum_j {}' \nabla_i \phi_{ij}, \tag{14}$$

where the prime denotes the condition $j \neq i$. The corresponding magnetic field is

$$b_i = - 2p\phi_0 \hat{z} \sum_j {}' \delta^2(r_i - r_j). \tag{15}$$

Thus, the phase of the Jastrow factor is equivalent to each electron seeing a flux tube of strength $-2p\phi_0$ on all other electrons; the minus sign indicates that the flux tube points in the $-z$ direction, opposite to the direction to the external field $B = B\hat{z}$.

This interpretation raises the following questions. (i) The effective vector potential does not take care of *all* of the phases in the unprojected wave function Ψ_ν^{up}, because there are additional vortices and antivortices in Φ_{ν^*}. What about their effect? (ii) How does the projection into the lowest LL affect the above considerations? The feature that $2p$ vortices are strictly bound to electrons prior to the projection is lost upon projection into the lowest LL. For example, for $\nu > 1/3$, where composite fermions manifestly carry two vortices prior to projection, only one vortex can be bound to each electron *after* projection. The projection thus obscures the physics of composite fermions. Is there any way of seeing an effective magnetic field directly with the projected wave functions?

Even though the effective magnetic field is revealed most clearly in the unprojected wave functions, the robustness of the concept to perturbations has been confirmed in a model independent manner by numerous facts. (i) Experiments clearly show a remarkably close correspondence between the FQHE and the IQHE, thus providing a nontrivial global confirmation of the effective magnetic field concept. (ii) Direct measurements of the cyclotron radius of the charge carrier[25] are consistent with B^*. (iii) Exact diagonalization studies show that the low-energy spectrum of interacting electrons at B has a one-to-one correspondence with the low-energy spectrum of noninteracting fermions at B^*.[11] (iv) The wave functions of interacting electrons at $B(\nu)$ are closely related to the wave functions of noninteracting electrons at $B^*(\nu^*)$, as seen in Eq. (3). From these observations, it is clear that the concept of effective magnetic field is more generally valid than the derivation based on the unprojected wave functions Ψ^{up} would suggest.

We now proceed to confirm Eq. (6) by calculating the Berry phase explicitly for a closed loop of composite fermion at $\nu = 1/3$ and $\nu = 2/5$ for the lowest LL projected wave functions. The answers are fully consistent with the effective magnetic field principle.

III. BERRY PHASE FOR A SINGLE CF QUASIPARTICLE

To confirm the effective magnetic field concept in a Berry phase calculation, one can envision creating a localized composite-fermion wave packet and determining the Berry phase associated with a closed loop enclosing an area A. Consider first the ground state at $\nu = n/(2pn+1)$, which maps into $\nu^* = n$ filled quasi-Landau levels of composite fermions at an effective magnetic field given by $B^* = B/(2pn \pm 1)$. From the analogous case of $\nu = n$, where n Landau levels are fully occupied, it is obvious that it is not possible to make a wave packet here without creating excitations. Therefore, one is forced to consider excitations. At $\nu = n$ we can straightforwardly make a wave packet if we put an additional electron in the lowest unoccupied LL, which can then be moved in any desired trajectory. That is what we will do with composite fermions.

We will refer to as the "composite-fermion quasiparticle" (CFQP) a composite fermion in the otherwise empty CF-quasi-LL, which is the image of the electron state which has n completely occupied LL's and a single electron in the $(n+1)^{st}$ LL. Similarly, the hole left behind when a composite fermion is removed from the topmost CF-quasi-LL will be termed "composite-fermion quasihole" (CFQH). The state at $\nu = n/(2pn \pm 1)$ is to be thought of as the "vacuum." Relative to the "vacuum" state, the CFQP or CFQH has a charge excess or deficiency in a spatially localized region. It ought to be stressed that even a single CFQP or a CFQH describes a strongly correlated state of many interacting electrons.

Now take a CFQP in a loop enclosing an area A. Because it is nothing but a composite fermion, the phase is predicted to be the same as in Eq. (6):

$$\Phi^* = - 2\pi \frac{B^*A}{\phi_0} = - 2\pi \frac{eBA}{(2pn+1)hc}. \tag{16}$$

This is what we will first confirm.

The calculation of Berry phase requires microscopic wave functions which are constructed starting with the wave function of a quasiparticle at $\nu^* = n$, using the standard framework of the CF theory. One problem is to figure out where to place the electrons in the corresponding IQHE problem, so, when the wave function is transformed to that of composite fermions, we get the CFQP's at the desired location. To do so, we implement the mapping from ν^* to ν in a manner that *preserves distances* (to zeroth order). We first construct a quasiparticle wave function at B^*, multiply it by Φ_1^{2p}, where

$$\Phi_1 = \prod_{j<k=1}^N (z_j - z_k) \exp\left[- \frac{1}{4l_1^2} \sum_i |z_i|^2 \right], \tag{17}$$

with $l_1^2 = \hbar c/eB_1 = \hbar c/e\rho\phi_0$, and finally project the product into the lowest electronic LL. This mapping preserves the size of the disk containing the quantum Hall droplet, because while the Jastrow factor pushes the particles out, the Gaussian pulls them in precisely by an amount to cancel the two effects. It is easy to check that the density is not changed in going from $\nu^* = n$ to $\nu = n/(2pn+1)$ in this manner. (See the article by Jain in Ref. 9 for more details.)

At ν^*, the single-particle orbitals in the lowest LL are given by

$$\zeta_m(z) \equiv \frac{z^m}{\sqrt{2\pi 2^m m!}} \exp\left[- \frac{1}{4l^{*2}}|z|^2 \right], \tag{18}$$

where $l^* = (2pn+1)^{1/2}l$ is the magnetic length at B^*. To put a CFQP at η, we first construct the electronic wave function at

ν^* with an electron in the relevant Landau level (otherwise empty) at η in a familiar coherent state. The coherent state at η in the lowest LL is given by

$$\bar{\phi}_\eta^{(0)}(\boldsymbol{r}) = \sum_{m=0}^{\infty} \bar{\zeta}_m(\eta)\zeta_m(z) = \exp\left[\frac{\bar{\eta}z}{2l^{*2}} - \frac{|\eta|^2}{4l^{*2}} - \frac{1}{4l^{*2}}|z|^2\right].$$

(19)

One can elevate the coherent state to higher Landau levels by repeated application of the LL raising operator $a^\dagger \equiv (2\,\partial/\partial z - \bar{z}/2)/\sqrt{2}$, which leads to the coherent-state wave function in the $(n+1)^{\text{st}}$ LL, apart from a constant factor:

$$\bar{\phi}_\eta^{(n)}(\boldsymbol{r}) = (\bar{z} - \bar{\eta})^n \exp\left[\frac{\bar{\eta}z}{2l^{*2}} - \frac{|\eta|^2}{4l^{*2}} - \frac{1}{4l^{*2}}|z|^2\right].$$

(20)

It is convenient to define

$$\phi_\eta^{(n)}(\boldsymbol{r}) = (\bar{z} - \bar{\eta})^n \exp\left[\frac{\bar{\eta}z}{2l^{*2}} - \frac{|\eta|^2}{4l^{*2}}\right],$$

(21)

so

$$\bar{\phi}_\eta^{(n)}(\boldsymbol{r}) = \phi_\eta^{(n)}(\boldsymbol{r})\exp\left[-\frac{1}{4l^{*2}}|z|^2\right].$$

