

**Monte Carlo method for a quantum measurement process by a single-electron transistor**

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We derive the quantum trajectory or stochastic (conditional) master equation for a single superconducting Cooper-pair box (SCB) charge qubit measured by a single-electron transistor (SET) detector. This stochastic master equation describes the random evolution of the measured SCB qubit density matrix which both conditions and is conditioned on a particular realization of the measured electron tunneling events through the SET junctions. Hence it can be regarded as a Monte Carlo method that allows us to simulate the continuous quantum measurement process. We show that the master equation for the “partially” reduced density matrix [Y. Makhlin *et al.*, Phys. Rev. Lett. **85**, 4578 (2000)] can be obtained when a “partial” average is taken on the stochastic master equation over the fine grained measurement records of the tunneling events in the SET. Finally, we present some Monte Carlo simulation results for the SCB/SET measurement process. We also analyze the probability distribution  $P(m, t)$  of finding  $m$  electrons that have tunneled into the drain of the SET in time  $t$  to demonstrate the connection between the quantum trajectory approach and the “partially” reduced density matrix approach.

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**I. INTRODUCTION**

The single-electron transistor (SET) is a highly charge-sensitive electro-meter and has been suggested as a readout device for solid-state charge qubits<sup>1,2</sup> or spin qubits<sup>3,4</sup> (through a measurement of a spin-dependent charge transfer). The problem of a charge qubit subject to a measurement by a SET has been extensively studied in Refs. 1 and 2. We refer to the approach of these papers as the master equation method of the “partially” reduced density matrix. In this approach, one takes a trace over environmental (detector) microscopic degrees of the freedom but keeps track of the number of electrons,  $m(t)$ , that have tunneled through the SET into the drain during time  $t$  in the “partially” reduced density matrix. If experimentally the number of accumulated electrons or current passing through the SET is measured, this approach can provide us with information about the initial qubit state. But, the system dynamics in this approach is still deterministic, i.e., this approach is still in an ensemble and time average sense.

A Monte Carlo method<sup>5</sup> which allows one to follow each electron tunneling event has been successfully applied to simulate transport properties of a SET or more complicated single electronics circuits. This method gives physical insight into the processes taking place in the simulated system. But to our knowledge, it has not yet been formally applied to quantum measurement problems by a SET detector. In this paper, we provide such an investigation. We derive the *quantum-jump* stochastic master equation (or quantum trajectory equation) for a single superconducting Cooper-pair box (SCB) charge qubit (generalization to other charge qubit case is simple) continuously measured by a SET. This stochastic master equation describes the random evolution of the measured SCB qubit density matrix which both conditions and is conditioned on a particular realization of the measured electron tunneling events through the SET junctions. We can regard it as a Monte Carlo method that allows

us to simulate the continuous quantum measurement process of a charge qubit by a SET. This quantum trajectory approach (or Bayesian formalism) was introduced recently<sup>6–8</sup> to describe a charge qubit measured by a low-transparency point contact detector. Here we present the *quantum-jump* stochastic master equation for the SET detector. Especially, we show that the master equation for the “partially” reduced density matrix (a “partial” course-grain description) presented in Refs. 1 and 2 can be obtained by taking a “partial” average on the stochastic master equation over the fine grained measurement records of the tunneling events in the SET. Finally, we present some Monte Carlo simulation results for the SCB/SET measurement process. We also analyze an important ensemble quantity for an initial qubit state readout experiment,  $P(m, t)$  the probability distribution<sup>1,2</sup> of finding  $m$  electrons that have tunneled into the drain of the SET in time  $t$ . This analysis demonstrates further the connection between the quantum trajectory approach presented here and the “partially” reduced density matrix approach in Refs. 1 and 2.

**II. MODEL HAMILTONIAN**

The Hamiltonian of the SCB/SET system is described in Refs. 1 and 2 as

$$\mathcal{H} = \mathcal{H}_{\text{SET}} + \mathcal{H}_L + \mathcal{H}_R + \mathcal{H}_I + \mathcal{H}_T + \mathcal{H}_{\text{qb}} + \mathcal{H}_{\text{int}}. \quad (1)$$

Briefly,

$$\mathcal{H}_{\text{SET}} = E_{\text{SET}}(N - N_g)^2 \quad (2)$$

describes the charging energy of the SET. The charge on the middle island is  $eN$ , and the induced charge  $eN_g$  is determined by the gate voltage  $V_g$  and other voltages in the circuit,

$$\mathcal{H}_r = \sum_{ks} \epsilon_{ks}^r c_{ks}^{\dagger} c_{ks}^r, \quad (3)$$

where  $r=L,R,I$ , describes microscopic degrees of freedom of noninteracting electrons in the two leads (left and right) and the middle island of the SET, respectively. To make the charge transfer explicit, two ‘‘macroscopic’’ operators,  $e^{\pm i\phi}$  and  $e^{\pm i\psi}$  are included in the tunneling Hamiltonian<sup>1,2</sup> in the SET,

$$\mathcal{H}_T = \sum_{kk's} T_{kk's}^L c_{ks}^{\dagger} c_{k's}^I e^{-i\phi} + \sum_{k'k's} T_{k'k's}^R c_{k's}^{\dagger} c_{k's}^I e^{-i\phi} e^{i\psi} + \text{h.c.} \quad (4)$$

