Split-ring resonators and localized modes

A. B. Movchan and S. Guenneau
Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, United Kingdom and Physics Department, Imperial College, Prince Consort Road, London SW7 2AZ, United Kingdom
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The paper presents asymptotic analysis of an eigenvalue problem for the Helmholtz operator in a periodic structure involving split-ring resonators [originally proposed in J. B. Pendry *et al.*, IEEE Trans. Microwave Theory Tech. **47**, 2075 (1999)]. The eigensolutions are sought in the form of Bloch waves. The main emphasis is given to the study of localized modes within such a structure and to the control of low-frequency bandgaps on the corresponding dispersion diagram. Asymptotic results are explicit and are in good agreement with numerical simulations.

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I. INTRODUCTION

During recent years the models involving split-ring resonators (SRRs) and their applications in the design of left-handed materials (LHM) were the subjects of publications and intensive discussions in Physical Review Letters.[1–4] The original concept was proposed by Pendry and his colleagues in Ref. 5. In the electromagnetic theory, materials with optical resonances are interpreted in the sense of negative effective dielectric permittivity for certain frequency intervals (photonic bandgap frequencies), as described in the classical work.[6] The paper[5] presents analysis of a crystal consisting of SRRs, also referred to as "double C" resonators because of their shape. An effective medium, associated with an array of such resonators, shows a negative effective permeability region (photonic bandgap region) in a neighborhood of the resonance frequency. In models of electromagnetism, composite structures that possess simultaneously effective negative electric permittivity and effective negative magnetic permeability are studied in Refs. 7–9.

Homogenization theory for arrays of infinitely conducting parallel fibers was first proposed in Ref. 10 and subsequently extended in Refs. 11 and 12. The SRRs were used further in the homogenization analysis in Ref. 13 and LHM simulations in Ref. 2. This last paper generated interesting discussion[3,4] involving analysis of anisotropy of the composite structure with SRRs. Experimentally, it was confirmed[1] that SRR composite structures possessing negative effective refractive index can be successfully used for focusing a signal generated by a point source. Most recently, SSRs were shown to possess high-frequency magnetic response, providing an artificial magnetic device composed of nonmagnetic conductive resonant elements.[14] Also of interest is the analytic model based on Bessel expansions proposed in Ref. 15 for infinitely thin, infinitely conducting circular split rings with *ad libitum* number of cuts.

One of important issues consists of analysis of localized eigenmodes existing within SRRs, and the present article shows that such analysis allows for an explicit analytical treatment by an asymptotic method. It is also possible to introduce a new physical interpretation associated with problems of continuum mechanics for sonic crystals that exhibit spectral gaps within a certain range of frequencies. For those familiar with the terminology of elasticity theory it will also be clear that our formulation is equivalent to a scalar antiplane shear problem. The composite structures considered here include locally resonant elements, and corresponding effective dynamic media may possess complex effective elastic moduli within a certain frequency range (stop band frequencies), as outlined in Ref. 16. In terms of mathematical modeling, this work is also related to a rigorous analysis[17] of boundary value problems in disintegrating domains.

By varying the size, material parameters, and geometry of structural elements, including SRRs, we can control the values of stop band frequencies. This enables us to design microstructured materials shielding acoustic signals and antiplane shear waves. Further extension to full vector problems provides applications in the design of seismic wave filters.

II. SPECTRAL PROPERTIES OF A DOUBLY PERIODIC ARRAY OF SRRs

First, we present illustrative numerical results for a spectral problem for the Helmholtz operator within a doubly periodic square array of SRRs. Homogeneous Neumann boundary conditions are prescribed on the contour of each resonator and the standard Floquet-Bloch conditions are set on the boundary of an elementary cell of the periodic structure. The full mathematical formulation is given in Sec. III, and the composite structure is shown in Fig. 3.