(22)

As an example, consider the system at $\nu = 1/(2p+1)$, which is related to the CF filling $\nu^* = 1$. The electron wave function at $\nu^* = 1$ with fully occupied lowest LL and an additional electron in the second LL at η is

$$\Phi_1^\eta = \begin{vmatrix} \phi_\eta^{(1)}(\boldsymbol{r}_1) & \phi_\eta^{(1)}(\boldsymbol{r}_2) & \cdots \\ 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ . & . & \cdots \\ . & . & \cdots \\ z_1^{N-2} & z_2^{N-2} & \cdots \end{vmatrix} \exp\left(-\sum_j |z_j|^2/4l^{*2}\right).$$

(23)

This leads to the (unnormalized) wave function for a CFQP at $\nu = 1/(2p+1)$:

$$\Psi_{1/(2p+1)}^\eta = \mathcal{P}_{LLL} \begin{vmatrix} \phi_\eta^{(1)}(\boldsymbol{r}_1) & \phi_\eta^{(1)}(\boldsymbol{r}_2) & \cdots \\ 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ . & . & \cdots \\ . & . & \cdots \\ z_1^{N-2} & z_2^{N-2} & \cdots \end{vmatrix}$$

$$\times \prod_{i<k=1}^{N} (z_i - z_k)^{2p}\exp\left(-\sum_j |z_j|^2/4l^2\right).$$

(24)

Here, we have used

$$\frac{1}{l^{*2}} + \frac{2p}{l_1^2} = \frac{1}{l^2},$$

(25)

which is equivalent to Eq. (9). This wave function is similar to that considered by Kjønsberg and Leinaas,[19] but not identical.

Figure 2 shows the excess density due to the presence of a single localized CFQP for $\nu = 1/3$. The localized excess profile is clearly observed in the intended position indicated by the arrow in the lower panel; the profile has a smoke-ring-like shape since the quasiparticle is in the second CF-quasi-LL. (The coherent wave packet for an electron in the second LL also has a similar shape.) A deficit of the charge along the boundary is also discernible.

In a similar way we can construct the wave function for a CFQP of the state at $\nu = n/(2pn+1)$ for arbitrary n and p. For example, the wave function at $\nu = 2/(4p+1)$ corresponding to $n = 2$ is given explicitly by

$$\Psi_{2/(4p+1)}^\eta = \mathcal{P}_{LLL} \begin{vmatrix} \phi_\eta^{(2)}(\boldsymbol{r}_1) & \phi_\eta^{(2)}(\boldsymbol{r}_2) & \cdots \\ \bar{z}_1 & \bar{z}_2 & \cdots \\ \bar{z}_1 z_1 & \bar{z}_2 z_2 & \cdots \\ \vdots & \vdots & \cdots \\ \bar{z}_1 z_1^{N/2-2} & \bar{z}_2 z_2^{N/2-2} & \cdots \\ 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ \vdots & \vdots & \cdots \\ z_1^{N/2-1} & z_2^{N/2-1} & \cdots \end{vmatrix}$$

$$\times \prod_{i<k=1}^{N} (z_i - z_k)^{2p} \exp\left(-\sum_j |z_j|^2/4l^2\right). \quad (26)$$

There are two methods for performing projection into the lowest LL. In one method,[26] (i) normal ordering the factor multiplying the Gaussian factor $\exp(-\Sigma_j |z_j|^2/4l^2)$ by bringing all $\bar{z}_i$ to the left of z_i and (ii) make the replacement $\bar{z}_i \rightarrow 2\,\partial/\partial z_i$ with the understanding that the partial derivatives do not act on the Gaussian factor $\exp(-\Sigma_j |z_j|^2/4l^2)$. We employ a slightly different other projection method, described in Ref. 10, which has many advantages in the numerical calculation, especially for large systems. In the CF theory, the unprojected wave function has the form

$$\Psi^{\text{up}} = \begin{vmatrix} \psi_1(z_1) & \psi_1(z_2) & \cdots \\ \psi_2(z_1) & \psi_2(z_2) & \cdots \\ . & . & \cdots \\ . & . & \cdots \\ \psi_N(z_1) & \psi_N(z_2) & \cdots \end{vmatrix}$$

$$\times \prod_{i<k=1}^{N} (z_i - z_k)^{2p} \exp\left(-\sum_j |z_j|^2/4l^2\right). \quad (27)$$
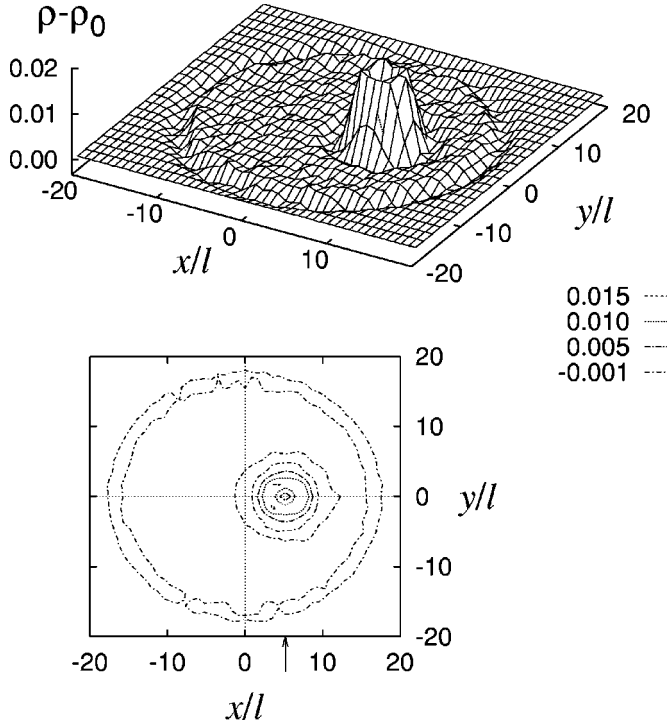
Such wave functions can be rewritten in the form

FIG. 2. Excess charge density relative to the ground state in the presence of a single CFQP. We used $\nu=1/3$, $N=50$, and $\eta/l=0.3R_d\approx5.2$ with disk size $R_d\equiv\sqrt{2N/\nu}$. The resulting position of the CFQP is in good agreement with the intended position, which is indicated by the arrow in the contour plot (lower panel).

$$\Psi^{up} = \exp\left(-\sum_j |z_j|^2/4l^2\right)\begin{vmatrix}\psi_1(z_1)J_1^p & \psi_1(z_2)J_2^p & \cdots \\ \psi_2(z_1)J_1^p & \psi_2(z_2)J_2^p & \cdots \\ . & . & \cdots \\ . & . & \cdots \\ \psi_N(z_1)J_1^p & \psi_N(z_2)J_2^p & \cdots\end{vmatrix},$$

(28)

with $J_j\equiv\Pi_{k\neq j}(z_j-z_k)$. Then the projected wave function is given by projecting each element of the determinant separately into the lowest Landau level by the method described above.

In order to test the robustness of the results to how the projection is carried out, we have studied wave function projected by the two ways as well as the unprojected one for the CFQP at $\nu=1/3$. The results were independent of the employed state as long as the position of the CFQP is far from the boundary of the system.

