# Superconductor-insulator transition driven by local dephasing

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We consider a system where localized bound electron pairs form an array of "Andreev"-like scattering centers coupled to a fermionic subsystem of uncorrelated electrons. By means of a path-integral approach, that describes the bound electron pairs within a coherent pseudospin representation, we derive the effective action for the collective phase modes that arise from the coupling between the two subsystems once the fermionic degrees of freedom are integrated out. This effective action has features of a quantum phase model in the presence of a Berry phase term and exhibits a coupling of phase fluctuations to those of the density of bound pairs and the amplitude of the fermion pairs. Due to the competition between the local and the hopping-induced nonlocal phase dynamics it is possible, by tuning the exchange coupling or the density of the bound pairs, to trigger a transition from a phase-ordered superconducting to a phase-disordered Mott insulating state. We discuss the different mechanisms that control the occurrence and eventual destruction of phase coherence both in the weak and strong coupling limit, restricting ourself to homogeneous phases.

## I. INTRODUCTION

The problem of interacting Cooper pairs and/or bosons, together with the possibility of a quantum control of long-range phase coherence in low-dimensional systems has received considerable interest since the experiments[1] on homogeneous lead and bismuth films, which exhibit a transition from a superconducting to an insulating phase as a function of thickness.

It is known, since the work of Abrahams *et al.*,[2] that no true metallic behavior can be expected for two-dimensional (2D) noninteracting electrons at $T=0$, because all the states will be localized by an arbitrarily small amount of disorder. When one includes Coulomb interaction, the situation is less clear, but the common belief is that a metallic phase should still not appear at $T=0$ in the presence of disorder — though no rigorous proof is available.[3] Yet, in the presence of attractive interactions, one expects a superconducting state both at $T=0$ and finite temperature, even in presence of a finite amount of disorder, due to its relative ineffectiveness for a transition of Kosterlitz-Thouless type (the degree of relevance being given by the Harris criterion[4]).

The existence of a superconducting state at $T=0$ in 2D systems is then considered to be directly linked to that of an insulating state, with no intermediate metallic phase present. Hence, one should be able to observe a direct superconductor-insulator transition (SIT) at zero temperature in 2D as a function of disorder, interaction strength, magnetic field, or any other external parameter that can drive the system away from the superconducting phase.

There are different theoretical scenarios that are commonly discussed in connection with the salient features of such quantum phase transitions: (i) dissipative models, considering a network of Josephson-coupled superconducting grains shunted by resistors,[5–10] where the competition between the Josephson intergrain coupling and the charging energy yields an increase of the phase fluctuations of the superconducting order parameter and hence leads to a phase-disordered state. Dissipation thereby plays the role of suppressing quantum phase fluctuations and thus competes with the charging mechanism. (ii) Bose-Hubbard models, where the superconducting phase is due to the Bose-Einstein condensation of charge-2e bosons and the insulating phase is due to a proliferation of vortices and localization of pairs.[11–14]

While the SIT was long thought to be a paradigm of the above theories, recent experiments have cast some doubt on that. They revealed what seems to be a low-temperature metallic state that is intertwined in such a SIT and thus requires a more appropriate theoretical description. In magnetic-field tuned experiments on Mo-Ge samples,[15] in granular superconductors (Ga films[16] and Pb films[17]), and in Josephson junction arrays[18] a metallic phase has been observed. Moreover, the recent experiments of Mason and Kapitulnik,[19] in which a metallic phase has been observed sandwiched between the superconductor and the insulating phase, suggest that perhaps two phase transitions accompany the loss of phase coherence in 2D superconductors: (i) a superconductor to a "Bose metal" and (ii) a Bose metal to an insulator. A Bose metal in this context, is thought of as just a gapless nonsuperfluid liquid with metalliclike transport[20].

These experimental results have led to reconsideration of the whole issue of the SIT. One recent proposal to handle such a new viewpoint of the SIT has been to reexamine the standard onsite charging model. By including nearest-neighbor charging terms it was shown that the resulting uniform Bose metal state lacks any trace of either phase or charge order,[21] due to a competition between the order parameters that describe the onset of charge order and phase coherence. A different point of view[22] is that in the quantum-disordered regime a cancellation happens between the exponentially long quasiparticle scattering time and the exponen-

tially small quasiparticle population, which ultimately leads to a finite dc conductivity. Finally, intrinsic as well as extrinsic sources of dissipation have been suggested as potentially relevant for the occurrence of a nonsuperfluid metallic phase in proximity to the superconducting phase.[19,23,24] In this context, the dissipation is a relevant perturbation[25] that, depending on the strength of the coupling to the dissipative source, can drive the system from a *superconductor-to-insulator transition* to a *superconductor-to-metal transition*.[19,26] Several other ideas supported a scenario where the system could break into superconducting and insulating "islands" weakly linked via percolating paths before going into a metallic phase, dominated by vortex dissipation.[26] This "puddle" scenario matches with the view to describe the 2D superconductor-metal transition via superconducting islands embedded in thin metal films.[27] Detailed analysis of the puddlelike model considered strongly fluctuating superconducting grains embedded in a metallic matrix,[28] predicting a *metal-to-superconductor transition* with a metallic phase just above the transition dominated by Andreev reflections between the superconducting grains. Finally, recent investigations directed the attention to new phases with Bose metallic features, where the dissipation is dynamically self-generated, as in the quantum-phase glass model, where disorder in the distribution of the tunneling amplitudes and quantum fluctuations destroy phase coherence.[20]

Along similar lines as those dealt with in the different approaches discussed in the literature, our aim here is to investigate a system where the breakdown of superconductivity situates itself in between the case of a "bosonic" and a "fermionic" mechanism for superconductivity suppression. We focus on determining the possible ground states for interacting Cooper pairs in close relation to the classical notions of superfluidity and localization of bosons and, on a more general basis, how the phase coherence can be tuned to a phase disordered state. More specifically, we consider within the framework of a boson-fermion model (BFM), a system where localized bound electron pairs (hard-core bosons) form an array of "Andreev"-like scattering centers coupled to a fermionic subsystem of itinerant uncorrelated electrons (quantum pair exchange). This scenario goes beyond that of pure phase models widely discussed in the literature. Here one is dealing with bosonic degrees of freedom (localized bound pairs) as well as fermionic ones (itinerant electrons) for which, due to the emergence of pair correlations, one has to account for both amplitude as well as phase modes. The possibility to tune from short- to long-range phase coherence arises from the following competing effects:

(i) on the one hand, the short-range interaction between bosons and fermion pairs (holes) induces a local phase locking in a configuration with a quantum superposition between bosons and electron pairs, leaving the common phase undetermined.

(ii) on the other hand, the itinerancy of the electrons tends to lock and rigidly extend in space these initially arbitrary local phases. As a result, phase coherence develops over longer distances by suppressing the quantum fluctuations of the local phase, which thus involves the dynamics of amplitude fluctuations.

Out of this competition one recovers either a superfluid state in the regime of a small scattering rate or a phase-

disordered state in the limit where the pair exchange dominates and the local quantum phase fluctuations do not allow for long-range phase coherence. As we shall show, the physics described by this scenario strongly depends on the concentration of fermions and bosons, on the coupling strength, as well as on temperature. The purpose of this study is to analyze how a transition from a superconductor to an insulator may occur in such a scenario.

The outline of the paper is as follows. In Sec. II we will introduce the BFM and describe its main qualitative features and the phenomena that can be described by it. In Sec. III, we shall develop a path-integral representation of this model and derive an effective coarse-grained action that, after integrating out the fermionic degrees of freedom, is able to describe the low-energy dynamics of the phase and amplitude modes. In Sec. IV we will discuss the phase diagram based on the simplest approximation of such a coarse-grained effective action in terms of a phase-only action and explore the transition from a superconductor to a phase-disordered state as a function of the coupling and the density of bosons. In Sec. V, we will study the physical features that arise from the intrinsic Berry phase term in our effective action and from the hard-core nature of the bosons, represented by quantum pseudospin $1/2$ variables. In Sec. VI, we compare the salient features of this BFM scenario with similar scenarios, such as the negative-U Hubbard model, the Bose-Hubbard model, and Josephson junction arrays. In the Discussion, Sec. VII, we review the main results obtained in this paper and indicate further developments planed for the near future.

## II. THE BOSON-FERMION MODEL

The boson-fermion model (BFM) has in recent years attracted considerable attention as a model capable of capturing basic physical properties in many body systems with strong interaction, giving rise to the formation of resonant pair states of bosonic nature inside a reservoir of fermions. Such a scenario was initially proposed by one of us (J.R.) as an alternative to the scenario of the hypothetical and yet to be experimentally verified *bipolaronic superconductivity*. It was meant to describe the intermediary coupling regime between the adiabatic and antiadiabatic limits in polaronic systems, where an exchange between localized bipolarons and pairs of uncorrelated electrons can be assumed to take place (for a recent intuitive justification of such a scenario see, for instance, Ref. 29.)

This boson-fermion scenario has, however, a much wider range of applicability than that for which it was initially proposed and seems to apply to very different physical situations, such as hole pairing in semiconductors,[30] isospin singlet pairing in nuclear matter,[31] d-wave hole and antiferromagnetic triplet pairing in the positive-U Hubbard model[32] (and possibly also in the t-J model), entangled atoms in squeezed states in molecular Bose Einstein condensation (BEC) in traps,[33] and superfluidity in ultracold fermi gases induced by a Feshbach resonance.[34]

The BFM is reminiscent of an anisotropic Kondo lattice model in terms of a pseudospin $1/2$ but characterizing localized electron pairs instead of localized impurity spins in the Kondo analog. The Hamiltonian is given by

$$H = (D - \mu) \sum_{i\sigma} c_{i\sigma}^+ c_{i\sigma} + (\Delta_B - 2\mu) \sum_i \left(\rho_i^z + \tfrac{1}{2}\right)$$

$$+ \sum_{i \neq j, \sigma} t_{ij} (c_{i\sigma}^+ c_{j\sigma} + \text{H.c.}) + g \sum_i (\rho_i^+ \tau_i^- + \rho_i^- \tau_i^+).$$

The pseudospin operators $[\rho_i^+, \rho_i^-, \rho_i^z]$ denote the local-bound electron pairs (bosons) and $[\tau_i^+ = c_{i\uparrow}^+ c_{i\downarrow}^+, \tau_i^- = c_{i\downarrow} c_{i\uparrow}, \tau_i^z = 1 - \tau_i^+ \tau_i^-]$, the itinerant pairs of uncorrelated electrons. $[c_{i\sigma}^+, c_{i\sigma}]$ stand for the creation and annihilation operators of the itinerant electrons (fermions) and $g$ is the strength of the *boson$\Leftrightarrow$fermion pair* exchange interaction. The hopping integral for the itinerant electrons, which is assumed to be different from zero only for nearest-neighbor sites, is given by $t$ with a band half-width equal to $D = zt$, $z$ denoting the coordination number of the underlying lattice. The energy level of the bound electron pairs is denoted by $\Delta_B$. The number of the ensemble of bosons and fermions being conserved, $n_{tot} = n_{F\uparrow} + n_{F\downarrow} + 2n_B$, implies a common chemical potential $\mu$ for both subsystems. $n_B$, $n_{F\uparrow,\downarrow}$ indicate the occupation number per site of the hard-core bosons and of the electrons with up- and down-spin states.