A finite element program was written to compute the eigenvalues and to generate the corresponding eigenfields. In Fig. 1 we present the dispersion diagrams for eigenfrequencies ω as functions of the Floquet-Bloch parameter k. Along the horizontal axis we have the values of modulus of $\mathbf{k}$, where $\mathbf{k}$ stands for the position vector of a point on a triangular contour ΓMK within the irreducible Brillouin zone. The negative values on the horizontal axis correspond to $-|\mathbf{k}|$ as we approach the origin along the $K\Gamma$ direction. Figure 1(a) is constructed for a doubly periodic array of circular voids, whereas Fig. 1(b) displays a dispersion diagram for a doubly periodic array of SRRs. Deliberately in Fig. 1(b), we consider the range of frequencies that covers the first bandgap of the dispersion diagram of Fig. 1(a). We show that the presence of SRRs gives additional low-frequency bandgaps and additional eigenmodes associated with the localized standing waves. These eigenmodes will be described in the frame of

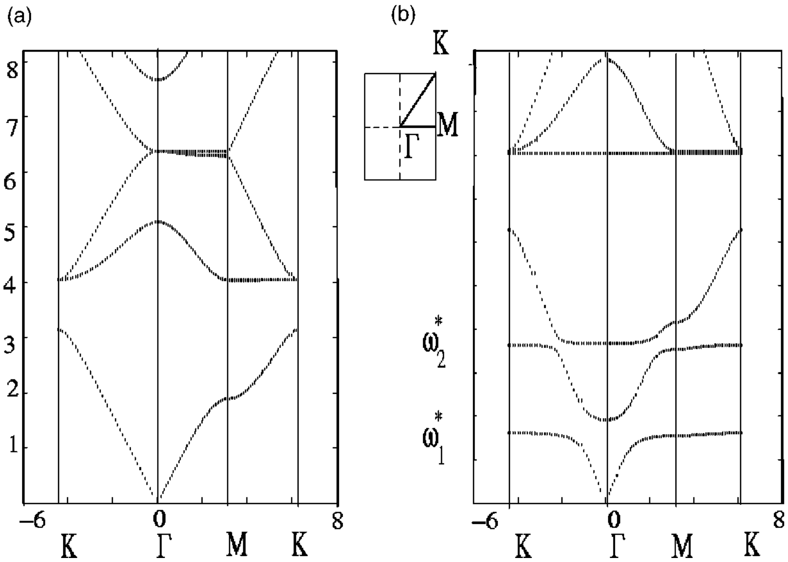

FIG. 1. Bandgap diagram for a doubly periodic array representing the normalized radian frequency ω versus $|\mathbf{k}|$: (a) The case of an array of circular voids. (b) The case of an array of double C defects.

the asymptotic model of the text below. The numerical values for the frequencies of the localized standing waves are $\omega_1^*=0.806$ and $\omega_2^*=1.817$ and these values are consistent with those predicted by the asymptotic model of Sec. V.

The eigenfunctions corresponding to these eigenfrequencies are shown in Fig. 2: Fig. 2(a) contains the contour line diagram corresponding to the eigenfunction associated to ω_1^*, whereas Fig. 2(b) shows the eigenfunction corresponding to ω_2^*. As shown in Fig. 2, the interior of the SRR consists of a ring and a disk-shaped body connected together by a thin ligament. As predicted by the asymptotic model [Sec. V (iii)], the first eigenfunction [Fig. 2(a)] describes a vibration of the whole set (the ring plus the disk-shaped body in the middle), as a rigid solid, connected by a thin ligament to a fixed rigid foundation. The second eigenfunction [Fig. 2(b)], corresponding to ω_2^*, describes vibration of the central disk as a rigid solid connected by a thin ligament to a fixed rigid foundation.

III. FORMULATION OF THE SPECTRAL PROBLEM

We consider a doubly periodic array of "defects" represented by a cloud of voids (see Fig. 3) $\Omega_1, \Omega_2, \ldots, \Omega_n$ embedded in an elementary cell $Y=]0;1[\times]0;1[$. Let $u(x,y)$ satisfy the Helmholtz equation

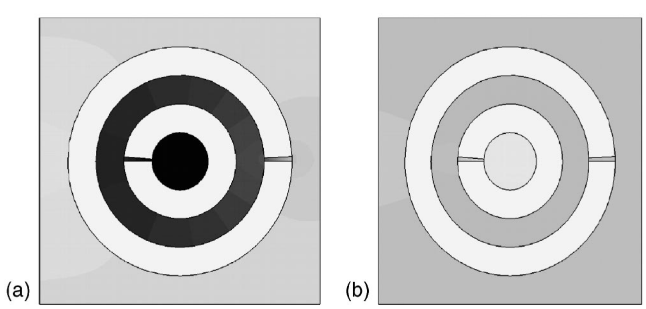

FIG. 2. (a) The eigenfunction corresponding to the eigenfrequency $\omega_1^*=0.806$. (b) The eigenfunction corresponding to the eigenfrequency $\omega_2^*=1.817$.