A. Berry phase

The Berry phase associated with a path $\mathcal{C}$ is given by

$$\Phi^* = \oint_c d\theta \frac{\left\langle\Psi^\eta\left|i\frac{d}{d\theta}\Psi^\eta\right.\right\rangle}{\langle\Psi^\eta|\Psi^\eta\rangle},$$

(29)

where Ψ^η is the wave function containing a single CFQP at η. For convenience, we take $\eta=Re^{-i\theta}$, and $\mathcal{C}$ refers to the
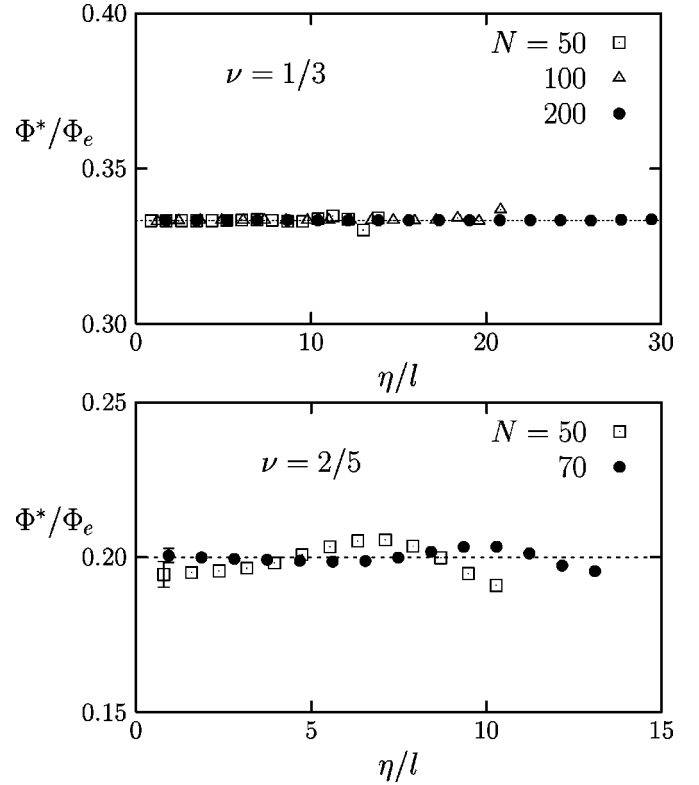

FIG. 3. The Berry phase Φ^* for a single CFQP at $\nu=1/3$ (upper panel) and $\nu=2/5$ (lower panel) as a function of η. Here N is the total number of composite fermions, l is the magnetic length, and $\Phi_e\equiv-2\pi BA/\phi_0$ is the Berry phase acquired by an electron moving in an empty space. The error bars from Monte Carlo sampling which are smaller than the symbol size are not shown explicitly. The deviation at the largest η/l for each N is due to proximity to the edge.

circular path with R fixed and θ varying from 0 to 2π in the counterclockwise direction. (Note that while the CFQP moves in the counterclockwise direction in the x-y plane, the complex coordinate η executes a clockwise loop in the complex plane.) The integrand in Eq. (29) involves $2N$-dimensional integrals over the CF coordinates, which we evaluate by the Monte Carlo method. Approximately 4×10^8 iterations are performed for each point. For $\nu=1/3$ we have studied systems with $N=50$, 100, and 200 particles, and for $\nu=2/5$ we study systems with $N=50$ and 70 particles. Projected wave functions are used in both cases.

The resulting values of Berry phase are displayed as a function of the radius of the circular motion in Fig. 3. For both $\nu=1/3$ and $\nu=2/5$ the Berry phase exhibits well-defined values, which agree well with those in Eq. (16) predicted by the effective magnetic field description. The deviation for large η is a boundary effect, caused by the finiteness of the system. The overall behavior for $\nu=1/3$ is consistent with the result in Ref. 19. The effective magnetic field thus survives projection into lowest LL.

B. Fractional local charge

Above we derived the Berry phase as coming from the combination of two terms, due to an electron and $2p$ vortices

going around a closed loop. This actually provides a derivation of the local charge of the CFQP, where the local charge, denoted by $-e^*$, is defined to be charge excess associated with it relative to the uniform ground state. The Berry phase of a CFQP is also the AB phase for a charge $-e^*$, which gives, using Eq. (16),

$$-2\pi\frac{e^*BA}{hc} = -2\pi\frac{eBA}{(2pn+1)hc}. \tag{30}$$

Thus the local charge of a CFQP is

$$-e^* = -\frac{e}{2pn+1}. \tag{31}$$

The local charge can be derived in many other ways.[6] Another way is to add the charge of the constituents of the CFQP—namely, the electron and the vortices.[9] The charge of a vortex[6] is νe, which is the occupation of a single orbital at filling ν. The local charge of the CFQP, a bound state of an electron and $2p$ vortices, is thus $-e^* = -e + 2p\nu e = -e/(2pn+1)$. One can also show that the addition of one electron creates $2pn+1$ CFQP's, which again implies that the local charge associated with each CFQP is $-e/(2pn+1)$. The fact that the local charge is independent of details (relying only on incompressibility) provides insight into why the Berry phase of the CFQP is robust to projection into the lowest LL.

C. CF quasiparticle outside the disk

In the previous sections, we have considered only the situation when the CFQP is inside a QHS droplet. It is interesting to ask what happens when a CFQP is located outside the droplet. Far from the droplet, the CFQP no longer has any other CF's nearby and therefore there is no screening hole. Its local charge therefore is the same as a bare charge $-e$ due to the absence of the screening cloud. How about the Berry phase acquired by the CFQP? Should it be the same as Φ_e, which is the Berry phase for an electron moving in the free space in a uniform external magnetic field B?

Before proceeding further within the CF theory, we should reexamine how the actual CFQP position is related to the parameter η in the wave function. The condition that the position of the CFQP be given by η was derived under the assumption that the CFQP is surrounded by other uniform CF's; we can no longer expect that the CFQP position is given by η when it is off the droplet. Far from the droplet, the CFQP will experience the *bare* external magnetic field B rather than $B^* = B/(2pn+1)$. Accordingly, the use of effective magnetic length l^* in the coherent state leads to the actual position of the CFQP given by $\xi \approx \eta/(2pn+1)$. This is verified in Fig. 4, which plots the actual location ξ obtained numerically as a function of the parameter η. The numerical calculation was performed at $\nu = 1/3$ for $N = 50$ composite fermions. When the parameter η exceeds the droplet size $R_d \equiv l\sqrt{2N/\nu}$, the position ξ deviates from the (dashed) line $\xi = \eta$. As η is increased further, ξ approaches the (solid) line $\xi = \eta/(2pn+1)$.