The effective Hamiltonian of the decoupled qubit, written in the charge eigen basis of the number  $n$  of extra Cooper pair on the island of SCB, is

$$H_{\text{qb}} = \frac{1}{2}(E_{\text{ch}}\hat{\sigma}_z - E_J\hat{\sigma}_x), \quad (5)$$

where  $\hat{n} = (1 - \hat{\sigma}_z)/2$  with eigenvalues  $n=0$  or  $1$ . The capacitive Coulomb coupling between the charge on the SET island and that on the SCB qubit is represented by

$$\mathcal{H}_{\text{int}} = 2E_{\text{int}}N\hat{n}. \quad (6)$$

We will consider the case that the leading tunneling process in the SET are sequential transitions between two adjacent charge states  $N$  and  $N+1$  (say,  $N=0$  and  $N+1=1$  states to represent the extra charge on the SET island). This would be the case if the applied transport voltage across the SET is not too high and the temperature is low (for simplicity, we consider the zero temperature case). Since, effectively, only two adjacent charge states  $N=0$  and  $N+1=1$  are considered, the charge transfer operators  $e^{\pm i\phi}$ , in this case, satisfy  $e^{-i\phi}|N\rangle = 0 = e^{i\phi}|N+1\rangle$ ,  $e^{i\phi}|N\rangle = |N+1\rangle$ , and  $e^{-i\phi}|N+1\rangle = |N\rangle$ . The other set of charge transfer operators satisfy  $e^{\pm i\psi}|m\rangle = |m\pm 1\rangle$ , where  $m$  represents the number of electrons that have tunneled into the right lead (drain) of the SET.

### III. MEASUREMENT RECORDS AND CONDITIONAL DENSITY MATRIX

To be able to describe the measured qubit in a pure state continuously, one needs to have the maximum knowledge about the change of its state. When the qubit interacts with (is measured by) the SET, this information is lost to the SET. For example, each time when an electron tunnels onto or off the SET island, it will cause a change (e.g., a phase shift) of the qubit state. One can recover this information lost, provided that a detailed measurement record from the SET is available. The transport of electrons through the SET occurs via real states of the central island, from  $N \rightarrow N+1 \rightarrow N$ . The information of detecting the  $m$ th electron just tunneling into the drain only tells us that the island state now is in the  $|N\rangle$  state. Thus knowing the ‘‘partially’’ reduced density matrix  $\rho_N(m, t)$  at every time<sup>1,2</sup> does not provide us with the full information.

One can imagine that in the transport process, electrons may spend different times in the intermediate  $|N+1\rangle$  state, causing different phase shifts to the qubit. If the record of the

times when electrons tunneling onto and off the island is not available from the measurement results of the SET, our knowledge about the precise qubit state decreases. When this happens, averaging the random dwelling times of electrons on the island over a period of time or over an ensemble of systems will then lead to the decoherence of the qubit. Hence, one needs to have a measurement record which records whether or not an electron tunnels onto or off the central island of the SET at each time interval  $dt$ . This time interval  $dt$  should be much smaller than the typical qubit system evolution or response time so that no information is lost as far as the qubit system evolution is concerned. In this sense, effectively the qubit is continuously monitored or measured.

For this purpose, we introduce  $dN_{Lc}(t)$  and  $dN_{Rc}(t)$  to represent, in the *quantum-jump* case,<sup>7</sup> the number (either zero or one) of tunneling events seen in infinitesimal time  $dt$  through the left and right junctions of the SET, respectively. Throughout the paper, the subscript or superscript  $c$  indicates that the quantity to which it is attached is conditioned on previous measurement results. If no tunneling electron is detected, the result is *null*, i.e.,  $dN_{Lc}(t)=0$  and  $dN_{Rc}(t)=0$ . If there is *detection* of a tunneling electron in time  $dt$ , then  $dN_{Lc}(t)=1$  or  $dN_{Rc}(t)=1$ . We can think of  $dN_{Rc}(t)$  as the increment in the number of electrons  $N_{Rc}(t) = \sum dN_{Rc}(t)$  passing through the right junction of the SET into the drain in the infinitesimal time  $dt$ . It is the variable  $N_R(t) = m(t)$ , the accumulated electron number transmitted through the SET in the drain, which is used in Refs. 1 and 2. Since the nature of detection results is classical and that of electrons tunneling through the SET is stochastic,  $dN_{Lc/Rc}(t)$  should represent a classical random process. The measurement record in each single run of experiment is the set of times  $\{t_L^{(i)}\}$  and  $\{t_R^{(i)}\}$  when electrons tunnel onto or off the SET island, respectively [i.e., ones of  $dN_{Lc}(t)$  and  $dN_{Rc}(t)$  over the entire detection time see, e.g., Figs. 1(i) and 1(j)].