$$\Delta u + \frac{\rho\omega^2}{\mu} u = 0 \tag{1}$$

in $Y\setminus\cup_j\overline{\Omega_j}$. Here, u represents an amplitude of an out-of-plane time-harmonic displacement within an elastic medium with the mass density ρ and the shear modulus μ; ω stands for the radian frequency of vibration. We also assume that u satisfies homogeneous Neumann boundary conditions on the contours of voids

$$\frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega_j, j = 1, \ldots, n \tag{2}$$

and the Floquet-Bloch condition

$$u(\mathbf{x} + m e^1 + n e^2) = u(\mathbf{x}) e^{i(k_1 m + k_2 n)} \tag{3}$$

within the doubly periodic array. Here m and n are integers and k_1, k_2 represent components of the Floquet-Bloch vector $\mathbf{k}$; $\mathbf{e}^j$ are the unit basis vectors of the Cartesian coordinate system in $\mathbb{R}^2$.

In the sequel, we normalize the wave velocity $v=\sqrt{\mu/\rho}$ to 1. This leads to a dimensionless normalized radian frequency $\omega d/v$, where $d=1$ denotes the pitch of the array. In the same spirit, we normalize the Bloch vector to $\mathbf{k}d$.

We would like to consider a particular important case when the voids $\Omega_1, \ldots, \Omega_N$ are of the shape of the letter C and they are located with respect to each other, as shown in Fig. 3. Formally,

$$\Omega_j = \{a_j < |x| < b_j\} \setminus \overline{\Pi_\varepsilon^{(j)}}, \tag{4}$$

where a_j and b_j are given constants and $\Pi_\varepsilon^{(j)}$ is a thin ligament between the "ends of the letter C" [see formula (7)].

IV. A MULTISTRUCTURE

Let M be a multistructure defined as follows:

$$M = \bigcup_{j=1}^{N} \overline{\{R_j \cup \Pi_\varepsilon^{(j)}\} \cup D_N}, \tag{5}$$

where R_j are concentric rings

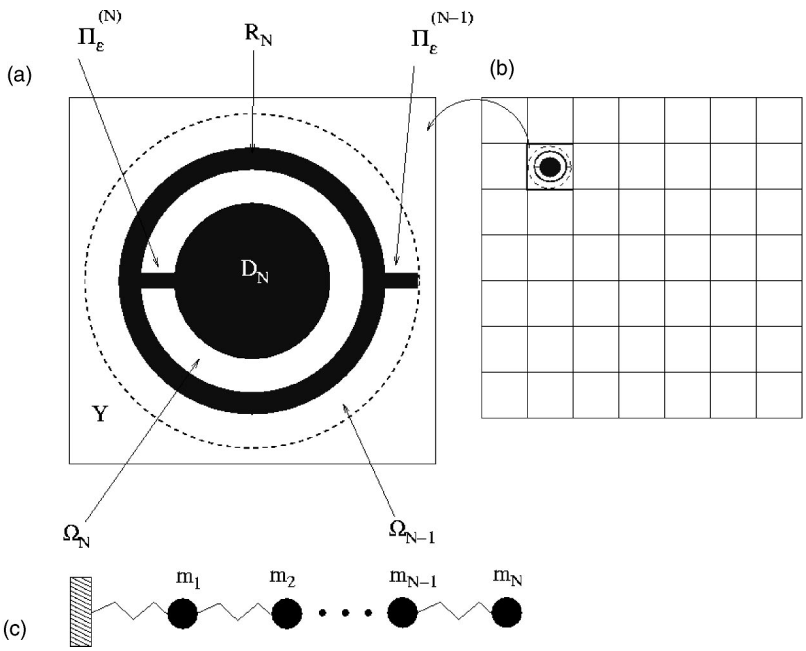

FIG. 3. The microstructure. (a) An elementary cell containing a SRR. (b) A doubly periodic array of defects. (c) A SRR is equivalent to a mechanical system of masses connected by soft elastic springs.

$$R_j = \{x{:}b_j < |x| < a_{j-1}\} \tag{6}$$

and D_N is the interior of a "ring" formed by $\overline{\Omega_N \cup \Pi_\varepsilon^{(N)}}$.