The exchange coupling between the bosons and the fermion pairs can be considered as an effective Andreev-like scattering leading to local states that are quantum superpositions of the form

$$|\psi_{loc}\rangle_i = \int d\phi_i [\cos(\theta_i/2)\cos(\phi_i)\rho_i^+ + \sin(\phi_i)][\cos(\phi_i)$$

$$+ \tau_i^+ \sin(\theta_i/2)\sin(\phi_i)]|0\rangle. \tag{1}$$

Such states evolve gradually out of the system of localized dephased bosons and essentially uncorrelated free fermions, which characterize the high-temperature phase of this model, when the temperature is decreased below a certain $T^* \simeq g$ where resonant pairing (not bound pairs!) starts to be induced in the fermionic subsystem. These pair states already have built-in features that are reminiscent of those that characterize Cooper pairing of fermions as well as superfluidity of bosons. The phases of the two coherent states, which correspond to the two subsystems, are the same and hence are locked together, but are averaged over all angles as a consequence of the conserved particle number on any given site, $n_{tot} = 2$. Roughly speaking, the ground state of the system is then given by a product state $\Pi_i |\psi_{loc}\rangle_i$ with $cos\ \theta_i = 1$ and that exhibits no phase correlations on any finite length scale. Let us next consider the effect of fermion hopping between adjacent sites. This will give rise to density fluctuations on each of those individual sites and thus help to stabilize an arbitrary but finite average value of the phases $\{\phi_i\}$ over finite length and time scales. In this way the localized bosons and fermion pairs acquire itinerancy[35–37] that eventually leads to a superfluid state in both subsystems,[38] provided that the effect of the local correlations between the bosons and the fermion pairs can be sufficiently diminished, but remaining still sufficiently strong to guarantee the formation of pairing in the fermionic subsystem. Achieved or not this situation will depend on the relative importance of the local exchange coupling versus the fermion hopping rate (given by the ratio

$g/t$), as well as on the concentration of the bosons as shown by exact diagonalization study[39] on finite-size clusters of this BFM.

What we shall attempt in the present study is to describe this physics in terms of an effective action for the phase and amplitude fields of the bosonic fields. In order to achieve this we shall put the discussion on a level that is more familiar, namely, that one of Josephson junction arrays and Bose-Hubbard models. For that purpose let us briefly sketch the analogy that exists (up to a certain point) between the BFM and those systems that have been widely discussed in the literature. A physically possible realization of this BFM scenario can be imagined in the form of a network of superconducting grains embedded in a metallic environment and where the only mechanism of interaction between the grains and the fermionic background is that of Andreev reflections. Via such a mechanism an electron (hole) is reflected on the grain as a hole (electron) leaving behind a surplus of two holes (electrons) in the fermionic subsystem and of two electrons (holes) in the grain. If the grains are such that they have a large charging energy, the fluctuations of the number of pairs on them are energetically unfavorable and hence are largely suppressed. We then have a situation where the state of the grains switches essentially between zero to double occupancy with respect to the average occupation, any time an electron (hole) is reflected at the interface of the grain. Thus, the quantum dynamics of the single grain can be directly represented by a pseudospin $\frac{1}{2}$, in order to account for the doublets that represent the two possible states of the grains.

For such a possible experimental setup, the effective sites in the BFM have to be considered as defining a regular array of grains and having the same periodicity as the underlying lattice on which the fermions move with a hopping amplitude $t$. Moreover the size of the grains has to be such that it is much smaller than the distance between them. A pictorial view of such an experimental setup is given in Fig. 1. The analogy between the BFM and the array of superconducting grains, which scatter pairs of fermions in a metallic matrix via Andreev-like reflections, may ultimately serve as an experimental device on which to test and analyze the theoretical issues that will be discussed in this paper.

## III. PATH-INTEGRAL REPRESENTATION: DERIVATION OF THE EFFECTIVE ACTION

### A. Generalities

Let us now construct an effective action, which describes the BFM, with the aim to extract the dynamical properties of the low-energy degrees of freedom of the phase and amplitude modes for the bosons and fermion pairs. We start by expressing the partition function in terms of a coherent-state path-integral representation,[40] where the fermionic part is formulated by means of the usual Grassmann variables and the bosonic part is described by a pseudospin-coherent state representation:[40,41]

$$Z = \prod_i \int D\theta_i D\phi_i D\bar{\Psi}_i D\Psi_i e^{-A[\bar{\Psi}_i, \Psi_i, \theta_i, \phi_i]}, \tag{2}$$

where

(a)

(b)
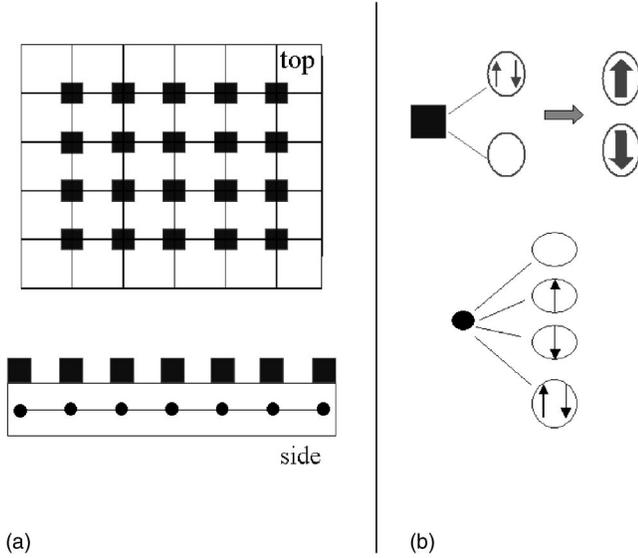
FIG. 1. (a) Schematic 1D representation of the BFM on a lattice (top and side views). The bosonic and fermionic particles move on two different arrays having the same periodicity; the fermions are indicated by circles and the bosons by squares on the respective arrays. (b) The single site configurations for the pseudospin and fermionic variables.

$$A[\bar{\Psi}_i, \Psi_i, \theta_i, \phi_i] = \int d\tau \sum_i \left[is(1 - \cos \theta_i)\partial_\tau \phi_i + (\Delta_B\right.$$

$$\left. - 2\mu)\cos \theta_i\right] + \sum_{\langle ij \rangle} \bar{\Psi}_j(\tau) G_{ij}^{-1} \Psi_i(\tau). \quad (3)$$

$\tau$ denotes the imaginary Matsubara time variable and a Nambu spinor representation for the Grassmann variables, related to that of the original fermionic operators by

$$\bar{\Psi}_i = \begin{pmatrix} c_{i\uparrow} \\ \bar{c}_{i\downarrow} \end{pmatrix} \quad \Psi_i = (\bar{c}_{i\uparrow} \; c_{i\downarrow}). \quad (4)$$

The pseudospin is described by a bosonic field that, in spherical coordinates, is given by $s_i = s(\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i)$ (see Fig. 2). $\theta_i$ describes the polar angle of the vector $s_i$ with respect to the north pole of the $z$ axis, while $\phi_i$ is the azimuthal angle, which defines the angular position of the basal plane projection of this vector. The first term of the action $A$ is the Wess-Zumino term,[41,42] ensuring the correct quantization of the quantum-pseudospin variable. For any path, parametrized by $\phi(\tau)$ and $\theta(\tau)$, the contribution of this term is equal to $i s$ times the surface area of the sphere between this path and the north pole. For closed paths this has exactly the form of the Berry phase.[43] The second term is linked to the density of the bosons $n_B(\tau)$ through the $\cos \theta(\tau)$ dependence of the pseudospin. Finally, the last contribution of the action contains the coupling between the fermionic and bosonic subsystem through the Green's function $G_{ij}$, determined by
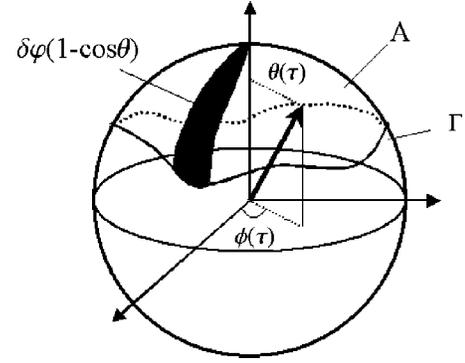


FIG. 2. Spherical representation of the pseudospin **s** including the Berry phase factor for one possible trajectory $\Gamma$. The Berry phase term is $\exp[is\oint(1-\cos[\theta])\,\partial\phi/\partial\tau] = e^{isA}$, where $A$ is the area of the surface enclosed in the trajectory $\Gamma$. The black part indicates the differential portion of the surface on the sphere taken with respect to the north pole.

$$\begin{pmatrix} K_1 & L \\ L^* & K_2 \end{pmatrix} G_{ij}(\tau - \tau') = \delta(\tau - \tau'), \quad (5)$$

where $K_1 = (-\partial_\tau + \mu)\delta_{ij} + t_{ij}$, $K_2 = (-\partial_\tau - \mu)\delta_{ij} - t_{ij}$, and $L = g \sin \theta_i(\tau)e^{i\phi_i(\tau)}\delta_{ij}$, $L^*$ being the conjugate field of $L$. Integrating out the fermionic part, one obtains the action in terms of, exclusively, the bosonic fields

$$A = -Tr \ln G^{-1} + \int d\tau \sum_i \left[is(1 - \cos \theta_i(\tau))\partial_\tau \phi_i(\tau) + (\Delta_B\right.$$

$$\left. - 2\mu)\cos \theta_i(\tau)\right], \quad (6)$$

where the trace has to be carried out over all internal as well as space-time indices.

Up to this point no approximation has been made in the derivation of the action that describes the coupling between the bosonic and fermionic degrees of freedom. It is important to note that the variation of the bosonic variable $\theta(\tau)$ describes both the density fluctuations of the bosonic subsystem (via the projection of the pseudospin vector on the $z$ axis, i.e., $\cos \theta(\tau)$, being the longitudinal component) and the amplitude fluctuations of the fermionic pair field [via the transverse part as projection of the pseudospin vector onto the basal plane, i.e. $\sin \theta(\tau)$] (see Fig. 2). The variable $\phi(\tau)$ determines the rotational degrees of freedom of the pseudospin vector, expressing its phase dynamics.