The CF theory also makes a prediction for the Berry phase of a single CFQP outside the droplet. Since the en-
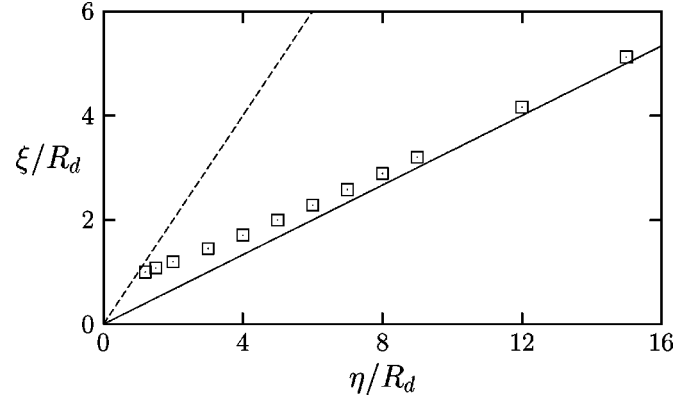

FIG. 4. Actual location ξ of a CFQP (in units of the disk size $R_d \equiv l\sqrt{2N/\nu}$) as a function of the parameter η for $\nu = 1/3$ and $N = 50$ when the CFQP is outside the quantum Hall superfluid droplet. The dashed line is $\xi = \eta$ and the solid line is $\xi = \eta/(2pn+1)$, the position the CFQP would have in the absence of any other composite fermions (in the case of $\nu = 1/3$, we have $\xi = \eta/3$; see text).

closed area is not filled uniformly with CF's, we can no longer use the *uniform* effective magnetic field in Eq. (9). Instead, we must use Eq. (6). Outside the droplet, the number of enclosed composite fermions is $N_{\text{enc}} = N - 1$ and the enclosed area is $A = \pi\xi^2$, yielding the Berry phase

$$\frac{\Phi^*}{\Phi_e} = 1 - \frac{4p(N-1)}{(\xi/l)^2}. \tag{32}$$

The second term, the contribution from the composite fermions on the QHS droplet, is of order $\mathcal{O}(\xi^{-2})$.

Figure 5 demonstrates clearly that the prediction in Eq. (32), denoted by the dashed line in the figure, explains nicely the behavior of Φ^* when the CFQP is outside the droplet. Indeed, the numerical data begin to deviate when the CFQP approaches the boundary of the droplet and eventually give a definite value $1/(2pn+1)$ inside the system (the Berry phase for a CFQP inside the disk are not shown in Fig. 5). Because the *local* charge of a CFQP becomes $-e$ just outside the droplet, the long tail $\sim\mathcal{O}(\xi^{-2})$ in Φ^*/Φ_e is not explained by
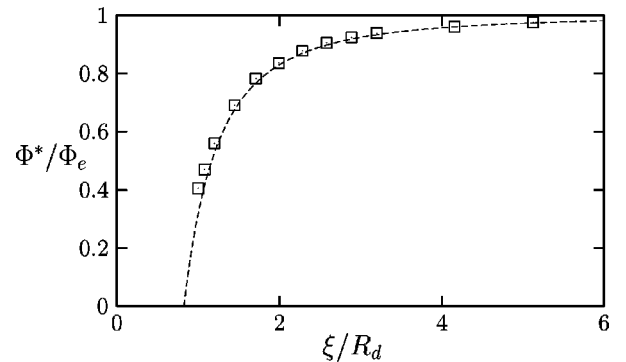

FIG. 5. Berry phase Φ^* acquired by a CFQP when it is outside the quantum Hall droplet. The system has $N = 50$ composite fermions at the filling $\nu = 1/3$. The dashed line is the prediction of the CF theory, and the squares are calculated from the microscopic wave functions.

the alternate interpretation of the Berry phase in terms of a charge of $-e^*$ moving under the external magnetic field B.

IV. TWO CF QUASIPARTICLES: FRACTIONAL STATISTICS

We have confirmed Eq. (6) for a single CFQP in a closed loop, when the other composite fermions make a uniform background state. How about the situation when the density is *not* uniform? The simplest question one may ask is, how does the Berry phase change when we change the number of enclosed particles in a loop by a number of order unity? Following Ref. 14, Eq. (6) predicts

$$\Delta\Phi^* = 2\pi 2p\Delta\langle N_{enc}\rangle. \tag{33}$$

To be specific, we will add a single CFQP inside the loop, which, counting the correlation hole around it, carries an excess of $\Delta\langle N_{enc}\rangle = 1/(2pn+1)$ electrons, which gives:

$$\Delta\Phi^* = 2\pi\frac{2p}{2pn+1} \equiv 2\pi\theta^*, \tag{34}$$

with

$$\theta^* = \frac{2p}{2pn+1}. \tag{35}$$

A fractional value of θ^* is often interpreted through an assignment of a fractional statistics to the CFQP's. Note that the fractionally quantized value for θ^* is a direct consequence of the fractional quantization of the local charge. It should also be stressed that θ^* is a much more subtle quantity than B^*, sensitive to order unity changes in the enclosed particle number. Equation (6) is surely correct in a mean-field sense, but it is by no means obvious that it captures O(1) effects accurately.

The meaning of fractional statistics is complicated in the QHS context by the presence of a magnetic field, which produces its own phase for any closed loop, even when it does not include another CFQP. (Of course *all* loops enclose other composite fermions; here we think of only the excitations as the CFQP's.) The fractional statistics is defined as the *difference* in the phase for a given closed path when one CFQP is added to the interior. It is a small perturbation on a large effect. Even though we have derived the fractional statistics as an immediate corollary of the effective magnetic field principle,[14] the value of θ^* had been derived prior to the CF theory from general arguments,[4,7] assuming incompressibility at a fractional filling; the earlier values (if evaluated with **B** in the $+z$ direction) differ from the one quoted here by 1 (mod 2). The reason for this deviation is that our theory deals with quasiparticles that obey fermionic exchange statistics, so an additional factor (-1) arises from the mere position exchange of two CF quasiparticles in the microscopic wave function, whereas the previous ones assume bosonic exchange statistics. In Ref. 19 the result was shifted by unity. We prefer not to do that. The quasiparticles that we work with are the actual quasiparticles (as would be obtained in an exact diagonalization study if, say, two impurities were placed to localize two quasiparticles), and therefore the

phases given below are what an actual experiment would measure.

Our goal is to confirm Eq. (35) in a microscopic calculation. The statistics parameter is given by

$$\theta^* = \oint_{\mathcal{C}}\frac{d\theta}{2\pi}\frac{\left\langle\Psi^{\eta,\eta'}\left|i\frac{d}{d\theta}\Psi^{\eta,\eta'}\right.\right\rangle}{\langle\Psi^{\eta,\eta'}|\Psi^{\eta,\eta'}\rangle} - \oint_{\mathcal{C}}\frac{d\theta}{2\pi}\frac{\left\langle\Psi^{\eta}\left|i\frac{d}{d\theta}\Psi^{\eta}\right.\right\rangle}{\langle\Psi^{\eta}|\Psi^{\eta}\rangle}, \tag{36}$$

where Ψ^{η} is the wave function containing a single CFQP at η, and $\Psi^{\eta,\eta'}$ has two CFQP's at η and η'. Here we take $\eta = Re^{-i\theta}$, and $\mathcal{C}$ refers to the path with R fixed and θ varying from 0 to 2π in the counterclockwise direction, as in the calculation of a single-CFQP Berry phase. For convenience, we will take $\eta' = 0$ and denote the microscopic numerical value of θ^* by $\tilde{\theta}^*$ (the reason will be clear below).

Before proceeding further, we mention a curious fact to illustrate the fragility of fractional statistics. Laughlin had proposed the following wave function for two quasiparticles at $\nu = 1/m$ (m odd):

$$\Psi_L^{\eta} = \exp\left(-\sum_j |z_j|^2/4l^2\right)\prod_{j=1}^{N}(\partial_{z_j} - \eta)\prod_{j<k=1}^{N}(z_j - z_k)^m,$$

$$\Psi_L^{\eta,\eta'} = \exp\left(-\sum_j |z_j|^2/4l^2\right)\prod_{j=1}^{N}(\partial_{z_j} - \eta)(\partial_{z_j} - \eta')$$
$$\times\prod_{j<k=1}^{N}(z_j - z_k)^m. \tag{37}$$

Kjønsberg and Myrheim[12] found that the Berry phase calculation of the statistics of the quasiparticle using this wave function does not produce a well-defined answer. It was realized by Kjønsberg and Leinaas[19] that the more accurate wave function of the CF theory produces a well-defined value for θ^*. What makes it all the more surprising is that both the wave functions of Laughlin and Jain produce the correct local charge for a single quasiparticle. It is not understood why one of them fails to produce proper statistics, but the example underscores how the statistics may be sensitive to rather subtle correlations in the wave function.