At first, one may expect that at the end of each time interval  $dt$ , there are four possible measurement outcomes,  $dN_{Lc}(t)[1-dN_{Rc}(t)]$ ,  $dN_{Rc}(t)[1-dN_{Lc}(t)]$ ,  $[1-dN_{Lc}(t)][1-dN_{Rc}(t)]$ , and  $dN_{Lc}(t)dN_{Rc}(t)$ . It is important to realize that a null result (e.g.,  $dN_{Lc} = dN_{Rc} = 0$ ) in a time interval  $dt$  is still a measurement result or outcome. Let us consider the case in the sequential tunneling dominated regime that the probability of electrons tunneling onto and off the SET island within the same infinitesimal time interval  $dt$  is rather small. In fact, the respective probability of  $dN_{Lc}(t)$  or  $dN_{Rc}(t)$  equal to unity is proportional to  $dt$  [see Eqs. (27) and (28)]. Thus the product of  $dN_{Lc}(t)dN_{Rc}(t)=1$  occurs with probability proportional to  $dt^2$ . Since we shall keep only terms to order  $dt$  in the master equations, we can neglect the case that  $dN_{Lc}(t)$  and  $dN_{Rc}(t)$  both equal one within the same infinitesimal time interval. The possible measurement outcomes then become  $dN_{Lc}(t)$ ,  $dN_{Rc}(t)$ , and  $[1-dN_{Lc}(t)-dN_{Rc}(t)]$ . The first two terms, in this case, represent that an electron tunneling event through, respectively, the left and right junctions of SET is measured at the end of the time interval  $[t, t+dt)$ . While the last term  $[1-dN_{Lc}(t)-dN_{Rc}(t)]$  represents that no tunneling event is observed in  $[t, t+dt)$ . Thus the conditioned density matrix  $W_c(t+dt)$  to order  $dt$  at

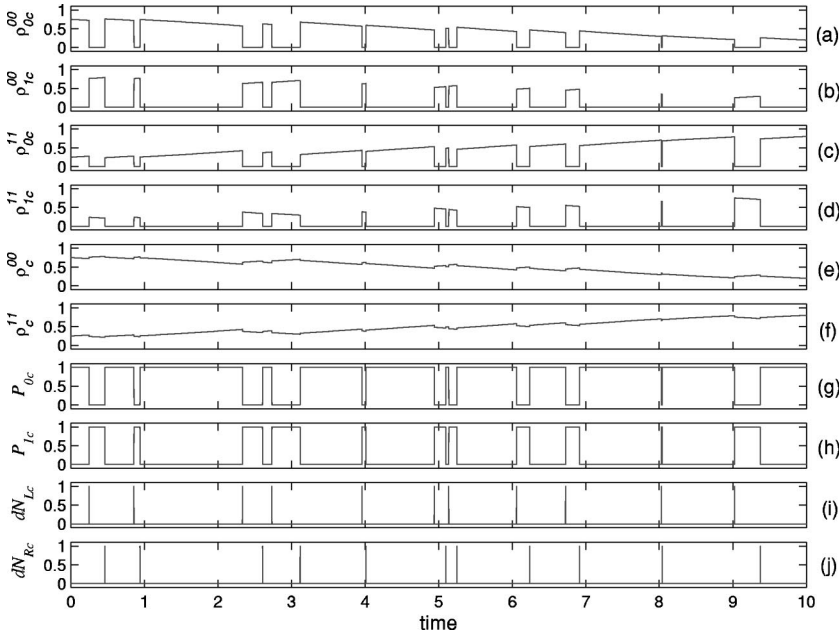


FIG. 1. Set of typical quantum trajectories and corresponding detection record for the initial conditions: qubit state in  $(\sqrt{3}|0\rangle + |1\rangle)/2$  and the SET island in  $N=0$  state. Other parameters are  $E_{\text{int}}=1.5E_{\text{ch}}=1500E_J$ ,  $\Gamma_L=(3/4)\Gamma_R=3E_{\text{ch}}/\hbar$ ,  $\alpha_L=\alpha_R=0.03$  and the time is in unit of  $\hbar/E_{\text{ch}}$ .

the end of the time interval  $[t, t+dt)$  can be written as

$$\begin{aligned}
 W_c(t+dt) = & dN_{Lc}(t) \frac{W_{L1c}(t+dt)}{\text{Tr}[W_{L1c}(t+dt)]} \\
 & + dN_{Rc}(t) \frac{W_{R1c}(t+dt)}{\text{Tr}[W_{R1c}(t+dt)]} + [1 - dN_{Lc}(t) \\
 & - dN_{Rc}(t)] \frac{W_{0c}(t+dt)}{\text{Tr}[W_{0c}(t+dt)]}, \quad (7)
 \end{aligned}$$