For extracting the relevant terms that control the low-energy dynamics of the coupled phase and amplitude modes and performing an expansion, which is meaningful in terms of the phase variable, it is judicious to take the following steps: (i) gauge away the phase dependence from the term $L$ that permits one to separate the trace into a part that does not depend on the phase of the bosonic field and another part that contains only spatial and time variations of $\phi_i(\tau)$,[44] and (ii) rewrite the term $L$ after the gauge transformation, as a sum of two pieces, one not dependent on time (which is linked to the average density of bosons) and another term containing the fluctuations with respect to its mean value.
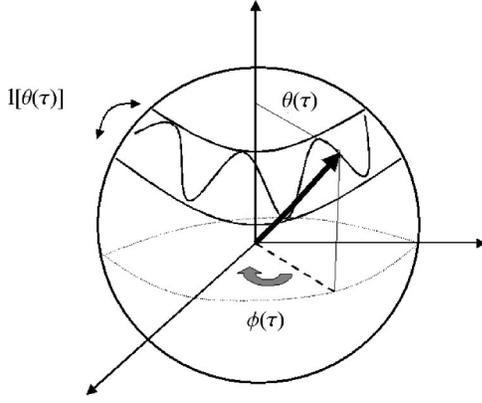
FIG. 3. Representation of a possible path for the local pseudospin motion. The field $l[\theta(\tau)]$ indicates the undulation of the pseudospin vector along the polar direction as it arises from the fluctuations of the average boson density ($\langle \cos \theta \rangle$) and of the pairing amplitude ($\langle \sin \theta \rangle$), while it precesses around the $z$ axis due to the time evolution of the phase variable ($\phi(\tau)$).

The first operation is performed by applying to the operators under the trace the rotation $U_i = e^{i \phi_i(\tau)\sigma_3/2}$, where $\sigma_i$ denote the Pauli matrices. Hence,

$$Tr \ln G^{-1} = Tr \ln U\widetilde{G}^{-1}U^{-1} = Tr \ln\widetilde{G}^{-1}, \qquad (7)$$

where

$$\widetilde{G}_{ij}^{-1} = \left\{ -\partial_\tau \sigma_0 + \left[ \frac{i}{2}\partial_\tau \phi_i(\tau) - \mu \right]\sigma_3 + g \sin \theta_i(\tau)\sigma_1 \right\}\delta_{ij}$$
$$+ t_{ij}e^{i[\phi_i(\tau)-\phi_j(\tau)]\sigma_3/2}. \qquad (8)$$

Since the bosonic density is fixed in average, we now separate the part that depends on the polar angle in a time-independent contribution and its time-dependent correction. That is, the term $\sin \theta_i(\tau)$ is decomposed into its average value $\langle \sin \theta_i \rangle$ (which is determined by fixing the density of bosons due to the spherical constraint) plus a time-dependent contribution $l(\theta(\tau))$ which contains the fluctuations around its average value (Fig. 3) as follows:

$$\sin \theta_i(\tau) = \langle \sin \theta_i \rangle + l(\theta(\tau)). \qquad (9)$$

Thus this local field, due to the constraint, will describe both the time-dependent variation of the density as well as of the pairing amplitude.

Next, let us write the Green's function in the usual form as

$$\widetilde{G}_{ij}^{-1} = G_{0_{ij}}^{-1} + \Sigma_{ij} \qquad (10)$$

with $G_{0_{ij}}^{-1} = [-\partial_\tau\sigma_0 - \mu\sigma_3 + \widetilde{g}\sigma_1]\delta_{ij}$ and $\Sigma_{ij} = T_{ij} + D_i + K_i$, where

$$D_i = \frac{i}{2}\frac{\partial \phi(\tau)}{\partial \tau}\sigma_3, \qquad (11)$$

$$K_i = \widetilde{g}\,l(\theta(\tau))\sigma_1, \qquad (12)$$

$$T_{ij} = t_{ij}e^{-i(\phi_i(\tau)-\phi_j(\tau))\sigma_3/2}, \qquad (13)$$

$$\widetilde{g} = g\langle\sin[\theta]\rangle. \qquad (14)$$

From this point onward *we shall assume the average density of bosons to be homogeneously distributed* and thus given by $n_B = \frac{1}{2}(1+\langle\cos \theta\rangle)$. This implies that the bare coupling $g$ is renormalized to $\widetilde{g}$ as a consequence of the spherical constraint of the pseudospin variable. Moreover, since $\langle\sin \theta\rangle = \sqrt{1-\langle\cos \theta\rangle^2}$ and the density of bosons is fixed in average via a suitable choice of the chemical potential and of the on-site bosonic energy, one can treat the exchange coupling as an external boson-density tunable parameter. The variation in the $\theta$ variable is then simply related to the variation of the bosonic density such that if $\theta$ varies in the range $[0,\pi]$ then $n_B$ varies in the interval $[0,1]$.

Before expanding the trace, let us write down explicitly the expression of the zero-order Green's function, as it will be frequently used in the following steps:

$$G_{0_i}^{\alpha\beta}(\tau) = \frac{1}{\beta}\sum_{\omega_n} G_{0_i}^{\alpha\beta}(\omega_n)\exp[-i \omega_n \tau] \qquad (15)$$

with

$$G_{0_i}^{\alpha\beta}(\omega_n) = \begin{pmatrix} \dfrac{-i\omega_n + \mu}{\omega_n^2 + \omega_0^2} & \dfrac{\sqrt{\omega_0^2 - \mu^2}}{\omega_n^2 + \omega_0^2} \\[12pt] \dfrac{\sqrt{\omega_0^2 - \mu^2}}{\omega_n^2 + \omega_0^2} & \dfrac{-i\omega_n - \mu}{\omega_n^2 + \omega_0^2} \end{pmatrix} \qquad (16)$$

and where we introduced $\omega_0 = \widetilde{g}$.

### B. Second-order loop expansion

We now evaluate the contribution of the self-energy $\Sigma_{ij}$ to the effective action. This is done in the usual way by making a loop expansion in the trace.[44] We shall construct that expansion up to second order in the time and space derivatives of the phase variable and in the terms that contain both the fluctuations of the density and the amplitude. For that purpose we use the standard identity

$$Tr \ln\widetilde{G}^{-1} = Tr \ln[G_0^{-1} + \Sigma] = Tr \ln G_0^{-1} + Tr \ln[1 + G_0\Sigma] \qquad (17)$$

and then expand the second term of this expression up to second order in $\Sigma$, such as to keep all the contributions up to quadratic order in the gradient of the phase. This gives

$$Tr \ln\widetilde{G}^{-1} \cong Tr \ln G_0^{-1} + Tr[G_0\Sigma] - \frac{1}{2}Tr[G_0\Sigma]^2. \qquad (18)$$

The first term of this expression is just a constant and does not contribute to the dynamics. In the second term

$$Tr[G_0\Sigma] = Tr[G_0T] + Tr[G_0D] + Tr[G_0K] \qquad (19)$$

the only parts different from zero are $Tr[G_{0i}K_i]$ and $Tr[G_{0i}D_i]$. $Tr[G_{0i}T_{ij}]$ gives no contribution once one makes the trace over the site indices. $Tr[G_{0i}D_i]$ introduces a contribution that is proportional to the chemical potential multiplied by the time derivative of the phase $\sim i\mu\partial_\tau\phi$. We will see below that this contribution describes an effective *offset charge* (in terms of the terminology of a similar Josephson

junction array scenario), which arises from the total fermionic and bosonic static-density distribution via their dependence on the chemical potential. $Tr[G_{0i}K_i]$ describes the lowest order fluctuations of the bosonic (fermion pair) density due to the presence of the spontaneous pair and/or hole creation out of the condensate. Its direct evaluation gives a contribution

$$A_1 = \int d\tau_1 d\tau_2 Tr_{ps}(G_{0i}(\tau_1, \tau_2)K_i(\tau_2, \tau_1)) \qquad (20)$$

where $Tr_{ps}$ represents the trace over the internal pseudospin index. With $K_i(\tau_2, \tau_1) = K_i(\tau_1)\delta(\tau_1 - \tau_2)$ and integrating over the Matsubara times gives

$$A_1 = \int d\tau Tr_{ps}(G_{0i}(0)K_i(\tau)) = 2gG_{0_i}^{12}(0) \int d\tau l(\theta(\tau)) =$$

$$- \frac{g \tanh[\beta\omega_0]}{\omega_0} \int d\tau l(\theta(\tau)), \qquad (21)$$

or in a compact form

$$A_1 = E_1 \int d\tau l(\theta(\tau))$$

$$E_1 = - \frac{g \tanh[\beta\omega_0]}{\omega_0}. \qquad (22)$$

Let us next come to the evaluation of the terms that contribute to the quadratic order in this loop expansion of the trace. The parts that are nonzero in these terms are the following:

$$A_2 = Tr[G_{0i} K_i G_{0i} D_i],$$

$$A_3 = Tr[G_{0i} K_i G_{0i} K_i],$$

$$A_4 = Tr[G_{0i} D_i G_{0i} D_i],$$

$$A_5 = Tr[G_{0i} T_{ij} G_{0j} T_{ji}]. \qquad (23)$$

The term $A_2$ contains processes where the fermionic background locally couples at different times to the fluctuations of the bosonic density and to the phase fluctuations

$$A_2 = Tr[G_{0\ i} K_i G_{0\ i} D_i] = \int d\tau_1 d\tau_2 Tr_{ps}[G_{0\ i}(\tau_1$$

$$- \tau_2)K_i(\tau_2)G_{0\ i}(\tau_2 - \tau_1)D_i(\tau_1)]. \qquad (24)$$

Since $G_{0i}(\tau_1 - \tau_2)$ depends exclusively on the time differences, we introduce the new variables $\tau = (\tau_1 - \tau_2)/2$ and $\eta = (\tau_1 + \tau_2)/2$, after which the integral over the Matsubara time variables becomes

$$A_2 = \frac{1}{2} \int d\tau d\eta Tr_{ps}[G_{0i}(\tau)K_i(\eta - \tau)G_{0i}(-\tau)D_i(\tau + \eta)]. \qquad (25)$$

As we are interested in the gradient expansion in the bosonic phase and density, related to $\phi(\tau)$ and $\theta(\tau)$, we keep only the

lowest order in their time derivatives. This is done by considering the expansion of the product $K_i(\eta - \tau)D_i(\tau - \eta)$ around $\tau = 0$ in order to separate the parts that depend exclusively on the local variable of the Green's function and those that are linked to the phase and density fluctuations. The expansion then reads as follows:

$$K_i(\eta - \tau)D_i(\eta + \tau) \simeq K_i(\eta)D_i(\eta) + \tau\left[ -\frac{\partial K_i(\eta)}{\partial \eta}D_i(\eta) \right.$$

$$\left. + \frac{\partial D_i(\eta)}{\partial \eta}K_i(\eta) \right] + O(\tau^2) \qquad (26)$$