In the CF theory, the wave function for two CFQP's at $\nu = 1/(2p+1)$ is a natural extension of that containing a single CFQP. The electron wave function at $\nu^* = 1$ with fully occupied lowest LL and two additional electrons in the second LL at η and η' is

$$\Phi_1^{\eta,\eta'} = \begin{vmatrix} \phi_\eta^{(1)}(\mathbf{r}_1) & \phi_\eta^{(1)}(\mathbf{r}_2) & \cdots \\ \phi_{\eta'}^{(1)}(\mathbf{r}_1) & \phi_{\eta'}^{(1)}(\mathbf{r}_2) & \cdots \\ 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ . & . & \cdots \\ . & . & \cdots \\ z_1^{N-3} & z_2^{N-3} & \cdots \end{vmatrix} \exp\left(-\sum_j |z_j|^2/4l^{*2}\right).$$

(38)

This leads to the (unnormalized) wave function for two CFQP's at $\nu=1/(2p+1)$:

$$\Psi_{1/(2p+1)}^{\eta,\eta'} = \mathcal{P}_{LLL} \begin{vmatrix} \phi_\eta^{(1)}(\mathbf{r}_1) & \phi_\eta^{(1)}(\mathbf{r}_2) & \cdots \\ \phi_{\eta'}^{(1)}(\mathbf{r}_1) & \phi_{\eta'}^{(1)}(\mathbf{r}_2) & \cdots \\ 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ . & . & \cdots \\ . & . & \cdots \\ z_1^{N-3} & z_2^{N-3} & \cdots \end{vmatrix}$$

$$\times \prod_{i<k=1}^{N} (z_i - z_k)^{2p} \exp\left(-\sum_j |z_j|^2/4l^2\right). \quad (39)$$

The extension to the general filling $\nu=n/(2pn+1)$ is again straightforward. For reference, we give an explicit expression of the two-CFQP wave function at $\nu=2/5$:

$$\Psi_{2/5}^{\eta,\eta'} = \mathcal{P}_{LLL} \begin{vmatrix} \phi_\eta^{(2)}(\mathbf{r}_1) & \phi_\eta^{(2)}(\mathbf{r}_2) & \cdots \\ \phi_{\eta'}^{(2)}(\mathbf{r}_1) & \phi_{\eta'}^{(2)}(\mathbf{r}_2) & \cdots \\ \bar{z}_1 & \bar{z}_2 & \cdots \\ \bar{z}_1 z_1 & \bar{z}_2 z_2 & \cdots \\ \vdots & \vdots & \cdots \\ \bar{z}_1 z_1^{N/2-3} & \bar{z}_2 z_2^{N/2-3} & \cdots \\ 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ \vdots & \vdots & \cdots \\ z_1^{N/2-1} & z_2^{N/2-1} & \cdots \end{vmatrix}$$

$$\times \prod_{i<k=1}^{N} (z_i - z_k)^2 \exp\left(-\sum_j |z_j|^2/4l^2\right). \quad (40)$$

The statistics parameter $\tilde{\theta}^*$ for $\nu=1/3$ and $\nu=2/5$ was shown in Ref. 20, reproduced in Fig. 6 for completeness. $\tilde{\theta}^*$ takes a well-defined value for large separations. At $\nu=1/3$ it approaches the asymptotic value of $\tilde{\theta}^*=-2/3$, which is consistent with that obtained in Ref. 19 without lowest LL projection. The calculation at $\nu=1/3$ explicitly demonstrates that $\tilde{\theta}^*$ is independent of whether the projected or the unprojected wave function is used. Assuming the same is true for other fractions, we have performed the calculation at $\nu=2/5$ without the projection. (The calculation of the statis-
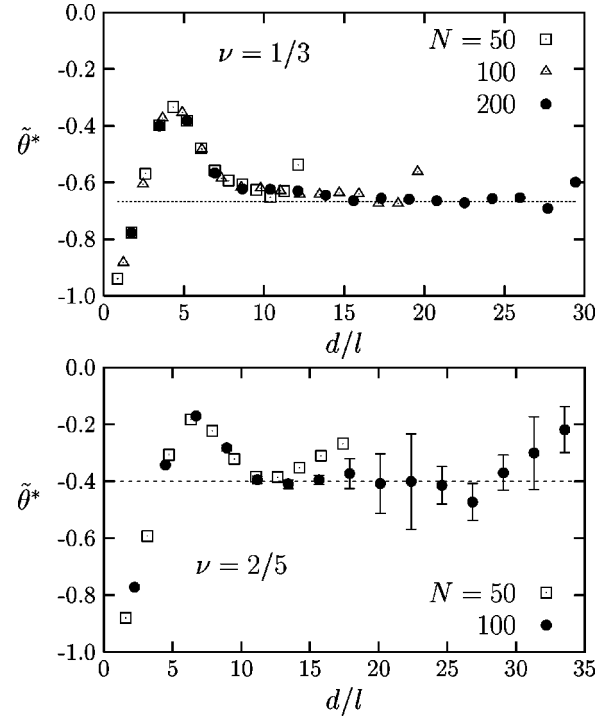

FIG. 6. The statistical angle $\tilde{\theta}^*$ for the CFQP's at $\nu=1/3$ (upper panel) and $\nu=2/5$ (lower panel) as a function of $d\equiv|\eta-\eta'|$. Here N is the total number of composite fermions, and l is the magnetic length. The error bar from Monte Carlo sampling is not shown explicitly when it is smaller than the symbol size. The deviation at the largest d/l for each N is due to proximity to the edge. This figure was shown earlier in Ref. 20 and is reproduced here for completeness.

tics, a small difference between two large quantities, requires much greater accuracy than the calculation of B^* considered in the previous section. The use of projected wave functions is in principle possible, but very costly in terms of computation time.) At $\nu=2/5$ the system size is smaller and the statistical uncertainty bigger, but the asymptotic value is clearly seen to be $\tilde{\theta}^*=-2/5$. At short separations there are substantial deviations in $\tilde{\theta}^*$; it reaches the asymptotic value only after the two CFQP's are separated by more than ~10 magnetic lengths. Such deviations are presumably due to a significant overlap between CFQP's when they are close. (In contrast, the effective magnetic field is well defined for arbitrarily small closed loops.)

A. Sign puzzle

The microscopic value $\tilde{\theta}^*$ obtained above has the same magnitude as θ^* in Eq. (35) *but the opposite sign*. The sign discrepancy, if real, cannot be reconciled with Eq. (6) and would cast doubt on the fundamental interpretation of the CF physics in terms of an effective magnetic field.