For certain range of frequencies, this multistructure will behave in a way similar to a set of masses connected by soft elastic springs, as shown in Fig. 3(c).

V. ASYMPTOTIC APPROXIMATION

Here we show that the multistructure M can be described within the discrete lattice approximation where evaluation of eigenvalues is a straightforward task.

(i) *Discrete lattice*. This approximation allows us to gain a physical insight on vibration modes for low frequencies. We know that the multistructure introduced here consists of several bodies R_j connected by thin ligaments $\Pi_\varepsilon^{(j)}$. As shown in Ref. 18 for sufficiently thin $\Pi_\varepsilon^{(j)}$, the low eigenfrequencies are close to those associated with the set of masses connected by harmonic springs (see Fig. 3).

The frequencies corresponding to standing modes of such a discrete structure can be evaluated as solutions of a certain polynomial equation. In particular, for a double C device proposed by Pendry,[5] the discrete analogue of the multistructure consists of two rigid bodies separated by a "thin neck."

(ii) *Thin ligaments*. In this section we derive an asymptotic approximation of the field within thin ligaments $\Pi_\varepsilon^{(j)}$. For the sake of simplicity we omit the superscript j. Let

$$\Pi_\varepsilon = \{(x,y){:}a < x < b, \langle y\rangle < \varepsilon h/2\}, \tag{7}$$

where ε is a small nondimensional parameter, and a, b, and h are given constants. Assume that the function $v(x,y,t)$ satisfies the wave equation within Π_ε

$$\mu \nabla^2 v(x,y) - \rho \frac{\partial^2}{\partial t^2} v(x,y) = 0 \tag{8}$$

together with the homogeneous Neumann boundary conditions

$$\left.\frac{\partial v}{\partial y}\right|_{y=\pm\varepsilon h/2} = 0. \tag{9}$$

This system can be supplied with appropriate boundary conditions at the ends $x=a$ and $x=b$ (for the moment these conditions are not important). Let us introduce the scaled variable

$$\xi = \frac{y}{\varepsilon}, \tag{10}$$

so that $\xi \in (-h/2, h/2)$ within Π_ε, and $\partial^2 v/\partial y^2 = (1/\varepsilon^2) \times (\partial^2 v/\partial \xi^2)$. The rescaled wave equation in Π_1 is

$$\left\{\mu\left(\frac{1}{\varepsilon^2}\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial x^2}\right) - \rho \frac{\partial^2}{\partial t^2}\right\} v = 0. \tag{11}$$

The field v is approximated in the form

$$v \sim v^{(0)}(x,y,t) + \varepsilon^2 v^{(1)}(x,y,t). \tag{12}$$

To leading order we obtain [see (9) and (11)]

$$\frac{\partial^2 v^{(0)}}{\partial \xi^2} = 0, \; |\xi| < h/2, \tag{13}$$

$$\left.\frac{\partial v^{(0)}}{\partial \xi}\right|_{\xi=\pm h/2} = 0. \tag{14}$$

Hence, $v^{(0)} = v^{(0)}(x,t)$ (it is ξ-independent). Assuming that $v^{(0)}$ is given, we derive that the function $v^{(1)}$ satisfies the following model problem on the scaled cross section of Π_1

$$\mu \frac{\partial^2 v^{(1)}}{\partial \xi^2} = -\mu \frac{\partial^2 v^{(0)}}{\partial x^2} + \rho \frac{\partial^2 v^{(0)}}{\partial t^2}, \quad |\xi| < h/2,$$

$$\left.\frac{\partial v^{(1)}}{\partial \xi}\right|_{\xi=\pm h/2} = 0. \tag{15}$$

The condition of solvability for the problem (15) has the form

$$\mu \frac{\partial^2 v^{(0)}}{\partial x^2} - \rho \frac{\partial^2 v^{(0)}}{\partial t^2} = 0, \quad a < x < b. \tag{16}$$