Due to the symmetry of the Green's functions (even for $\tau \rightarrow -\tau$), the linear contribution of the series expansion of $K_i(\eta - \tau)D_i(\tau - \eta)$ cancels in the effective action after integrating over the time. Hence, to the lowest order one obtains,

$$A_2 = \frac{1}{2} \int d\tau d\eta Tr_{ps}(G_{0i}(\tau)K_i(\eta)G_{0i}(-\tau)D_i(\eta))$$

$$= \frac{g}{2} \int d\tau [G_{0_i}^{12}(\tau)G_{0_i}^{11}(-\tau) + G_{0_i}^{11}(\tau)G_{0_i}^{12}(-\tau)$$

$$- G_{0_i}^{22}(\tau)G_{0_i}^{12}(-\tau) - G_{0_i}^{12}(\tau)G_{0_i}^{22}(-\tau)]$$

$$\times \int d\eta \frac{i}{2} \frac{\partial \phi(\eta)}{\partial \eta} l(\theta(\eta)) \qquad (27)$$

or in a compact form,

$$A_2 = E_2 \int d\eta\, i\frac{\partial \phi(\eta)}{\partial \eta} l(\theta(\eta)), \qquad (28)$$

where the coefficient $E_2$, after integrating over the Matsubara time, is given by

$$E_2 = \frac{\tilde{g}^2}{4} \frac{\mu(\beta \omega_0 - \sinh[\beta \omega_0])}{\omega_0^3(1 + \cosh[\beta \omega_0])}. \qquad (29)$$

Let us next consider the term $A_3$, which expresses the coupling of the fermionic field to the fluctuations of bosonic density at different times. In order to extract the lowest-order gradient contributions, we follow the same procedure as that just used above, giving us

$$A_3 = \frac{1}{2} \int d\tau\, d\eta Tr_{ps}[G_{0i}(\tau)K_i(\eta - \tau)G_{0i}(-\tau)K_i(\tau + \eta)]. \qquad (30)$$

Performing again the expansion in time up to quadratic order in the time derivatives of $K_i(\tau)$, we have

$$K_i(\eta - \tau)K_i(\eta + \tau) \simeq K_i(\eta)^2 + \tau^2\left[ -\frac{\partial K_i(\eta)}{\partial \eta}^2 \right.$$

$$\left. + \frac{\partial^2 K_i(\eta)}{\partial \eta^2}K_i(\eta) \right].$$

and hence can rewrite $A_3$ in the following way:

$$A_3 = \frac{1}{2} \int d\tau \, d\eta Tr_{ps}[G_{0i}(\tau)K_i(\eta)G_{0i}(-\tau)K_i(\eta)]$$

$$+ \int d\tau \, d\eta \; \tau^2 Tr_{ps}\left[G_{0i}(\tau)\frac{\partial K_i(\eta)}{\partial \eta}G_{0i}(-\tau)\frac{\partial K_i(\eta)}{\partial \eta}\right].$$

(31)

Carrying out the trace over the internal indices gives

$$Tr_{ps}(G_{0i}(\tau)K_i(\eta)G_{0i}(-\tau)K_i(\eta)) = [G_{0_i}^{22}(\tau)G_{0_i}^{11}(-\tau)$$

$$+ 2 \; G_{0_i}^{12}(\tau)G_{0_i}^{12}(-\tau) + G_{0_i}^{11}(\tau)G_{0_i}^{22}(-\tau)][\tilde{g}^2 l(\theta(\eta))^2]$$

(32)

so that

$$A_3 = E_{3a} \int d\eta \; l(\theta(\eta))^2 - E_{3b} \int d\eta \frac{\partial \, l(\theta(\eta))}{\partial \, \eta}2.$$    (33)

By evaluating the integrals over the Matsubara times we obtain

$$E_{3a} = \frac{\tilde{g}^2}{2} \frac{\beta\omega_0\tilde{g}^2 + \mu^2\sinh[\beta\omega_0]}{\omega_0^3(1 + \cosh[\beta \, \omega_0])}$$

$$E_{3b} = \frac{\tilde{g}^2}{2}\left\{\frac{sech[(\beta\omega_0/2)]^2(-6\beta\mu^2\omega_0 + \beta^3(-\mu^2\omega_0^3 + \omega_0^5))}{24\omega_0^5}\right.$$

$$\left. + \frac{6\mu^2\sinh[\beta\omega_0])}{24\omega_0^5}\right\}.$$

(34)

Applying the same procedure for the evaluation of $A_4$ (which has the same functional form as $A_3$ but involving the coupling between the fermionic degrees of freedom and the phase velocity at different time) one gets an analogous expression, providing we discard terms of higher order in the derivatives than $[(\partial\phi(\tau))/\partial\tau]^2$

$$A_4 \simeq \frac{1}{2} \int d\tau \, d\eta Tr_{ps}(G_{0i}(\tau)D_i(\eta)G_0(-\tau)D_i(\eta))$$

$$= \frac{1}{8}C_0 \int d\eta \left(\frac{\partial \phi(\eta)}{\partial \eta}\right)^2.$$    (35)

Here

$$C_0 = \int d\tau\{-[G_{0_i}^{11}(\tau)G_{0_i}^{11}(-\tau) - 2 \; G_{0_i}^{12}(\tau)G_{0_i}^{12}(-\tau)$$

$$+ G_{0_i}^{22}(\tau)G_{0_i}^{22}(-\tau)]\},$$    (36)

which after evaluating the integration over the Matsubara times gives,

$$C_0 = \frac{sech[(\beta\omega_0/2)]^2(\beta\mu^2\omega_0 + \omega_0^2\sinh[\beta\omega_0])}{2\omega_0^3}.$$    (37)

Finally, we come to the evaluation of the last term $A_5$, which involves the coupling between phase fluctuations on different sites and the process of single particle hopping. This contribution will yield terms that are quadratic in the time derivative of the phase (*charging like*, in the terminol-

ogy of a similar Josepson junction array scenario) and, moreover, will generate an effective hopping-induced intersite phase coupling that turns out to be similar to the Josepshon coupling in arrays of superconducting grains.

Considering again the lowest-order gradient expansion contributions we have

$$A_5 = Tr[G_{0i} \, T_{ij} \, G_{0j} \, T_{ji}] \simeq \frac{1}{2}\int d\tau \, d\eta Tr_{ps}$$

$$\times \left\{ G_{0i}(\tau)T_{ij}(\eta)G_{0j}(-\tau)T_{ji}(\eta) - \tau^2 \right.$$

$$\left. \times \left[G_{0i}(\tau)\frac{\partial \, T_{ij}(\eta)}{\partial \, \eta}G_{0j}(-\tau)\frac{\partial \, T_{ij}(\eta)}{\partial \, \eta}\right]\right\}$$

$$= P_1 + P_2$$    (38)

The terms $P_{1,2}$ are given by

$$P_1 = -t_{ij}^2 \int d\tau \, (G_{0_i}^{12}(\tau)G_{0_i}^{12}(-\tau)) \int d\eta \, \cos[\phi_i(\eta) - \phi_j(\eta)]$$

$$P_2 = -\frac{t_{ij}^2}{8}\int d\tau \, \tau^2[G_{0\,i}^{11}(\tau)G_{0\,j}^{11}(-\tau)$$

$$+ G_{0\,i}^{22}(\tau)G_{0\,j}^{22}(-\tau)]\int d\eta \left[\frac{\partial \, \phi_i(\eta)}{\partial \, \eta} - \frac{\partial \, \phi_j(\eta)}{\partial \, \eta}\right]^2.$$

(39)

By carrying out the integration over the Matsubara times one obtains for the hopping-induced intersite amplitude and phase coupling

$$P_1 = E_J \int d\eta \, \cos[\phi_i(\eta) - \phi_j(\eta)]$$

$$E_J = \frac{t_{ij}^2 \, \tilde{g}^2(-\beta\omega_0 sech[\beta\omega_0/2]^2 + 2 \, \tanh[\beta\omega_0/2])}{8\omega_0^3}.$$    (40)

Similarly, by calculating the coefficient of the term $P_2$, one obtains the strength of a *mutual capacitance* term (in the terminology of a similar Josephson junction array scenario) for neighboring effective sites

$$P_2 = \frac{1}{8}C_1 \int d\eta \left[\frac{\partial \, \phi_i(\eta)}{\partial \, \eta} - \frac{\partial \, \phi_j(\eta)}{\partial \, \eta}\right]^2$$

$$C_1 = -t_{ij}^2 \int_{-\beta/2}^{\beta/2} d\tau \, \tau^2[G_i^{11}(\tau)G_i^{11}(-\tau) + G_i^{22}(\tau)G_i^{22}(-\tau)]$$

$$= -t_{ij}^2\left[\frac{\exp[\beta\omega_0](-6\beta\tilde{g}^2\omega_0 + \beta^3\omega_0^3(\mu^2 + \omega_0^2))}{12(1 + \exp[\beta\omega_0])^2\omega_0^5}\right.$$

$$\left. + \frac{6\tilde{g}^2\sinh[\beta\omega_0])}{12(1 + \exp[\beta\omega_0])^2\omega_0^5}\right].$$    (41)

The final effective action is then given by the sum of all the terms evaluated above that are grouped together in form of three different contributions:

$$S = \int d\tau [S_\phi + S_{\phi-\theta} + S_\theta]. \tag{42}$$

$S_\phi$, $S_\theta$, $S_{\phi-\theta}$ are the contributions arising from, exclusively, (i) the phase dynamics, (ii) the fluctuations of the bosonic density and of the amplitude $l[\theta]$, and (iii) the coupling between them. They are given by

$$S_\phi = \sum_i \frac{1}{8}(C_0 + zC_1)\left(\frac{\partial \phi_i(\tau)}{\partial \tau}\right)^2 - \sum_{\langle i,j\rangle}\left[\frac{1}{8}C_1\frac{\partial \phi_i(\tau)}{\partial \tau}\frac{\partial \phi_j(\tau)}{\partial \tau}\right.$$
$$\left. + E_J\cos[\phi_i(\tau)-\phi_j(\tau)]\right] + i\sum_i \mu\frac{\partial \phi(\tau)}{\partial \tau},$$

$$S_\theta = \sum_i \left\{E_1 l(\theta_i(\tau)) + E_{3a}l(\theta_i(\tau))^2 - E_{3b}\left[\frac{\partial l(\theta_i(\tau))}{\partial \tau}\right]^2\right\},$$

$$S_{\phi-\theta} = \sum_i \left\{i\,s\frac{\partial \phi_i(\tau)}{\partial \tau}[1-\cos(\theta_i(\tau))] + i\,E_2\frac{\partial \phi_i(\tau)}{\partial \tau}l(\theta_i(\tau))\right\}. \tag{43}$$

We expand the Berry phase contribution (the first term in $S_{\phi-\theta}$) up to second order in $l(\theta(\tau))$ and subsequently redefine this field as $l(\theta(\tau))=a+\bar{l}(\theta(\tau))$, with the time-independent constant $a$ chosen in such a way as to eliminate the terms linear in $\bar{l}$ in the the action $S_\theta$, i.e., $a=-(E_1/2E_{3a})$. We then find two contributions to the action that are linear in $\partial\phi(\tau)/\partial\tau$: one that is time independent and can be absorbed into the chemical potential and another one that is quadratic in $\bar{l}(\theta(\tau))$. With this, $S_{\phi-\theta}$ can be rewritten as follows:

$$S_{\phi-\theta} = \sum_i i\frac{\partial \phi_i(\tau)}{\partial \tau}[q_i + \bar{q}_i(\tau)], \tag{44}$$

$$q_i = \mu/g + 1 - n_B, \tag{45}$$

where we have rescaled the chemical potential and introduced the local offset-charge variable $q_i$ that sets the average number of particles on the island made up by the bosonic and fermionic sites, while $\bar{q}_i(\tau)$ describes the time-dependent fluctuations of the local offset-charge and to the lowest the order is a function of $\bar{l}(\theta(\tau))$ and $\bar{l}(\theta(\tau))^2$.