where  $W_{L1c}(t+dt)$ ,  $W_{R1c}(t+dt)$ , and  $W_{0c}(t+dt)$  are the unnormalized density matrices, given that an electron tunneling event through left or right junction of the SET island, or no tunneling event is measured at the end of the time interval  $[t, t+dt)$ . Equation (7) simply states that when  $dN_{Lc}=1$  and  $dN_{Rc}=0$ , the normalized conditioned density matrix is  $W_{L1c}(t+dt)/\text{Tr}[W_{L1c}(t+dt)]$ , and so on. Self-consistently, the ensemble averages  $E[dN_{Lc}(t)]$  and  $E[dN_{Rc}(t)]$  of the classical stochastic processes  $dN_{Lc}(t)$  and  $dN_{Rc}(t)$  should equal, respectively, the probabilities (quantum average) of electrons tunneling through the left and right junctions of the SET island in time  $dt$ , i.e.,  $E[dN_{Lc}(t)]=\text{Tr}[W_{L1c}(t+dt)]$  and  $E[dN_{Rc}(t)]=\text{Tr}[W_{R1c}(t+dt)]$ .

Formally, we can write the currents through the junctions as

$$I_{Lc}(t) = e[dN_{Lc}(t)/dt], \quad (8)$$

$$I_{Rc}(t) = e[dN_{Rc}(t)/dt]. \quad (9)$$

The question now is to find expressions for  $W_{L1c}(t+dt)$ ,  $W_{R1c}(t+dt)$ , and  $W_{0c}(t+dt)$  in the model. To do this, we derive the unconditional master equation and then use it to find  $W_{L1c}(t+dt)$ ,  $W_{R1c}(t+dt)$ , and  $W_{0c}(t+dt)$ .

#### IV. STOCHASTIC MASTER EQUATION

Following the same assumptions and approximations in Refs. 1 and 2 and similar derivations in Refs. 7 and 9, we first derive the master equation of “partially” reduced density matrix. By tracing out the microscopic degrees of freedom of the left and right leads and the island of the SET, but keeping the electron transfer operators explicitly<sup>1,2</sup> (so that we can keep track of effects of electron tunneling events on the system density matrix), we obtain the Born-Markov master equation for the “partially” reduced density matrix operator  $\mathcal{W}(t)$  of the SCB/SET system (consisting of the qubit, and the island and drain of the SET) as

$$\begin{aligned}
 [d\mathcal{W}(t)/dt] = & - (i/\hbar)[\mathcal{H}_{\text{qb}} + \mathcal{H}_{\text{int}}, \mathcal{W}(t)] + \Gamma_L \mathcal{D}[e^{i\phi}(1 - \hat{n})]\mathcal{W}(t) \\
 & + \Gamma'_L \mathcal{D}[e^{i\phi}\hat{n}]\mathcal{W}(t) + \Gamma_R \mathcal{D}[e^{-i\phi}e^{i\psi}(1 - \hat{n})]\mathcal{W}(t) \\
 & + \Gamma'_R \mathcal{D}[e^{-i\phi}e^{i\psi}\hat{n}]\mathcal{W}(t) - (\Gamma_L + \Gamma'_L) \\
 & \times [\hat{n}, [\hat{n}, e^{i\phi}\mathcal{W}(t)e^{-i\phi}]]/2 - (\Gamma_R + \Gamma'_R) \\
 & \times [\hat{n}, [\hat{n}, e^{-i\phi}e^{i\psi}\mathcal{W}(t)e^{-i\psi}e^{i\phi}]]/2, \quad (10)
 \end{aligned}$$

where  $\mathcal{D}$  is defined for arbitrary operators  $B$  and  $\mathcal{W}$  as

$$\mathcal{D}[B]\mathcal{W} = \mathcal{J}[B]\mathcal{W} - \mathcal{A}[B]\mathcal{W}, \quad (11)$$

$$\mathcal{J}[B]\mathcal{W} = B\mathcal{W}B^\dagger, \quad (12)$$

$$\mathcal{A}[B]\mathcal{W} = (B^\dagger B\mathcal{W} + \mathcal{W}B^\dagger B)/2. \quad (13)$$

The rates  $\Gamma_{L/R}$  and  $\Gamma'_{L/R}$  represent the tunneling rates (in the left or right junction) with and without the presence of the extra Cooper pair on the island of the SCB (i.e.,  $n=1$  or  $n=0$ ), respectively. They are determined by the chemical potentials  $\mu_{L/R}$  of the leads and the induced charge  $N_g$  on the SET's island,

$$\Gamma_L = (2\pi\alpha_L/\hbar)[\mu_L - (1 - 2N_g)E_{\text{SET}}], \quad (14)$$

$$\Gamma_R = (2\pi\alpha_R/\hbar)[(1 - 2N_g)E_{\text{SET}} - \mu_R], \quad (15)$$

$$\Gamma'_L = \Gamma_L - (4\pi\alpha_L E_{\text{int}}/\hbar), \quad (16)$$

$$\Gamma'_R = \Gamma_R + (4\pi\alpha_R E_{\text{int}}/\hbar), \quad (17)$$

where  $\alpha_{L/R} = R_Q/(4\pi^2 R_{L/R})$ ,  $R_Q = h/e^2$  is the resistance quantum, and  $R_{L/R}$  represents the resistance of the left or right junction.