To gain insight into the issue, consider two composite fermions in the otherwise empty lowest LL, for which various quantities can be obtained analytically. When there is only one composite fermion at $\eta=Re^{-i\theta}$, it is the same as an electron, with the wave function given by

$$\chi^{\eta} = \exp\left[\frac{\bar{\eta}z}{2l^2} - \frac{R^2}{4l^2} - \frac{|z|^2}{4l^2}\right]. \tag{41}$$

For a closed loop,

$$\oint_{\mathcal{C}} \frac{d\theta}{2\pi} \frac{\left\langle \chi^{\eta} \left| i\frac{d}{d\theta} \chi^{\eta} \right.\right\rangle}{\langle \chi^{\eta} | \chi^{\eta} \rangle} = -\frac{R^2}{2l^2} = -\frac{\pi R^2 B}{\phi_0}. \tag{42}$$

Two composite fermions, one at η and the other at $\eta' = 0$, are described by the wave function

$$\chi^{\eta,0} = (z_1 - z_2)^{2p}(e^{\bar{\eta}z_1/2} - e^{\bar{\eta}z_2/2})e^{-(R^2 + |z_1|^2 + |z_2|^2)/4}. \tag{43}$$

Here, we expect $\theta^* = 2p$. However, an explicit evaluation of the Berry phase shows, neglecting $O(R^{-2})$ terms,

$$\oint_{\mathcal{C}} \frac{d\theta}{2\pi} \frac{\left\langle \chi^{\eta,0} \left| i\frac{d}{d\theta} \chi^{\eta,0} \right.\right\rangle}{\langle \chi^{\eta,0} | \chi^{\eta,0} \rangle} = -\frac{R^2}{2l^2} - 2p, \tag{44}$$

which gives $\widetilde{\theta}^* = -2p$ for large R. Again, it apparently has the "wrong" sign.

A calculation of the density for $\chi^{\eta,0}$ shows that the actual position of the outer composite fermion is not $R = |\eta|$ but R', given by

$$\frac{R'^2}{l^2} = \frac{R^2}{l^2} + 4 \times 2p \tag{45}$$

for large R. This can also be seen in the inset of Fig. 2 of Ref. 20. The correct interpretation of Eq. (44) therefore is

$$\oint_{\mathcal{C}} \frac{d\theta}{2\pi} \frac{\left\langle \chi^{\eta,0} \left| i\frac{d}{d\theta} \chi^{\eta,0} \right.\right\rangle}{\langle \chi^{\eta,0} | \chi^{\eta,0} \rangle} = -\frac{R'^2}{2l^2} + 2p, \tag{46}$$

which produces $\theta^* = 2p$. The $O(1)$ correction to the area enclosed thus makes a nonvanishing correction to the statistics. (It is noted that the CFQP at $\eta = 0$ is also a little off center and executes a tiny circular loop which provides another correction to the phase, but this contribution vanishes in the limit of large R.)

This exercise tells us that an implicit assumption made in the earlier analysis—namely, that the position of the outer CFQP labeled by η remains unperturbed by the insertion of another CFQP–leads to an incorrect value for θ^*. In reality, inserting another CFQP inside the loop pushes the CFQP at η very slightly outward.

To determine the correction at $\nu = n/(2pn + 1)$, we note that the mapping into composite fermions preserves distances to zeroth order, so Eq. (45) ought to be valid also at $\nu = n/(2pn + 1)$. This is consistent with the shift seen in Fig. 2 of Ref. 20 for the position of the CFQP calculated numerically directly from the wave function. Our earlier result

$$\oint_{\mathcal{C}} \frac{d\theta}{2\pi} \frac{\left\langle \Psi^{\eta,0} \left| i\frac{d}{d\theta} \Psi^{\eta,0} \right.\right\rangle}{\langle \Psi^{\eta,0} | \Psi^{\eta,0} \rangle} = -\frac{R^2}{2l^{*2}} - \frac{2p}{2pn + 1} \tag{47}$$

ought to be rewritten, using $l^{*2}/l^2 = B/B^* = 2pn + 1$, as

$$\oint_{\mathcal{C}} \frac{d\theta}{2\pi} \frac{\left\langle \Psi^{\eta,0} \left| i\frac{d}{d\theta} \Psi^{\eta,0} \right.\right\rangle}{\langle \Psi^{\eta,0} | \Psi^{\eta,0} \rangle} = -\frac{R'^2}{2l^{*2}} + \frac{2p}{2pn + 1}. \tag{48}$$

When the contribution from the closed path without the other CFQP, $-R'^2/2l^{*2}$, is subtracted out, θ^* of Eq. (35) is obtained. The neglect of the correction in the radius of the loop introduces an error which just happens to be twice the negative of the "correct" answer.

Before ending this subsection we note another subtle effect. A quasiparticle in the bulk induces a quasihole at the boundary, the charge of which is nonuniformly distributed over the edge when the bulk quasiparticle is off center. As the primary quasiparticle is taken around a loop, the "center" of the induced edge quasihole also executes a complete loop. The contribution of the latter to the Berry phase is neglected in the heuristic derivation of the statistics as well as in the analytical calculation of Arovas *et al.*,[5] but is explicitly included in the numerical calculations with a boundary. The consistency of the numerical results with the heuristic expectation indicates that the boundary effects are negligible, at least as long as the primary quasiparticles are sufficiently far from the edge.

B. Approach to the asymptotic value

In the previous section, it was shown that the asymptotic value of the statistic parameter is explained within the CF theory. The next question is how the asymptotic value is reached as the distance between two CFQP's is increased. In Fig. 6, particularly for $\nu = 1/3$, we can see that $\widetilde{\theta}^*$ approaches its asymptotic value very slowly even for $d \gtrsim 10l$. Is that slow convergence real or only a result of the fact that the actual position of the CFQP has slight corrections? Should the slow convergence persist for θ^*, which would cast doubt on the usefulness of the concept of fractional statistics.

To examine the origin of such long tail, we consider in more detail two composite fermions in the lowest CF-quasi-LL. For ^{2}CF ($p = 1$), we can explicitly calculate the statistics parameter in Eq. (36) through the use of the wave function in Eq. (43), leading to

$$\widetilde{\theta}^* = \frac{-\frac{R^2}{2l^2}\left[\frac{4R^2}{l^2} + 32 + e^{-R^2/2l^2}\left(\frac{R^4}{l^4} - \frac{20R^2}{l^2} + 64\right)\right]}{\left[\frac{R^4}{l^4} + \frac{16R^2}{l^2} + 32 - e^{-R^2/2l^2}\left(\frac{R^4}{l^4} - \frac{16R^2}{l^2} + 32\right)\right]}. \tag{49}$$

In the limit of $R \gg l$, $\widetilde{\theta}^*$ reduces to

$$\widetilde{\theta}^* = -2 + 16\left(\frac{l}{R}\right)^2 + \mathcal{O}\left(\frac{l}{R}\right)^4. \tag{50}$$

As observed for $\nu = 1/3$, we find that the deviation of $\widetilde{\theta}^*$ from the asymptotic value decays only algebraically. The density profile for two ^{2}CF's on the lowest CF-quasi-LL is straightforwardly computed to be

$$\rho^{\eta,0}(x) \propto e^{-(x-R)^2/2l^2}(x^4 + 8x^2 + 8)$$
$$- 2e^{-(R^2-Rx+x^2)/2l^2}[x^2(x-R)^2 + 8x(x-R) + 8]$$
$$+ e^{-x^2/2}[(x-R)^4 + 8(x-R)^2 + 8] \qquad (51)$$

along the x axis, with the outer composite fermion intended to be located at $(R,0)$. As discussed in the previous section, the actual position R' of the outer composite fermion is given by

$$\frac{R'}{l} = \frac{R}{l} + 4\left(\frac{l}{R}\right) - 32\left(\frac{l}{R}\right)^3 + \mathcal{O}\left(\frac{l}{R}\right)^5 \equiv R + \Delta R. \quad (52)$$