The effective action thus constructed for the BFM [Eqs. (42) and (43)] is a generalization of that for Josephson junction arrays with self and nearest-neighbor *capacitance, Josephson* coupling, and *offset charge* terms. The action for the BFM goes beyond that for such Josephson junction arrays in the following respects. We have extra terms that control the dynamics of the amplitude modes given by $S_\theta$ and an intrinsic Berry phase term that gives rise to a direct phase-amplitude coupling [Eq. (45)], where the dynamical amplitude fluctuations would correspond to a time-dependent *offset charge* term in an analogous Josephson junction array picture. Finally, the Berry phase term, being an intrinsic topological term, results in a Magnus force on vortices, as will be discussed in Sec. V.

If one considers just the linear part of the coupling between the time derivative of the phase and the time-dependent boson density or amplitude-fluctuating field together with the Gaussian part of the field $\bar{l}(\theta_i(\tau))$, the relative action would have the following structure:

$$S' = \int d\tau\left[\left(\frac{\partial \bar{l}(\theta_i(\tau))}{\partial \tau}\right)^2 + \omega_\alpha^2\bar{l}(\theta_i(\tau))^2 + \lambda_\alpha\frac{\partial \phi_i(\tau)}{\partial \tau}\bar{l}(\theta_i(\tau))\right], \tag{46}$$

where $\omega_\alpha$ is the frequency of the normal mode oscillation for the bosonic field $\bar{l}(\theta_i(\tau))$ and $\lambda$ is the constant controlling the coupling between the phase velocity and the amplitude of the bosonic field. By integrating out the $\bar{l}(\theta_i(\tau))$ mode, the effective action acquires a form with local correlations of the phase velocity at different times,[45] that is

$$S'_{eff} = \int d\tau d\tau' K(\tau-\tau')\left(\frac{\partial \phi_i(\tau)}{\partial \tau} - \frac{\partial \phi_i(\tau')}{\partial \tau'}\right)^2, \tag{47}$$

and where the form of the kernel $K(\tau-\tau')$ depends on the frequency $\omega_\alpha$ in the following way:

$$K(\tau) = \frac{1}{\beta}\sum_{i\omega_n}K(i\omega_n)e^{-i\omega_n\tau},$$

$$K(i\omega_n) = \frac{\lambda_\alpha^2}{2}\frac{\omega_n^2}{\omega_n^2 + \omega_\alpha^2}. \tag{48}$$

This term correlates the phase velocity at different times and, indirectly, the phase variable. It is similar, except for the $\tau$ dependence of the kernel, to the case studied in Ref. 23, where it has been shown to load to a different universality class for the superconductor-insulator transition. In the present case, it is the feedback of the boson density fluctuations that induce nonlocal time correlations in the local phase velocity that can eventually give rise to such features upon approaching the SIT boundary. An analysis on this issue is in progress and will be discussed elsewhere.

## IV. THE SUPERCONDUCTOR-INSULATOR PHASE BOUNDARY

In a very preliminary attempt to analyze the stability region of long-range phase superconducting coherence, let us now examine the effective action [Eqs. (42) and (43)] upon restricting ourselves to the phase-only part of it. This implies a SIT driven by a competition between the phase coherence induced by pair hopping and *the charging effect* due to the local boson density (or, equivalently, pair-field amplitude) fluctuations and the single-particle intersite processes. Within this framework our study is equivalent to that of Josephson junction arrays, except that the effective coupling constants entering in such an action depend in a highly nontrivial way on the parameters that characterize the original BFM Hamiltonian. This, as we shall see, will lead to features concerning the phase diagram with a SIT for the BFM when we examine it in terms of the boson-fermion exchange coupling $g/t$, the total density, and the boson concentration $n_B$.

We determine the phase diagram by means of the so-called coarse-graining approximation, which has been successfully applied for this kind of problem and which permits one to capture the relevant qualitative and quantitative features of such a Josephson junction arraylike action.[46,47] It is known for such systems that, as a consequence of the uncertainty relation between the phase $\phi_i$ and the total pair number operator $Q_i = i\partial/\partial\phi_i$, the system can switch from a phase-ordered to a disordered state. The essential part of this study will concern how the relevant parameters of the BFM-Hamiltonian influence the equivalent amplitudes of the Josephson coupling, the *capacitance*, and the *offset charge* terms.

Let us stress the subtle role played by the particle number in controlling the onset of the superfluid phase coherence. Once the total density per site $n_{tot} = n_f + 2n_b$ is fixed, i.e., assigning a definite value to the offset charge $q_i$, the relative concentration of bosons and fermion pairs has to adjust to this constraint. In other words, it is possible to fix the total density and tune the average density of bosons with respect to that of the fermion pairs. This control of the relative concentration of bosons and fermion pairs can be achieved by suitably changing the parameter $\Delta_B$, which acts like an external field for the boson density. Alternatively it is possible to change the total charge, made up by the sum of boson and fermion pairs on the local island, by varying the value of the offset charge $q_i$.

Our final goal is to show how a transition between a superconductor and a Mott insulator emerges out of the competition between the Josephson and charging energy, as a function of the above-mentioned parameters. Since we shall neglect the long-range part of the charging term, the Mott insulating state has an excitation gap for adding or removing bosons (fermion pairs) but does not have any breaking of the translational symmetry. There are three parameters that can be independently controlled and are relevant for the competition between charging and Josephson energy; that is $n_{tot}$, $n_B$, and $g/t$. We shall show how the superconductor to Mott insulator transition can be tuned by considering all the physically realizable conditions in order to investigate the role played by each of the above-mentioned parameters.

Exploring the possible microscopic realizations at zero temperature, by fixing the total average concentration and the bosonic one, one finds that the variation of the ratio between the Josephson coupling energy $E_J$ and the *charging* energy $E_C = C_{ii}^{-1}/2$, which controls the phase-density interplay, increases from zero, goes through a maximum, and then decreases to zero with growing $g/t$. Alternatively, if one fixes the coupling $g/t$ and the total concentration by varying the density of the bosons, one is able to control the effective coupling $\tilde{g} = 2g\sqrt{n_B(n_B-1)}$ that appears in $E_J$ and $C_{ij}$. It is thus immediately evident that in the BFM scenario there is a nontrivial interplay between the renormalization of the Josephson coupling and the charging effect. If one goes to the limit of empty ($n_B = 0$) and full bosonic occupation ($n_B = 1$), the effective coupling $\tilde{g} \to 0$, and the critical temperature, consequently, reduces to zero. Furthermore, by changing the total charge on the island, it is possible to go toward configurations with charge frustration so that the superfluid state is always favored for any finite value of the coupling $g/t$.

The general form of the action we then have to examine is given by

$$S_{phase} = \int_0^\beta \sum_i i \frac{\partial \phi_i(\tau)}{\partial \tau} q_i - E_J \sum_{ij} \alpha_{ij} \cos(\phi_i(\tau) - \phi_j(\tau))$$

$$+ \sum_{i,j} \frac{1}{8} \frac{\partial \phi_i(\tau)}{\partial \tau} C_{ij} \frac{\partial \phi_j(\tau)}{\partial \tau}. \tag{49}$$

The first term describes the effect of a static *offset charge* $q_i$ and, as said before, takes into account the amplitude of the total charge on the island given by both the bosons and the fermion pairs. The second term contains the physics of the pair-hopping processes and an analog of the Josephson tunneling processes, and has a coupling strength $E_J$ with $\alpha_{ij} = 1$ if $(i,j)$ are nearest-neighbors and zero otherwise. Finally, the third term describes the *charging* term arising from the local exchange of boson and fermion pairs and from quasiparticle hopping between nearest-neighbor sites. The strength of this *charging*-type interaction is given by $C_{ij} = (C_0 + zC_1)\delta_{i,j} - C_1 \sum_p \delta_{i,i+p}$, which represents an effective general *capacitance* matrix, with the vector $p$ running over the nearest-neighbors. $C_0$ denotes the *self-capacitance* and $C_1$ the *mutual capacitance* ($z$ being the coordination number).

In order to extract the phase diagram we now make use of the coarse-graining approximation.[46] The main idea of this approach is to introduce a Hubbard-Stratonovich auxiliary field that is conjugate to the average of $\langle e^{i\phi_i} \rangle$ and plays the role of an order parameter for the transition from a superconducting to an insulating state. Since the phase transition has a continuous character, one can expand the action in powers of the auxiliary field and determine the occurrence of phase coherence by looking at the coefficients of the quadratic term in the limit of long wavelengths and zero frequency.