The master equation (10) has a translational symmetry in  $m$  space. So by summing all possible values of  $m$  of the right reservoir (drain) states completely, a closed form of the master equation of  $W(t) = \sum_m \langle m | \mathcal{W}(t) | m \rangle$  can be obtained. The resultant equation is equivalent to Eq. (10) but with the replacements of  $e^{\pm i\psi} \rightarrow 1$  and  $W(t) \rightarrow \mathcal{W}(t)$ . One may expect to apply the similar sum procedure to the island states. However, since effectively only two extra adjacent charge states  $|N\rangle$  and  $|N+1\rangle$  are considered, a closed form of the master equation for the qubit density matrix operator alone,  $\rho(t) = \langle N | W(t) | N \rangle + \langle N+1 | W(t) | N+1 \rangle \equiv \rho_N(t) + \rho_{N+1}(t)$ , cannot be obtained without further approximations, where  $\rho_{N/N+1}(t)$  is each a  $2 \times 2$  operator in the qubit basis. One approach<sup>9</sup> is to assume extremely asymmetric tunnel junctions for the SET, i.e., one of the tunneling rates through the junctions is much larger than the other. In this case one can apply the adiabatic elimination procedure<sup>9</sup> to eliminate the degrees of freedom of the SET island to obtain the reduced density matrix for the qubit alone. But this asymmetric assumption is equivalent to treating the SET as effectively a single junction device, similar to a point contact detector. Here, however, we take the joint density matrix of the qubit and extra charge on the SET island as the system density matrix in Eq. (7). After evaluating  $W_c(t)$  [or  $\rho_N^c(t)$  and  $\rho_{N+1}^c(t)$ ] from the conditional master equation [see Eqs. (21) and (22)], we can then find the conditional qubit density matrix operator alone by writing  $\rho^c(t) = \text{Tr}_N[W_c(t)] = \rho_N^c(t) + \rho_{N+1}^c(t)$ .

Using the definition of the superoperator  $\mathcal{D}$  and the fact that the charge transfer operators are explicitly kept in each term in Eq. (10), one can then find<sup>7</sup> from there [or more precisely from the master equation for  $W(t)$ ] the unnormalized density matrices, given that an electron tunneling event through left or right junction of the SET island takes place at the end of the time interval  $[t, t+dt)$ , as

$$W_{L1c}(t+dt) = dt\{\Gamma_L \mathcal{A}[e^{i\phi}(1-\hat{n})]W_c(t) + \Gamma'_L \mathcal{A}[e^{i\phi}\hat{n}]W_c(t) - (\Gamma_L + \Gamma'_L)[\hat{n}, [\hat{n}, e^{i\phi}W_c(t)e^{-i\phi}]]/2\}, \quad (18)$$

$$W_{R1c}(t+dt) = dt\{\Gamma_R \mathcal{A}[e^{-i\phi}(1-\hat{n})]W_c(t) + \Gamma'_R \mathcal{A}[e^{-i\phi}\hat{n}]W_c(t) - (\Gamma_R + \Gamma'_R)[\hat{n}, [\hat{n}, e^{-i\phi}W_c(t)e^{i\phi}]]/2\}. \quad (19)$$

It is required that the unconditional (ensemble averaged) density matrix  $E[W_c(t+dt)] = W(t+dt) = W_{L1c}(t+dt) + W_{R1c}(t+dt) + W_0(t+dt)$ . Hence we find  $W_{0c}(t+dt)$ , from the master equation for  $W(t)$ , as

$$W_{0c}(t+dt) = W_c(t) - dt(i/\hbar)[\mathcal{H}_{\text{qb}} + \mathcal{H}_{\text{int}}, W_c(t)] - dt\{\Gamma_L \mathcal{A}[e^{i\phi}(1-\hat{n})] + \Gamma'_L \mathcal{A}[e^{i\phi}\hat{n}] + \Gamma_R \mathcal{A}[e^{-i\phi}(1-\hat{n})] + \Gamma'_R \mathcal{A}[e^{-i\phi}\hat{n}]\}W_c(t). \quad (20)$$

Substituting Eqs. (18)–(20) into Eq. (7) and replacing  $\text{Tr}[W_{0c}(t+dt)] = 1 - \text{Tr}[W_{L1c}(t+dt)] - \text{Tr}[W_{R1c}(t+dt)]$ , then keeping only the terms to order  $dt$  in the resultant equation,<sup>7</sup> and finally evaluating this equation in  $|N\rangle$  and  $|N+1\rangle$  states, respectively, we obtain the conditional master equation

$$d\rho_N^c(t+dt) = -[dN_{Lc}(t) + dN_{Rc}(t)]\rho_N^c(t) + dN_{Rc}(t) \times [\check{\Gamma}_R \rho_{N+1}^c(t)/\mathcal{P}_{R1c}(t) - dt\{(i/\hbar)[\mathcal{H}_{\text{qb}}, \rho_N^c(t)] + \check{\Gamma}_L \rho_N^c(t) - [\mathcal{P}_{L1c}(t) + \mathcal{P}_{R1c}(t)]\rho_N^c(t)\}, \quad (21)$$