At the same time, the inner composite fermion also shifts to $(R'',0) = (-\Delta R, 0)$. The Berry phase (divided by 2π) acquired due to the position shift of the composite fermions is

$$\Delta \theta^* = -\frac{B\Delta A}{\phi_0} = -\frac{1}{2\pi l^2}[\pi R'^2 + \pi R''^2 - \pi R^2]$$
$$= -4 + 16\left(\frac{l}{R}\right)^2 + \mathcal{O}\left(\frac{l}{R}\right)^4. \qquad (53)$$

The real statistical parameter $\theta^* = \tilde{\theta}^* - \Delta\theta^*$ is given by

$$\theta^* = 2 + \mathcal{O}\left(\frac{l}{R}\right)^4, \qquad (54)$$

with the $+\mathcal{O}(l/R)^2$ term canceling out. Thus, the power law tail in the difference between the CF value $\theta^* = 2$ and the microscopic value $\tilde{\theta}^*$ in Eq. (49) is not real, but caused by a shift in the positions of the CFQP's.

If the same argument holds for nonzero n and $p=1$, the additional Berry phase (divided by 2π) due to the position shift can be written as

$$\Delta\theta^* = -\frac{4}{2n+1} + \frac{16}{2n+1}\left(\frac{l}{d}\right)^2 + \mathcal{O}\left(\frac{l}{d}\right)^4 \qquad (55)$$

through the use of the effective magnetic field $B^*/B = 1/(2n+1)$. Adding the asymptotic value $\theta^* = 2/(2n+1)$ gives

$$\tilde{\theta}^* = -\frac{2}{2n+1} + \frac{16}{2n+1}\left(\frac{l}{d}\right)^2 + \mathcal{O}\left(\frac{l}{d}\right)^4. \qquad (56)$$

This heuristic prediction of the CF theory is plotted in Fig. 7 (dashed line) and agrees well with the long tail of the numerical behavior for large d/l.

C. Two nearby CF quasiparticles

We now turn to the situation when the two CFQP's are located very close to one another. When the distance becomes comparable to the size of the CFQP's, it is not possible to define the distance between the CFQP's in a meaningful manner, so we will consider here the dependence of fractional statistics on $d = |\eta - \eta'|$, which is a parameter entering the wave function.

The microscopic value $\tilde{\theta}^*$ for small d, as shown in Fig. 8, exhibits significant deviation from its asymptotic value. For
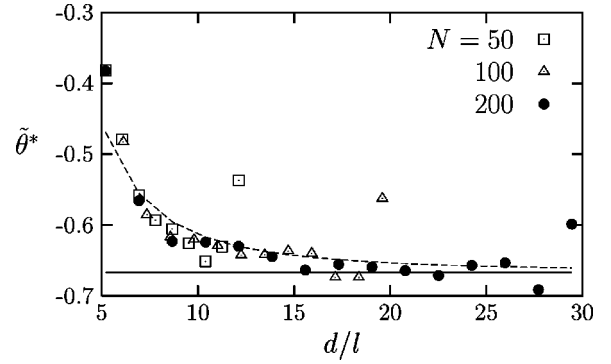

FIG. 7. The statistical angle $\tilde{\theta}^*$ for the CFQP's at $\nu = 1/3$ for large $d \equiv |\eta - \eta'|$. Here N is the total number of composite fermions, and l is the magnetic length. The dashed line is Eq. (56), which is in good agreement with the long tail of the numerical data. The points near the edge deviate significantly from the dashed line.

very small d it grows monotonically from -1 before undergoing a crossover to the asymptotic value. To gain insight into this behavior, we again resort to CF's in an otherwise empty lowest CF-quasi-LL. In the limit of $R \ll l$, Eq. (49) reduces to

$$\tilde{\theta}^* = -1 + \frac{1}{4}\left(\frac{R}{l}\right)^2 + \mathcal{O}\left(\frac{R}{l}\right)^4, \qquad (57)$$

showing a quadratic increase from -1. Similar behavior is displayed by two electrons in the second Landau level for small separations. For $\nu = 1$, the statistics parameter for electrons separated by a distance d is given by

$$\tilde{\theta}^* = -\frac{(d/l)^2}{2(1 - e^{-d^2/2l^2})}. \qquad (58)$$
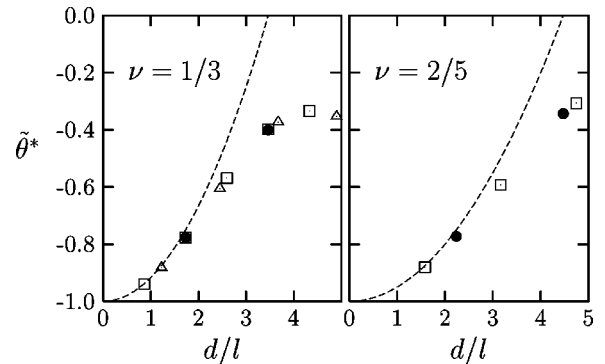
This yields in the limit $d \ll l$

FIG. 8. The statistical angle $\tilde{\theta}^*$ for the CFQP's at $\nu = 1/3$ (left panel) and $\nu = 2/5$ (right panel) for small $d \equiv |\eta - \eta'|$. The symbols are the same as in Fig. 7, and l is the magnetic length. The heuristic formula in Eq. (60) (dashed lines) agrees well with the actual result for small d.

$$\widetilde{\theta}^* = -1 + \frac{1}{4}\left(\frac{d}{l}\right)^2 + \mathcal{O}\left(\frac{d}{l}\right)^4, \qquad (59)$$

which is identical to that for composite fermions in lowest CF-quasi-LL.

We can expect similar behavior for CFQP's in higher CF-quasi-LL's. The only difference is that they feel an effective magnetic field $B^* = B/(2pn+1)$, which changes the length scale from l to l^*. It is expected that for small d, $\widetilde{\theta}^*$ is given by

$$\widetilde{\theta}^* = -1 + \frac{1}{4(2pn+1)}\left(\frac{R}{l}\right)^2 + \mathcal{O}\left(\frac{R}{l}\right)^4. \qquad (60)$$

Figure 8 presents the calculated $\widetilde{\theta}^*$ for small separation at $\nu = 1/3$ and 2/5 along with the heuristic expression of Eq. (60). As can be seen in Fig. 8, Eq. (60) gives a good account of the behavior at both fillings.

D. CF quasiparticles in different CF quasi-Landau levels

In this section we investigate another interesting question: What is the relative statistics for two CFQP's in different CF-quasi-Landau levels? This corresponds to the situation when a CFQP is inserted into an excited CF-quasi-Landau level. From the CF point of view, the statistics is related to the excess charge due to the presence of the additional CFQP as shown in Eq. (34). Since the local charge of the CFQP is independent of the quasi-Landau level to which it belongs, the resulting statistics is expected to be the same as that when both CFQP's are in the same CF quasi-Landau levels.