We briefly sketch the main steps of such an approximation and adapt it to the present scenario of the BFM. The partition function for $S_{phase}$ is given by the sum of all the possible paths of the phase variables in the imaginary time and in the real space

$$Z = \int \prod_i D\phi_i \exp\left[ \int_0^\beta \sum_i -i \frac{\partial \phi_i(\tau)}{\partial \tau} q_i + E_J \sum_{ij} \alpha_{ij} \cos(\phi_i(\tau) \right.$$

$$\left. - \phi_j(\tau)) - \frac{1}{8} \sum_{ij} \frac{\partial \phi_i(\tau)}{\partial \tau} C_{ij} \frac{\partial \phi_j(\tau)}{\partial \tau} \right].$$

To perform the Hubbard-Stratonovich transformation, one rewrites the *Josephson coupling* term as $(E_J/2) \sum_{ij} \exp[i\phi_i] \alpha_{ij} \exp[i\phi_j]$ and then, by using the usual Gaussian identity, introduces an auxiliary field $\psi_i(\tau)$. The partition function $Z$ then becomes

$$Z = \int \prod_i D\psi_i^* D\psi_i D\phi_i \exp\left\{ \int_0^\beta d\tau \left[ \left(\frac{-2}{E_J}\right) \sum_{ij} \psi_i^* \alpha_{ij}^{-1} \psi_j \right] \right\}$$

$$\times \exp\left\{ \int_0^\beta d\tau \left[ \sum_i i \frac{\partial \phi_i(\tau)}{\partial \tau} q_i - \sum_i (\psi_i e^{i\phi_i} - \psi_i^* e^{-i\phi_i}) \right. \right.$$

$$\left. \left. - \sum_{i,j} \frac{1}{8} \sum_{ij} \frac{\partial \phi_i(\tau)}{\partial \tau} C_{ij} \frac{\partial \phi_j(\tau)}{\partial \tau} \right] \right\}. \tag{50}$$

Starting from this effective action for the auxiliary field, we perform an expansion up to second order in $\psi$ and thus derive a Ginzburg-Landau-type free-energy functional that permits us to determine the boundary line between the superconducting and the insulating states. The corresponding effective action for these auxiliary fields $\psi, \psi^*$

$$S_\psi = \ln\left[\int \prod_i D\phi_i \exp\left(\int_0^\beta d\tau\left\{\sum_i i\frac{\partial \phi_i(\tau)}{\partial \tau}q_i\right.\right.\right.$$
$$- \sum_i [\psi_i(\tau)e^{i\phi_i(\tau)} - \psi_i^*(\tau)e^{-i\phi_i(\tau)}]$$
$$\left.\left.\left.- \sum_{i,j}\frac{1}{8}\frac{\partial \phi_i(\tau)}{\partial \tau}C_{ij}\frac{\partial \phi_j(\tau)}{\partial \tau}\right\}\right)\right] \quad (51)$$

when expanded to second order is given by

$$S_\psi = \int_0^\beta d\tau d\tau' \chi_{ij}(\tau, \tau')\psi_i^*(\tau)\psi_j(\tau) + O(\psi^4). \quad (52)$$

$$\chi_{ij}(\tau, \tau') = \langle e^{i[\phi_i(\tau)-\phi_j(\tau)']}\rangle_0 \quad (53)$$

denotes the two-time phase correlator, which is equal to the second derivative of $S_\psi$ with respect to the auxiliary field $\psi$ and its conjugate at different time and space positions, evaluated around their zero values. By performing the functional derivation, one gets the following expression for it:

$$\chi_{ij}(\tau, \tau') = \frac{\int \prod_i D\phi_i e^{i[\phi_i(\tau)-\phi_j(\tau)']}\exp[-S_0]}{\int \prod_i D\phi_i \exp[-S_0]} \quad (54)$$

with $S_0$ being the part of the action which contains exclusively the *charging* contributions, i.e.,

$$S_0 = \exp\left[\int_0^\beta d\tau\sum_i i\frac{\partial \phi_i(\tau)}{\partial \tau}q_i - \sum_{i,j}\frac{1}{8}\frac{\partial \phi_i(\tau)}{\partial \tau}C_{ij}\frac{\partial \phi_j(\tau)}{\partial \tau}\right]. \quad (55)$$

According to the scheme outlined above, we now develop the partition function $Z$ up to second order in the auxiliary fields, thus putting it into a familiar form

$$Z = \int \prod_i D\psi_i^* D\psi_i e^{-F_\psi} \quad (56)$$

with

$$F_\psi = \int_0^\beta d\tau d\tau'\sum_{ij} \psi_i^*(\tau)[\alpha_{ij}\delta(\tau-\tau') - \chi_{ij}(\tau, \tau')]\psi_i(\tau'). \quad (57)$$

The determination of the conditions for the boundary line between the superconducting and the insulating phase then reduces to the explicit evaluation of $\chi_{ij}(\tau, \tau')$. The result for that has been first obtained in Ref. 47, and we sketch below the main steps of this derivation.

In determining $\chi_{ij}(\tau, \tau')$ it is essential to treat the phase variable $\phi_i$ in a compact form. In order to separate the imaginary time evolution of the phase into a periodic part $\bar\phi_i(\tau)$ and an nonperiodic part, one introduces the following parametrization:

$$\phi_i(\tau) = \bar\phi_i(\tau) + \frac{2\pi n_i\tau}{\beta} \quad (58)$$

with $\bar\phi_i(0)=\bar\phi_i(\beta)$ and $n_i$ being an integer that counts how many times the phase winds over an angle that is a multiple of $2\pi$.

The use of such a relation allows one to express the sum over all $\phi_i$ as an integration over $\bar\phi_i$ plus a sum over all the possible integer values of the winding number $n_i$. After performing a number of suitable algebraic operations,[47] one ends up with the following expression for the two-time phase correlator:

$$\chi_{ij}(\tau) = \delta_{ij}\frac{\exp[-2C_{ii}^{-1}|\tau|]}{\sum_{\{n_i\}} \exp[-\sum_{ij} 2\beta C_{ij}^{-1}N_iN_j]}$$
$$\times\sum_{\{n_i\}} \exp[-\sum_{ij} 2\beta C_{ij}^{-1}N_iN_j - \sum_k 4C_{ik}^{-1}N_k\tau], \quad (59)$$

where $N_i=q_i+n_i$. This two-time phase correlator is local in space and its time dependence follows an exponential behavior, if one assumes that the static *offset charge* $q_i$ has a distribution that is homogeneous in space.

Having obtained the expression for the local two-time phase correlator in the time and space representation, we can now express the effective Ginzburg-Landau free-energy functional in the Fourier space in the following way:

$$F_\psi = \frac{1}{\beta L}\sum_{n,k} \psi_k^*(\omega_n)[\alpha_k^{-1} - \chi_k(\omega_n)]\psi_k(\omega_n). \quad (60)$$

Expanding the inverse matrix $\alpha_{ij}^{-1}$ in the form $\alpha_k^{-1}=(1/z)+k^2(a^2/z^2)+...$, this free-energy functional finally is written as

$$F_\psi = \frac{1}{\beta L}\sum_{k,n}\left[\frac{2}{zE_J} - \chi_0 + ak^2 + b\omega_n^2 + i\lambda\omega_n + ...\right]|\psi_k(\omega_n)|^2, \quad (61)$$

where $b$ and $\lambda$ are the coefficients of the expansion of $\chi_k(\omega_n)$ around the limit $\omega_n=0$.

The transition line is then given by the condition that the coefficient of the quadratic term vanishes in the limit of vanishing $k$ and $\omega$, that is,

$$1 - \frac{zE_J}{2}\chi_0(0) = 0. \quad (62)$$

For a quantitative analysis, one has to determine the explicit zero-frequency limit of the two-time phase correlator. As mentioned above, the inclusion of the time-dependent *offset charge* coming from the fluctuations of the bosonic density can modify the low-frequency behavior of the local phase correlations. Such a feature will be addressed in a future study.

In order to gain insight into the underlying physics at play here we evaluate the two-time phase correlator in the limit of a purely local *capacitance* (the so-called *self-charging* limit), by keeping only the on-site part in the original structure of the *capacitance* matrix. Under those conditions the two time phase correlator at finite frequency is given by

$$\chi_{ii}(\omega_n) = \frac{1}{Z_0}\sum_{\{n_i\}} F[N_i]\left[\frac{1}{C_{ii}^{-1}} - (i\omega_n - 4C_{ii}^{-1}N_i)^2\right]^{-1} \quad (63)$$

with $F[N_i] = \exp[-\Sigma_i\, 2\beta C_{ii}^{-1}N_i^2]$ and $Z_0 = \Sigma_{\{n_i\}}F[N_i]$.

With these expressions, one can finally cast Eq. (62) in the form

$$1 - \frac{zE_J}{4E_C}\frac{\sum_n \dfrac{\exp[-4\beta E_C(n+q)^2]}{1-4(n+q)^2}}{\sum_m \exp[-4E_C\beta(m+q)^2]} = 0, \quad (64)$$

expressed in terms of the *Josephson coupling* $E_J$ and the *charging energy* $E_C = C_{ii}^{-1}/2$, as given by the Eqs. (37), (40), and (41).

Given this defining equation for the boundary between the superconducting and the insulating phases, let us see now how the intrinsic dependence of the Josephson and charging energy on the exchange coupling and the density of bosons manifests itself in the competition between phase and density degrees of freedom. In the loop expansion given in the Sec. III, for the BFM we have obtained the amplitude of the intersite phase coupling and the *charging* effect as a function of temperature, the ratio $g/t$, and the bosonic density. It is the *nontrivial dependence* of those coupling constants on the parameters of the original microscopic Hamiltonian of the BFM that leads to the interesting features in the competition between the phase, the boson, and total density. Thus, upon increasing the pair exchange coupling it is expected that the *Josephson* coupling is reduced while the *charging* energy is increased because of the local transfer between fermion pairs and bosons as well as because of the single particle processes of fermions between nearest-neighbor sites.

We determine the critical line for the cases (i)–(iii) mentioned above in order to highlight the role played, on the one hand, by the coupling and, on the other hand, by the boson density and the total average concentration. In Fig. 4, we illustrate the transition line $T_\phi$, separating a phase coherent state from a phase-disordered one, as a function of the coupling strength and the effective boson density $n_B$. The evolution of the transition line is nonmonotonic as a function of $g/t$ and goes through a maximum at $g_{max} \sim t$. The quantum critical point, where the SIT occurs, is given by $g_{crit} \sim 2t$. The critical behavior close to the transition is that of a *XY* model in $d+1$ dimensions.

More interesting still is the behavior of $T_\phi$ as a function of the $n_B$. The variation of $T_\phi$ is qualitatively different for the different parameter regimes: (a) the weak coupling case for $g < g_{max}$, (b) the intermediate one with $g_{max} < g < g_{crit}$, and (c) the strong coupling limit for $g > g_{crit}$ (see Fig. 5). Going from the limit a to c we find that with increasing $n_B$ the
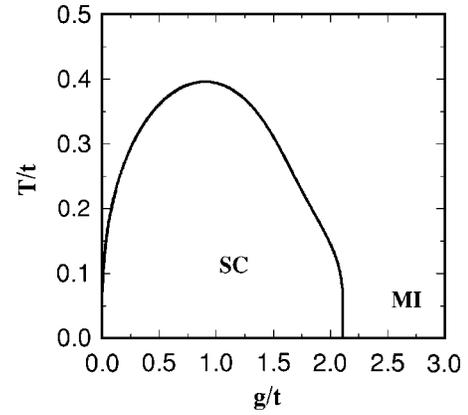


FIG. 4. Phase diagram as a function of $g/t$ describing the boundary between a Mott-insulating (MI) state with homogeneous charge distribution and a superconducting (SC) phase with long-range phase coherence for the case of fully symmetric limit, i.e., total density is given by one pair in average for site ($\Delta_B = 0, n_{F\uparrow} = n_{F\downarrow} = 2n_B = 1$).

critical line decreases to zero, goes through a maximum starting at a finite $T^\phi(n_B = 1)$, and finally shows a SIT at a critical density.