$$d\rho_{N+1}^c(t+dt) = -[dN_{Lc}(t) + dN_{Rc}(t)]\rho_{N+1}^c(t) + dN_{Lc}(t) \times [\check{\Gamma}_L \rho_N^c(t)/\mathcal{P}_{L1c}(t) - dt\{(i/\hbar)[\mathcal{H}_{\text{qb}} + 2E_{\text{int}}\hat{n}, \rho_{N+1}^c(t)] + \check{\Gamma}_R \rho_{N+1}^c(t) - [\mathcal{P}_{L1c}(t) + \mathcal{P}_{R1c}(t)]\rho_{N+1}^c(t)\}, \quad (22)$$

where  $\mathcal{P}_{Lc}(t)$  and  $\mathcal{P}_{Rc}(t)$  appearing in Eqs. (21) and (22) are due to the normalization requirement for the density matrix after each detection interval  $dt$  as in Eq. (7), and are given by

$$\mathcal{P}_{L1c}(t) = \Gamma_L \text{Tr}[\rho_N^c(t)] + (\Gamma'_L - \Gamma_L)\text{Tr}[\hat{n}\rho_N^c(t)], \quad (23)$$

$$\mathcal{P}_{R1c}(t) = \Gamma_R \text{Tr}[\rho_{N+1}^c(t)] + (\Gamma'_R - \Gamma_R)\text{Tr}[\hat{n}\rho_{N+1}^c(t)]. \quad (24)$$

The rates  $\check{\Gamma}_L$  and  $\check{\Gamma}_R$  are defined as

$$\check{\Gamma}_L \rho_N^c = \Gamma_L \rho_N^c + (\Gamma_L - \Gamma'_L)\{\hat{n}, \rho_N^c\}/2, \quad (25)$$

$$\check{\Gamma}_R \rho_{N+1}^c = \Gamma_R \rho_{N+1}^c + (\Gamma_R - \Gamma'_R)\{\hat{n}, \rho_{N+1}^c\}/2. \quad (26)$$

Self-consistently,  $E[dN_{Lc}(t)]$  and  $E[dN_{Rc}(t)]$  should equal their respective quantum averages, and from Eqs. (18) and (19) can be written as<sup>10</sup>

$$E[dN_{Lc}(t)] = \text{Tr}[W_{L1c}(t+dt)] = \mathcal{P}_{L1c}(t)dt, \quad (27)$$

$$E[dN_{Rc}(t)] = \text{Tr}[W_{R1c}(t+dt)] = \mathcal{P}_{R1c}(t)dt, \quad (28)$$

where  $\mathcal{P}_{Lc}(t)$  and  $\mathcal{P}_{Rc}(t)$  are defined in Eqs. (23) and (24). Equations (21)–(28) are the main results of the paper. One can use them to simulate the conditional (stochastic) qubit dynamics under continuous quantum measurements by the SET. We will present some simulation results in Sec. VI.

## V. CONNECTION TO “PARTIALLY” REDUCED DENSITY MATRIX

Next, we show that the master equation of the “partially” reduced density matrix, e.g., Eq. 2 of Ref. 2, can be obtained<sup>8</sup> by taking a “partial” average on Eqs. (21) and (22). First, performing a full ensemble average over the observed



stochastic process on Eqs. (21) and (22) by replacing  $E[dN_{Lc}(t)]$  and  $E[dN_{Rc}(t)]$  by their expected values Eqs. (27) and (28), and setting  $E[\rho_N^c(t)] = \rho_N(t)$ , we obtain the master equation for the reduced density matrix  $\rho_N(t)$  and  $\rho_{N+1}(t)$  as

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \rho_N(t) \\ \rho_{N+1}(t) \end{pmatrix} + \frac{i}{\hbar} \begin{pmatrix} [\mathcal{H}_{qb}, \rho_N(t)] \\ [\mathcal{H}_{qb} + 2E_{in}\hat{n}, \rho_{N+1}(t)] \end{pmatrix} \\ = \begin{pmatrix} -\check{\Gamma}_L & \check{\Gamma}_R \\ \check{\Gamma}_L & -\check{\Gamma}_R \end{pmatrix} \begin{pmatrix} \rho_N(t) \\ \rho_{N+1}(t) \end{pmatrix}. \end{aligned} \quad (29)$$