Hence, we have shown that to the leading order we can approximate the field v within the thin bridge Π_ε by the function $v^{(0)}$, which satisfies the wave equation in one-space dimension. In particular if $v^{(0)}$ is time-harmonic $[v^{(0)}=u^{(0)}\times(x)e^{i\omega t}]$ then the amplitude $u^{(0)}$ satisfies the ordinary differential equation

$$\mu \frac{d^2 u^{(0)}}{dx^2} + \omega^2 \rho u^{(0)} = 0, \quad a < x < b, \tag{17}$$

and therefore

$$u^{(0)}(x) = A_1 \cos\left(\frac{\omega}{c}x\right) + A_2 \sin\left(\frac{\omega}{c}x\right), \tag{18}$$

where $c=\sqrt{\mu/\rho}$.

(iii) *Double C*. We would like to consider a particular example when the multistructure is associated to the set of just two voids Ω_1 and Ω_2 as shown in Fig. 3(a). This geometry is described in the earlier work by Pendry *et al.*[5] In accordance with the asymptotic algorithm presented in Ref. 18, we propose an approximation for eigenfrequencies of two standing modes within the doubly periodic array described above. These eigenmodes correspond to the first eigenfrequencies of the following structures:

(1) The thin ligament $\Pi_\varepsilon^{(1)}$ connected to a rigid body $\Xi_1 = D_2 \cup \Pi_\varepsilon^{(2)} \cup R_1$ at one end; the other end of the thin ligament is fixed.

(2) The thin ligament $\Pi_\varepsilon^{(2)}$ connected to a rigid body $\Xi_2 = D_2$ at one end; the other end of the thin ligament is fixed.

The eigensolutions V_j corresponding to the vibrations of the model domains described above satisfy the following problems:

$$\mu V_j''(x) + \rho\omega^2 V_j(x) = 0, 0 < x < l_j, \tag{19}$$

$$V(0) = 0, \tag{20}$$

$$\mu\varepsilon h_j V'_{|x=l_j} = M_j \omega^2 V(l_j), \tag{21}$$

where εh_j and l_j are the thickness and the length of the thin ligament $\Pi_\varepsilon^{(j)}$, and M_j is the mass of the body Ξ_j. The solution of the problem (19)–(21) has the form

$$V_j(x) = A_j \sin\left(\frac{\omega}{c}x\right), \tag{22}$$

where $c=\sqrt{\mu/\rho}$ and the frequency ω is given as the solution of the following equation:

$$\varepsilon h_j \cot\left(\frac{\omega l_j}{c}\right) = \frac{M_j c}{\mu}\omega. \tag{23}$$

Looking at a first low frequency

$$\omega_j = \sqrt{\varepsilon}\Lambda_j, \tag{24}$$

we deduce an explicit asymptotic approximation

$$\Lambda_j^2 = \frac{h_j \mu}{l_j M_j}, \tag{25}$$

and hence

$$\omega_j \sim \sqrt{\frac{\varepsilon h_j}{l_j}}\sqrt{\frac{\mu}{M_j}}. \tag{26}$$

(iv) *A numerical estimate for the eigenfrequencies associated with the localized modes*. In the numerical example discussed in the sequel $\varepsilon h_1=0.012\,953\,4$, $\varepsilon h_2=0.010\,375\,2$, $l_1=l_2=0.1$, $M_2=\pi r_2^2$, $M_1=\pi[r_2^2+(b_1^2-a_1^2)]+\varepsilon h_2 l_2$, where r_2 is the radius of the disk D_2, and a_1, b_1 are the interior and exterior radii, respectively, of the ring R_1. In our case, $r_2=0.1$, $a_1=0.2$, and $b_1=0.3$, and hence $M_1 \sim 0.189\,533\,1$ and $M_2 \sim 0.031\,415\,9$. The formula (26) gives the following values for the first eigenfrequencies of the multistructures $\Pi_\varepsilon^{(1)} \cup \Xi_1$ and $\Pi_\varepsilon^{(2)} \cup \Xi_2$:

$$\omega_1 \sim 0.826\,702\,7, \;\; \omega_2 \sim 1.817\,286\,1. \tag{27}$$

The corresponding frequencies associated with the standing waves in the doubly periodic structure [see Fig. 3(b)] were obtained numerically, and they are $\omega_1^*=0.806$ and $\omega_2^*=1.817$. Formulas (26) and (27) give an excellent estimate for the eigenfrequency ω_2^*. However, we observe a discrepancy in the approximation of ω_1^*. The estimate for the eigenfrequency ω_1^* can be improved if the multistructure $\Pi_\varepsilon^{(1)} \cup \Xi_1$ (where Ξ_1 is regarded as a rigid body) is replaced by the multistructure $\Pi_\varepsilon^{(1)} \cup R_1 \cup \Pi_\varepsilon^{(2)} \cup D_2$ (where R_1 and D_2 are treated as rigid bodies connected by a thin ligament $\Pi_\varepsilon^{(2)}$). In this case the eigenfrequency ω_1 is approximated by the first positive eigenvalue of the problem

$$\mu V_1''(x) + \rho\omega^2 V_1(x) = 0, 0 < x < l_1, \tag{28}$$

$$V_1(0) = 0, \tag{29}$$

$$\mu\varepsilon h_1 V_1'(l_1) - \mu\varepsilon h_2 V_2'(0) = m_1 \omega^2 V_1(l_1), \tag{30}$$

$$\mu V_2''(x) + \rho\omega^2 V_2(x) = 0, 0 < x < l_2, \tag{31}$$

$$\mu\varepsilon h_2 V_2'(l_2) = m_2 \omega^2 V_2(l_2), \tag{32}$$

$$V_2(0) = V_1(l_1), \tag{33}$$

where $V_1(x)$, $V_2(x)$ are the eigenfunctions defined on $(0,l_1)$ and $(0,l_2)$, respectively, and the masses m_1, m_2 are defined by

$$m_1 = \pi(b_1^2 - a_1^2) = 0.157\,079\,6,$$

$$m_2 = M_2 = \pi r_2^2 = 0.031\,415\,9. \tag{34}$$

Taking into account that $\omega_1=O(\varepsilon)$ we deduce that it can be approximated as the first positive solution of the following algebraic equation:

$$m_1 m_2 l_1 l_2 \omega^4 - \varepsilon \mu \omega^2 \{l_1 h_2 m_2 + h_1 l_2 m_2 + h_2 m_1 l_1\} + \varepsilon^2 \mu^2 h_1 h_2 = 0, \quad (35)$$

so that $\omega_1 \sim 0.812\,249\,4$, which provides a reasonably accurate approximation of ω_1^*.

VI. CONCLUSIONS

Our paper addresses the issue of design of photonic bandgap structures. It is known in the literature[19] that periodic arrays of inclusions may be used to create filters, polarizers of electromagnetic (elastic) waves. Our paper outlines a method that can be used to control the stop bands in two-dimensional photonic/phononic bandgap structures.

Experimental observations reported in Ref. 16 show the presence of localized eigenstates for a composite elastic structure involving an array of spherical coated inclusions; the coating layers were sufficiently thin and the elastic material of the coating was much softer compared to the material of the host medium. As discussed in Ref. 16, the built-in localized resonances due to “defects” like coated spheres give rise to flat dispersion curves that are nearly **k**-independent. Analytical study of low frequency bandgaps and localized modes for arrays of coated inclusions is presented in Ref. 20.

In our paper we propose a more general multiscale configuration. It has been shown that one can use our method to “tune” the existing bandgaps on the dispersion diagram as well as to create new low-frequency bandgaps. The band diagram in Fig. 1(b) shows that some partial bandgaps (see the intervals *MK* on the horizontal axis) are fairly wide, which implies that such a microstructure can be used efficiently for the design of low-frequency photonic/phononic crystal waveguides.

As an example, we have presented the asymptotic estimates and numerical simulations for a microstructure involving a doubly periodic square array of SRRs.

Finally, we note that formula (26) brings the cornerstone towards a possible realization of acoustic subwavelength imaging via a heterogeneous slab consisting of Helmholtz resonators shaped as SSR. Focusing of elastic waves might be achieved through a slab of such a composite material in a way similar to the one discussed by Pendry.[21] This elastic composite would convey not only the propagating part of the acoustic signal, but also its evanescent part thanks to amplification that occurs through a resonant process. Indeed, an eigenmode will be excited at frequency given by (26) on the side of the slab remote from an object, and it is this resonance which provides the necessary amplification to the evanescent field to refocus it onto a perfect image.

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