Let us finally consider the case where the local charge on the island formed by the bosonic and fermionic site is varied by tuning the value of the offset charge $q_i$ while keeping the boson density fixed. Since we are considering only the self-charging case, the boundary line will again separate a superconducting from a homogeneous Mott-like state. As shown in previous studies on Josephson junction arrays,[47–49] the effect of the offset charge is to induce a charge frustration due to a degeneracy in the space of configurations, such as to reduce the effective charging energy and favoring the superconductivity for an arbitrarily small ratio of $E_J/E_C$. This is a consequence of the behavior of the zero-frequency correlation function in Eq. (64), which is dominated at low temperature by the integer-valued charge $n=0$ in the sum, that produces the smallest exponential and becomes infinitely large approaching the limit $q=1/2$ due to the divergence in the
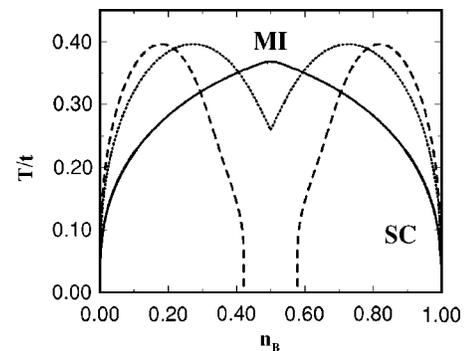


FIG. 5. Phase boundary lines between the superconducting (SC) and Mott-insulating (MI) state at different values of the coupling and upon varying the average density of bosons, keeping fixed the total average density to one pair for site. Solid, dotted, and dashed lines stand for values of the coupling constants for which $g < g_{max}$, $g > g > g_{max}$, and $g_{crit} > g_{crit}$, respectively.

denominator of $\exp[-4\beta E_C(n+q)^2]/[1-4(n+q)^2]$. This corresponds to an effective renormalization of the ratio between $E_J/E_C$, which tends to favor the superconducting phase by shrinking the Mott insulator region as one tunes the total density toward the limit $q=1/2$.

## V. ROLE OF BERRY PHASE TERM: AN INTRINSIC MAGNUS FORCE ON VORTICES

In Sec. IV we have analyzed the basic physics resulting from the effective action by neglecting (i) the influence of any feedback between the density and amplitude fluctuations (included in the field $l[\theta_i]$) on the phase dynamics, and (ii) the contribution from the Berry phase term, responsible for the correct quantization of the pseudospin variable, which is given by the integral over all the possible paths of $is(1-\cos[\theta_i])\partial_\tau\phi_i$. In this section, we discuss the consequences of the presence of such a Berry phase term in the case where the phase action has a vortex solution. The existence of a vortex solution is assured for 2D systems where the phase correlations are described by a $XY$-type dynamics. We shall show here that the Berry phase term will produce an intrinsic Magnus Force on the vortex, which is analogous to the Lorentz force for a charged particle, whose effective magnetic field depends on the spatial distribution of the $\theta$ field. Using the correspondence with the bosonic pseudospin variable, this implies a relation with the spatial-density distribution of the bosons. The Magnus force has been widely discussed in the context of normal BCS-type superconductors, where it has been shown that it arises from the Berry phase caused by the adiabatic motion of a vortex along a closed loop coming back to its starting position.[50] The adiabatic vortex motion on a loop in the superconducting state turns out to be affected by an effective magnetic field generated by the supercurrent arising from the gradient of the phase that encircles such a vortex. In the BFM scenario discussed here, we find that on top of the usual contribution due to the superfluid electrons, there is an intrinsic Berry phase term that will generate such a Magnus force and which has an intensity proportional to $2n_B-1$, where $n_B$ is the bosonic density. This Magnus force, arising solely from the quantum nature of the pseudospin variable, is clearly independent on any superconducting state and does not require a coherent superfluid current induced by the presence of the vortex itself. Moreover, since $n_B$ can be controlled externally, one has the possibility to tune the strength of the effective magnetic field acting on the vortex, and hence to alter its dynamical properties. This will result in a possible measurable effect on transport coefficients, such as the Hall coefficient, resistance, etc.

To be more explicit and following the procedures used in different approaches treating with the Magnus force problem, let us assume that one has a vortex centered at the position $R$. Let us furthermore consider that we are in the continuum and at zero temperature so that the Berry phase term in the Lagrangian now reads $L_B=\int d^2r \cos[\theta(r,t)]\partial_t\phi(r,t)$. The contributions linear in $\partial_t\phi(r,t)$ and having time-independent coefficients do not contribute to the dynamics. Moreover, since we consider that the phase $\phi$ and the variable $\theta$ are linked to a vortex solution centering at $R$, it is judicious to introduce a

new relative variable $r-R$, which defines the intrinsic position dependence of the variables with respect to the vortex center.

Starting from the above-mentioned expression for the Lagrangian $L_B$ and making a simple change in the time derivative we can rewrite it in the following form:

$$L_B \int d^2r \cos[\theta(r-R,t)]\partial_t\phi(r-R,t) \qquad (65)$$

$$=\frac{d\mathbf{R}}{dt}\int d^2r \cos[\theta(r-R,t)]\nabla_r\phi(r-R,t). \qquad (66)$$

At this point, one recognizes that the term $\int d^2r \cos[\theta(r-R,t)]\nabla_r\phi(r-R,t)$ plays the role of an effective vector potential $\mathbf{A}_B$ and $d\mathbf{R}/dt$ represents the velocity of the vortex. This Lagrangian is thus equivalent to that of a charged particle in presence of an effective magnetic field in the $z$ direction given by $\mathbf{B}=\nabla\times\mathbf{A}$. $\mathbf{B}$ hence creates a Magnus force $\mathbf{F}_M=-(d\mathbf{R}/dt\times\hat{\mathbf{z}})2\pi\cos[\theta_0]$ which is transverse and proportional to the vortex velocity and whose strength is linked to the magnitude of this effective magnetic field $B_z=-2\pi\cos[\theta_0]$. Its magnitude is given by the asymptotic value of the background bosonic density at large distance $\cos[\theta_0]$ for $r\rightarrow\infty$.

In case of a 2D superconductor described by the BFM scenario, the usual effective magnetic field $-2\pi\rho_s$ arising from the supercurrent of the electrons circling the vortex has thus to be supplemented by the above-mentioned contribution $B$ (intrinsic to such a scenario) and permits one, in principle, to modulate the total Magnus force upon changing the density of the bosons. Clearly, a full description of such an eventuality, which is beyond the scope of the present analysis, has to be studied in more detail and has to include the derivation of the effective action in the presence of the magnetic field such as to determine the superfluid density of the electrons $\rho_s$.

At this stage we simply want to point out that the sign of the magnetic field arising from the Berry phase term in the action of the BFM changes if one tunes the bosonic density to between zero and unity. In terms of the variable $\cos[\theta]$ this implies a variation in the range $[-1,1]$. In other words, the sign of the intrinsic Berry phase-induced Magnus force will change at $n_B=1/2$ when going from the limit of small density of bosons to high density. This extra contribution to the Magnus force has to be added of course to the conventional one known for standard superconductors.

## VI. COMPARISON WITH NEGATIVE-U AND BOSE-HUBBARD MODELS

A frequently asked question concerns the qualitative differences that exist between the BFM and similar scenarios, such as the negative-U Hubbard model, and pure bosonic systems, such as the Bose-Hubbard model and Josephson junction arrays.

Let us start with a comparison of the BFM and the negative-U Hubbard model, which has been studied in a great variety of different approaches and discussed especially

in connection with the BCS-BEC crossover. The main issue of such a comparison is the occurrence or not of a quantum critical point in the negative-U Hubbard scenario at a finite value of the coupling $U$, which introduces the pairing among the electrons. As we have shown in this paper, in the BFM there exists a critical value for the exchange coupling $g/t$ responsible for the pairing in the fermionic subsystem above which the system undergoes a SIT. We have seen that within a path-integral formalism, one can extract an effective action describing such a transition as being primarily due to the interplay between the *local* quantum pair exchange (between a boson and a fermion pair) and the *nonlocal* intersite pair hopping leading to a phase dynamics originating from the itinerancy of the fermions. Due to the nontrivial dependence of the strength of these two competing mechanisms on the microscopic parameters of the BFM, there occurs a SIT for a finite density as well as finite exchange coupling $g/t$. For the negative-U Hubbard scenario this is not the case and the SIT does not occur at any finite value of the ratio $U/t$.[51] This model merely describes a continuous crossover between a BCS superconductor and a BEC of tightly bound electron pairs as U is increased from 0 to $\infty$.

In order to better understand this difference between the BFM and the negative-U Hubbard model, let us consider two scenarios in equivalent situations, i.e., the half-filled band case for the negative-U Hubbard model and the fully symmetric case for the BFM (with the bosonic level lying in the middle of the fermionic band such that both the fermionic band as well as the bosonic level are half occupied). We then address the question how, in the strong coupling limit, the pair hopping is generated out of the basic configurations of states and processes that contribute in this regime.

For the case of the negative-U Hubbard model with fermions interacting via a local attractive potential, the ground-state wave function in the limit of $U \rightarrow \infty$ and at half filling is highly degenerate and is composed of all the possible configurations comprising equal distribution of zero and doubly occupied site. In this limit, one can perform a mapping of this model on the hard-core boson model described by the Hamiltonian

$$H_U = -\sum_{\langle ij \rangle} \frac{2t_{ij}^2}{U} b_i^+ b_j + \sum_{\langle ij \rangle} \frac{2t_{ij}^2}{U} n_i n_j - \tilde{\mu} \sum_i n_i. \quad (67)$$

$b_i^+(b_i)$ and $n_i = b_i^+ b_i$ stand for the creation (destruction) operators of hard-core bosonic particles (tightly bound electron pairs) and for their density operator, respectively. Due to the presence of a coupling, which is isotropic both in the boson hopping and in the charge interacting channel, one has a superconducting state for any finite value of the ratio $U/t$, and possibly a supersolid phase due to the symmetry of the charging interactions characterized by a coexistence of diagonal and off-diagonal long-range order. This implies that the quantum critical point is strictly pushed to $U \rightarrow \infty$.

In the BFM the phase space in the large $g/t$ limit is completely different. In the fully symmetric case of this model [corresponding to a total density equal twice the number of sites $(n_{tot}=2)$], the ground-state configuration is given by a

wave function where all the bosons are strongly coupled with pairs of fermions, thus resulting in a product state of local bonding states

$$|\psi_0\rangle = \prod_i \frac{1}{\sqrt{2}} (\rho_i^+ + \tau_i^+) |0\rangle. \quad (68)$$

This wave function can be viewed as a ferromagnetic Ising-type state in the sense that it is made up of a ferro-type bonding order and is not degenerate with respect to all the other possible local configurations. Moreover, by construction it contains bonding-bonding correlations on a long range (bonding solid), but has a zero-phase correlation length. The latter can be shown by evaluating the static correlation function for the bosons or, equivalently, the fermion pairs, which gives $\langle \psi_0 | \rho_i^+ \rho_j | \psi_0 \rangle = 0$ and $\langle \psi_0 | \tau_i^+ \tau_j | \psi_0 \rangle = 0$ for any distance $|i-j|$.