Then to keep track of the number of electrons  $m=N_R$  that have tunneled into the drain, we need to identify the terms in Eq. (29), which come from Eqs. (21) and (22) and have effects corresponding to an electron tunneling through the right junction of the SET. Only one such term, originating from  $dN_{Rc}(t)[\check{\Gamma}_R \rho_{N+1}^c(t)/\mathcal{P}_{Rc}(t)]$  in Eq. (21), survives in Eq. (29). It is in the upper right corner of the matrix on the right-hand side of Eq. (29). If  $m$  electrons have tunneled through the right junction of the SET at time  $t+dt$ , then the accumulated number of electrons in the drain at the earlier time  $t$ , due to the contribution of the *jump* term through the right junction, should be  $(m-1)$ . Hence, after writing out the number dependence  $m$  or  $(m-1)$  explicitly for the density matrix in Eq. (29), we obtain<sup>11</sup> the master equation for the “partially” reduced density matrix as

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \rho_N(m,t) \\ \rho_{N+1}(m,t) \end{pmatrix} + \frac{i}{\hbar} \begin{pmatrix} [\mathcal{H}_{qb}, \rho_N(m,t)] \\ [\mathcal{H}_{qb} + 2E_{in}\hat{n}, \rho_{N+1}(m,t)] \end{pmatrix} \\ = \begin{pmatrix} -\check{\Gamma}_L \rho_N(m,t) + \check{\Gamma}_R \rho_{N+1}(m-1,t) \\ \check{\Gamma}_L \rho_N(m,t) - \check{\Gamma}_R \rho_{N+1}(m,t) \end{pmatrix}. \end{aligned} \quad (30)$$

Making a Fourier transform  $\rho_{N/N+1}(k,t) = \sum_m e^{-ikm} \rho_{N/N+1}(m,t)$  on Eq. (30), we find that the resultant equation is exactly the same as Eq. 2 of Ref. 2. If the sum over all possible values of  $m$  is taken on the “partially” reduced density matrix [i.e., tracing out the detector states completely,  $\rho_N(t) = \sum_m \rho_N(m,t)$ ], Eq. (30) then reduces to the master equation of the reduced density matrix, Eq. (29). This procedure of reducing Eqs. (21) and (22) to Eq. (30) and then to Eq. (29), by successively disregarding information that distinguishes different states of the detector, provides a connection between the approach of Refs. 1 and 2 and the more detailed stochastic master equation used here. To further demonstrate this connection, we analyze in the next section an important ensemble quantity for an initial qubit state read-out experiment,  $P(m,t)$ , the probability distribution of finding  $m(=N_R)$  electrons that have tunneled into the drain in time  $t$ , considered in Ref. 1.

## VI. SIMULATION RESULTS AND DISCUSSIONS

Although the “partially” reduced density matrix approach<sup>1,2</sup> can provide the information about the initial qubit state, the system dynamics in this approach is still deterministic; i.e., this approach is still in an ensemble and time average sense. If a measurement is made on the qubit system

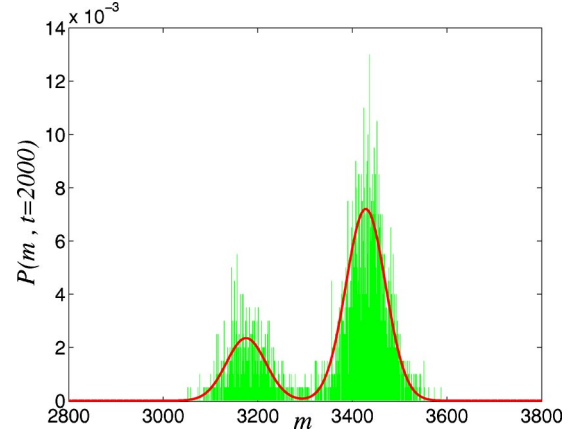


FIG. 2. (Color online) Probability distribution  $P(m, t=2000)$ . The initial conditions and parameters are the same as those in Fig. 1.

and the results are available, the state or density matrix is conditioned on the measurement results. If the subsequent system evolution after the measurement is concerned, the conditional or quantum trajectory approach should be employed. In particular, to describe the conditional dynamics of the qubit system in a single realization of continuous measurements, which reflects the stochastic nature of electrons tunneling through the SET junctions, we should use the conditional, stochastic master equations (21)–(28).

A set of typical quantum trajectories, generated using Eqs. (21)–(28), is shown in Figs. 1(a)–1(h) and its corresponding randomly distributed moments of detections are presented in Figs. 1(i) and 1(j), where  $\rho^{mm'} \equiv \langle n|\rho|n' \rangle$ . Due to Coulomb blockade, when an electron is on the SET island, Eqs. (21)–(28) ensure that no electron can tunnel onto the SET island, i.e., guarantee  $dN_{Lc}=0$  for the next time interval  $dt$ . Note that in this case, we still have two possible measurement outcomes of either  $dN_{Rc}=1$  or 0 in the next time interval  $dt$ . But only after a detection of  $dN_{Rc}=1$  in the next or the next several time interval(s), could another electron tunnel onto the SET island again. This is the measurement record shown in Figs. 1(i) and 1(j). They are in the order of exactly alternating  $dN_{Lc}=1$  and  $dN_{Rc}=1$  time sequence. The conditional evolutions of the qubit alone shown in Figs. 1(e) and 1(f) can be obtained from the sum of the joint state evolutions of Figs. 1(a) and 1(b), or Figs. 1(c) and 1(d), respectively. The probabilities,  $P_{0/1,c} = \text{Tr}_{qb}[\rho_{0/1,c}] = \rho_{0/1,c}^{00} + \rho_{0/1,c}^{11}$ , of the SET island state alone in Figs. 1(g) and 1(h) can be obtained by summing the evolutions in Figs. 1(a) and 1(c) or Figs. 1(b) and 1(d) respectively. The conditional evolutions in Figs. 1(a)–1(h) differ considerably from their ensemble average counterparts.