For an explicit calculation, we investigate the situation that a CFQP in the second CF-quasi-Landau level goes around a CFQP in the third CF-quasi-Landau level at the filling for $\nu = 1/3$. The wave function for two CFQP's is given by

$$\Psi_{1/3}^{\eta,\eta'} = P \begin{vmatrix} \phi_\eta^{(1)}(\mathbf{r}_1) & \phi_\eta^{(1)}(\mathbf{r}_2) & \cdots \\ \phi_{\eta'}^{(2)}(\mathbf{r}_1) & \phi_{\eta'}^{(2)}(\mathbf{r}_2) & \cdots \\ 1 & 1 & \cdots \\ z_1 & z_2 & \cdots \\ . & . & \cdots \\ . & . & \cdots \\ z_1^{N-3} & z_2^{N-3} & \cdots \end{vmatrix}$$

$$\times \prod_{i<k=1}^{N} (z_i - z_k)^2 \exp\left(-\sum_j |z_j|^2/4l^2\right). \qquad (61)$$

For simplicity, we set $\eta' = 0$.

Figure 9 demonstrates that the asymptotic value of the relative statistics of two CFQP's in two different CF-quasi-Landau levels is the same as for those in the same CF-quasi-Landau level. On the other hand, there is significant difference for small separations between two CFQP's. The behavior at small separations is believed to be sensitive to the local structure of each CFQP, because corrections to the statistics are caused by their overlap.

Jeon and Jain[18] noted that for two quasiparticles at the origin there is a qualitative difference between the wave
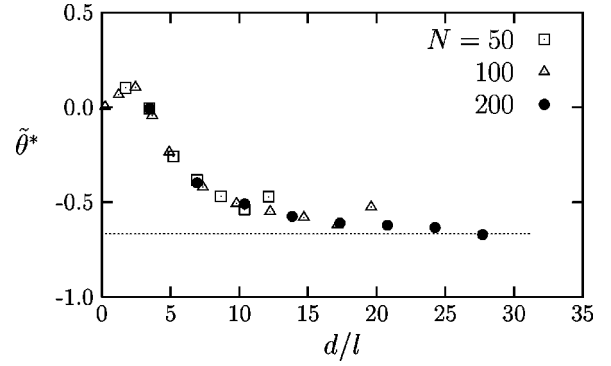

FIG. 9. The statistical angle $\widetilde{\theta}^*$ for the CFQP's in different CF-quasi-LL's at the filling $\nu = 1/3$ as a function of $d \equiv |\eta - \eta'|$. The CFQP at the origin is in the third CF-quasi-LL while the CFQP traversing a closed loop is in the second CF-quasi-LL. Here N is the total number of composite fermions, and l is the magnetic length.

functions constructed according to Laughlin's ansatz and the one used above based on the CF theory at $\nu = 1/3$. For CFQP's there are many candidates for two-quasiparticle states. The CF $[N-2,2]$ with both quasiparticles in the second CF-quasi-LL has lowest energy among the candidates as expected from the fact that it has the lowest effective cyclotron energy. As discussed in Ref. 18, Laughlin's wave function for two quasiparticles is more akin to the $[N-2,1,1]$ state of composite fermions, with one CFQP in the second CF-quasi-Landau level and the other in the third; both states have the same total angular momentum and their density profiles look alike. One might therefore have expected that the $[N-2,1,1]$ state would not display definite statistics. However, our result above demonstrates that even the $[N-2,1,1]$ state is fundamentally different from the one in Eq. (37).

E. Composite fermions: Fermions or anyons?

The fractional statistics of the CFQP's ought not to be confused with the fermionic statistics of composite fermions. The wave functions of composite fermions are single valued and antisymmetric under particle exchange; the fermionic statistics of composite fermions has been firmly established through a variety of facts, including the observation of the Fermi sea of composite fermions, the observation of FQHE at fillings that correspond to the IQHE of composite fermions, and also by the fact that the low-energy spectra in exact calculations on finite systems have a one-to-one correspondence with those of weakly interacting fermions.[9] The appearance of fractional statistics may seem at odds with the fermionic nature of composite fermions, but there is no contradiction. After all, any fractional statistics in nature *must* arise in a theory of particles that are either fermions or bosons when an "effective" description is sought in terms of a small number of collective degrees of freedom. The fractional statistics appears in the CF theory when all of the original particles at $\{z\}$ are integrated out (or treated in an average, mean-field sense) to formulate an effective description in terms of the few CFQP's at $\{\eta\}$. If we work with *all*

composite fermions, then Eq. (6) is sufficient.

F. Constraints on possible observation of fractional statistics

There are features that complicate a possible observation of fractional statistics. (i) The CFQP's are not *ideal* anyons. As seen in our calculations, the fractional statistics is sharply defined only asymptotically; in general there are corrections to it. Substantial deviation of θ^* from its asymptotic value is seen at separations of up to 10 magnetic lengths. Therefore, a measurement of θ^* must ensure that there is no overlap between the CFQP's at any time. One might expect that the interaction between the CFQP's will be repulsive which will automatically ensure that they do not come very close to one another. That turns out not to be the case, however. The interaction between the CFQP's is very weak and often *attractive*.[27] (ii) There is another important aspect through which the situation here differs from that for ideal anyons. For two ideal anyons, the Berry phase is zero for paths with zero winding number and $2\pi\theta^*$ for paths with unit winding. One therefore only needs to measure the Berry phase for a path that encircles another particle. In the case of the FQHE, on the other hand, the fractional statistics, itself an O(1) quantity, arises as a difference between two O(N) Berry phases, where N is the number of particles enclosed by the closed trajectory. For the reason listed in (i), N must necessarily be quite large. A precise measurement of the difference therefore requires an almost perfect control over the trajectory. Fluctuations in the trajectory on the order of the size of the CFQP will produce O($\sqrt{N}$) fluctuations in each Berry phase which will completely obscure the O(1) difference. (Our calculation actually provides an example where an im-

measurable error in the trajectory produces a finite correction to θ^*, changing its sign.) In fact, one may ask how quantum fluctuations in each O(N) quantity affect the O(1) difference and whether the O(1) difference can be defined in a rigorous manner.[28] (In this context, it is noted that the effective magnetic field is related to the total Berry phase associated with a path, an order-N quantity, and therefore robust to quantum mechanical fluctuations which are of smaller order.) (iii) There are many other features likely to be present in a real experimental situation that would be inimical to an observation of fractional statistics—for example, disorder and finite temperature, both of which generate particle-hole pairs which would provide a correction. (iv) The current flows at the edge of an incompressible FQHE system, where the fractional statistics is not well defined due to the absence of a gap.[30] This creates a problem for a detection of fractional statistics in a transport experiment. (v) It is not known how robust the fractional statistics concept is to perturbations. We have confirmed it for *noninteracting* composite fermions. However, it has been found that interactions between composite fermions can produce significant corrections to apparently topological quantities.[31] Also, the fact that certain quasiparticle wave functions at $\nu=1/m$ do not produce a sharp fractional statistics shows that it is not as robust as the fractional charge or the effective magnetic field. Whether it survives a more realistic calculation remains to be tested.

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[28] Such subtleties appear also in the definition of fractional charge. There is no fundamental particle with a fractional charge; the fractional charge is the difference between the net charge in a large enough area with or without a CFQP in it. In other words, it is also an order-1 difference between two large numbers. Will quantum fluctuations in the net charge, which are large compared to unity for a large area, make the definition of a fractional charge conceptually meaningless? Here, a possible way out (Ref. 29) is to consider an area enclosing the CFQP which is small enough that the fluctuations are small compared to the difference. Any experiment seeking to measure the fractional charge must consider such issues carefully.

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