Next, let us consider the low-energy configurations that are mixed into such a $g/t = \infty$ ground state when the kinetic energy operator $H_t = \sum_{i \neq j, \sigma} t_{ij} (c_{i\sigma}^+ c_{j\sigma} + \text{H.c.})$ is switched on. Applying $H_t$ to $|\psi_0\rangle$, there will occur states that are separated by an energy gap with respect to $|\psi_0\rangle$. The low-energy states to be considered as the relevant quantum mixed configurations are of a nonbonding nature, such as $C_{i\sigma}^+ |0\rangle = c_{i\sigma}^+ |0\rangle$ and $S_{i\sigma}^+ |0\rangle = \rho_i^+ c_{i\sigma}^+ |0\rangle$. In order to construct those states, let us begin by considering a local excitation on two adjacent sites $i, j$ given by $S_{i\sigma}^+ C_{j-\sigma}^+ |0\rangle$ in the background of bonding states. This configuration can be seen as a ferro-type order interrupted by two domain walls of a nonbonding nature. Now let us consider the dynamics of such objects and under which conditions they can be rendered itinerant in a way that leaves the number of bonding configurations unchanged. The problem is thus analogous to that of domain wall dynamics in an Ising-type system. It so happens (a detailed discussion of this is beyond the scope of the present study and will be given at a later stage elsewhere) that the local degeneracy of the non-bonding states will be removed by the action of the kinetic term. In this way it induces a global lowering of their excitation energy due to the dispersion of each domain wall, which is of the form

$$E_{DW}(k) = \frac{1}{2} [\epsilon_k + 6g \pm \sqrt{(\epsilon_k - 2g)^2 + 4t^2}], \quad (69)$$

and where $\varepsilon(k) = -2zt \cos k$. Only after the condensation of the domain wall-like excitations in the presence of the bonding-state background one can meet the conditions for setting up long-range boson and fermion-pair phase correlations.

This simple sketch of the nature of the excitations in the BFM implies that the onset of phase coherence cannot be activated for an arbitrarily small hopping amplitude because one has to overcome the energy gap between the ground state and the manifold of nonbonding states, which is of the order of $\sim 2g$. As we can see from the expression for the dispersion of the domain walls, this is achieved when $E_{DW}(k)$ becomes zero for $k=0$, i.e., when $g/t = \frac{1}{4}(2 + \sqrt{2 + 4z^2})$. For $z=2$ this gives $g/t \sim 2.0$, which determines when those defectlike excitations become gapless and thus induce a proliferation within the background of the bonding states. It is worth pointing out that this value for the critical exchange coupling

matches rather well the value obtained from our functional integral approach discussed in this paper (see Fig. 3).

The picture that hence emerges for the BFM is very similar to that of a *XY* model in a transverse field. There, the hopping term is responsible of the *XY* dynamics, while the local boson-fermion pair exchange provides the role of the effective transverse field. This itself is already a strong indication that this model is of a different nature from that of the negative-U Hubbard scenario, for which we know that it is, on the contrary, akin to an isotropic Heisenberg model.

A further essential difference between the BFM and the negative-U Hubbard model can be seen from their respective path-integral formulations. When the path-integral representation is considered for this model,[52] the first step is to make use of a Hubbard-Stratonovich transformation to rewrite the quartic interaction term in a bilinear form, where the fermionic operators are now coupled to random auxiliary fields. Limiting oneself to purely superconducting order, the usual procedure for manipulating such a bilinear action is to separate the complex auxiliary field into its modulus and a pure phase part before performing an expansion around the saddle-point solution for the amplitude of this auxiliary field. This way of proceeding allows one then to extract an effective action for the slow phase dynamics. This is distinctly different from the functional integral representation for the BFM, which we have presented above and where from the very beginning and throughout such a procedure the fermionic pair fields are coupled to physically real bosonic modes that have their own proper dynamics. As we have seen in Sec. III, the bosonic part of the action is treated within a coherent pseudospin representation where the dynamics of the pseudospin are parametrized by the time dependence of the spherical variables. The role of the dynamics within the spherical representation of the bosonic field and of the Berry phase term (which is a consequence of the quantum interference in the local pseudospin space) is a distinct feature of this BFM and presents specific differences with respect to the negative-U model. Such differences imply, in particular, that in the BFM case one has feedback effects between the amplitude and the phase fluctuations, which are totally absent in an equivalent description of the negative-U Hubbard model. Such features are important because of intrinsic dissipation effects that eventually can change the nature of the transition and possibly are relevant for the emergence of a 'bosonic metal' in proximity of the SIT, a feature which seems to be outside the framework of the negative-U Hubbard model.

Let us conclude this comparison of the BFM with similar models with a brief discussion on the Bose-Hubbard model and Josephson junction arrays. The main aspect that emerges as a common denominator in all those models, at least as far as the phenomenon of the SIT is concerned, is that the mechanism responsible for the degrading of the phase coherence is analogous and originates from the competition between the phase and charge degrees of freedom. In the Bose-Hubbard model this manifests itself as a competition between the boson hopping and their charge repulsion, while in the Josephson junction array scenario it appears as an interplay between the Josephson tunneling amplitude and the charging energy.

In spite of this, initial similarity between the BFM and those scenarios, we would like to stress that in a system where the dynamics is described by the BFM scenario, the interplay between the phase and amplitude fluctuations is intrinsically related to the coupling between the fermionic and bosonic degrees of freedom. This introduces amplitudes of the different processes at work in the form of a nontrivial dependence on the microscopic parameters of the starting Hamiltonian. Furthermore, as we have seen in the above discussion of the effective action of the BFM, this model contains features that go beyond those that characterize the pure Josephson-type dynamics to those that arise from the peculiar feedback between the fluctuations of the bosonic density (or amplitude pair field) and fluctuations of the phase. Last not least, the appearance of an intrinsic Berry phase term in the BFM scenario together with the effect of dissipation due to the fermionic dynamics can give rise to an unconventional phenomenology when topological phases play a role, and especially in the presence of vortices.

## VII. CONCLUSION

In this paper we examined the possibility of a superconductor-insulator transition in a system of localized bosons and itinerant fermions coupled together via a pair exchange term. An effective action was derived from such a microscopic model, which, after integrating out the fermionic fields, could be phrased in terms of amplitude and phase fluctuations of the bosons. In order to make the presentation more familiar we discussed the action in a terminology frequently employed in connection with the study of Josephson junction arrays. We stress that our system does not necessarily imply any charged fermions and bosons.

Considering the phase-only part of the effective action it is fully equivalent to the quantum-phase model for the Josephson junction arrays, discussed in terms of: (i) a *Josephson coupling* term, (ii) a *charging or capacitance* term, and (iii) an *offset charge* term. Equivalent to that in our scenario is (i) a boson hopping term, (ii) a term that takes into account the reduction in hopping amplitude due to a fluctuating local boson density arising from the intrinsic on-site exchange coupling between the bosons and the fermions, and (iii) a chemical potential term controlling the total concentration. In the present study, within already the lowest (phase-only) approach to the boson-fermion system, is the intricacy of the dependence of the effective Josephson coupling, the capacitance term, and the offset charge term on the parameters of the initial Hamiltonian, i.e., the exchange coupling and the density of bosons. It turns out that already within the phase-only part of the effective action, a superconductor-to-Mott insulator transition can be triggered not only by a change in the exchange coupling, but also by a variation of the boson density. The latter evidently presents an interest from the experimental point of view and can possibly be tested in such transition in optical lattices for ultracold fermi gases with Feshbach resonance pairing.

Apart from the phase-only part of the effective action we established the existence of an intrinsic Berry phase term that arises from the hard-core nature of the bosons and gives rise

to an additional Magnus force when the system is such that topological ground-state configurations, such as vortices, are stabilized. This again should be of potential interest for experiments since, in principle, one can change the sign of the Magnus force upon changing the density of bosons and thus change the motion of a vortex from one direction into the opposite one. These very preliminary results will be dealt with in greater detail in some future studies.

A further property of the Berry phase term is that it gives rise to a coupling between phase and amplitude fluctuations and is hence much more direct and relevant than similar terms in, for instance, scenarios based on the negative-U Hubbard model where they occur only at a much higher order in a corresponding loop expansion of the trace in the effective action. This again merits being investigated in some detail with the aim to study the retarded in time correlations introduced by such amplitude-phase coupling and its effect on the nature of the transition, in view of exploring the possibility of an intermediary bosonic metallic ground state.

A frequently asked question concerns the differences between the negative-U Hubbard scenario, mainly studied in connection with the BCS-BEC cross-over, and the presently studied boson-fermion model. Far from being able to give a complete account for the major differences, we found mainly two aspects that distinguish the physics of these models on a qualitative and robust level. One is that in the negative-U Hubbard model a superconductor-insulator transition cannot take place at any finite coupling U, nor can such a transition be triggered by the change in particle concentration. The second point is that the ground states in the strong coupling limit of the two models are quite different: a highly degenerate ground state for the negative-U Hubbard model with excitations being controlled by an isotropic Heisenberg model when the hopping term is switched on. Contrary to

that, for the boson fermion model the ground state is nondegenerate (corresponding to a ferro pseudomagnetic Ising-type system of singlets formed by bosons and fermion pairs), which, after switching on the fermion hopping term, gives rise to propagating domainlike structures. The topological structures appearing in the ground state might have measurable consequences in the transport properties near the superconductor-insulator transition.

Apart from the more basic features of the model discussed in this paper, there are a number of very subtle questions that pose themselves in connection with the nature of the insulating state. The boundary line of the phase diagram presented in the paper was determined by considering only the homogeneous Mott-insulator phase once the superconducting state is suppressed. In doing that, we have neglected long-range charge interactions (which arise when inverting the capacitance matrix) and kept only the local charging energy. We can expect, as also shown in systems of interacting bosons with short-range repulsion, that for a total concentration of about 1/2 pair per site, the long-range charge interaction will induce breaking of translational symmetry and the formation of a charge-density wave, possibly allowing for the occurrence of supersolid phases. Such eventualities have been studied both for bosonic models with short-range (first or next nearest-neighbor sites) repulsion[48] as well as for long-range interaction. The possibility of commensurate charge distribution, away from the situation of one pair per island, has been considered.[47] Moreover, the competing local and nonlocal phase correlations, which characterize the BFM, may give rise to exotic topological phases in the insulating state. These very subtle questions certainly merit a detailed analysis. They are beyond the scope of the present paper and will be addressed in some future work.

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