In this conditional or quantum trajectory approach, we are propagating in parallel the information of the conditioned (stochastic) state evolution and detection record of  $dN_{Lc}(t)$  and  $dN_{Rc}(t)$  in a single run of a continuous measurement process. One can see that in this approach the instantaneous system state conditions the measured electron tunneling events through the SET junctions [see Eqs. (27) and (28)], while the measured electron tunneling events themselves

condition the future evolution of the measured system [see Eqs. (21) and (22)] in a self-consistent manner. Each set of quantum trajectories (stochastic state evolutions), obtained from the stochastic master equations (21)–(28), mimics a single history of the qubit state in a single run of the continuous measurement experiment. The stochastic element in the quantum trajectories corresponds exactly to the consequence of the random outcomes of the detection record of the tunneling events in the SET. The macroscopic ensemble measurement properties can be calculated by using large ensembles of single electron tunneling events (fine grained measurement records).

If only one measurement value is recorded in each run of experiments [for example, the number of electrons  $m$  that have tunneled into the right lead (drain) in time  $t$ ] and ensemble average properties [for example,  $P(m, t)$ , the probability distribution of finding  $m=(N_R)$  electrons that have tunneled through the right junction into the drain in time  $t$ ] are studied<sup>1,2</sup> over many repeated experiments, the quantum trajectory approach will give the same result as the master equation approach of the “partially” reduced density matrix.<sup>8</sup> However, more physical insight can be gained in the approach of quantum trajectories. We demonstrate this feature below.

We consider the case that the SCB/SET system is in the so-called Zeno regime<sup>1,2,7,8</sup> where the mixing time is much larger than the measurement (localization) time. In this regime, a good quantum readout measurement for an initial qubit state in the charge-state basis can be performed by repeatedly measuring the number of electrons,  $m(t)=N_R(t)$ , that have tunneled through the right junction into the drain of the SET in the same detection time period  $t$ . One can then use the measurement results to construct the probability distribution  $P(m, t)$ . In Ref. 1, the probability distribution  $P(m, t)=\text{Tr}_{\text{qb}}[\rho_N(m, t)+\rho_{N+1}(m, t)]$  is obtained from the Fourier analysis of the partially reduced density matrix, Eq. (30). The result, obtained in this way, is plotted in solid line in Fig. 2. This probability distribution  $P(m, t)$  splits into two and their weights correspond closely to the initial qubit diagonal elements of  $\rho^{11}(0)=0.25$  and  $\rho^{00}(0)=0.75$ .

In the quantum trajectory approach,  $P(m, t)$  can be explicitly simulated through constructing the histogram of the

accumulated number of electrons  $N_{Rc}=\sum dN_{Rc}$  up to time  $t$  for many realizations of the detection records (generated together with their corresponding quantum trajectories), and then normalizing the distribution to one. The simulation of the normalized histogram shown in Fig. 2 using 2000 quantum trajectories and their corresponding detection records is already in good agreement with the plot in solid line. However, the possible individual realizations of measurement records and their corresponding quantum trajectories (e.g., in Fig. 1) do provide insight into, and aid in the interpretation of the ensemble average properties. This is one of the appealing features of the quantum trajectory approach.

For a charge qubit measured by a low-transparency point contact detector, this appealing feature of the quantum trajectories is illustrated in Ref. 8. Another advantage of the quantum trajectory approach (or Bayesian formalism) is that it may describe a quantum feedback process. It has been shown<sup>12</sup> that one may utilize the measurement output for the feedback control and manipulation of a qubit state.

## VII. CONCLUSION

To summarize, we have derived the stochastic master equation for the SCB/SET system, which can be regarded as a Monte Carlo method that allows us to simulate the continuous quantum measurement process of the SCB qubit by the SET. We have shown that by taking a “partial” average over the fine grained measurement records of the tunneling events in the SET, this stochastic master equation reduces to the master equation presented in Refs. 1 and 2. We have also presented numerical simulation for the dynamics of the qubit in a particular realization of the readout measurement experiment. We have shown that the probability distribution  $P(m, t)$  constructed from 2000 quantum trajectories and their corresponding detection records, is, as expected, in good agreement with that obtained from the Fourier analysis of the “partially” reduced density matrix.

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<sup>10</sup>The last term,  $\{-(\Gamma_L+\Gamma'_L)[\hat{n},[\hat{n},e^{i\phi}W_c(t)e^{-i\phi}]]/2\}dt$ , in Eq. (18) and that,  $\{-(\Gamma_R+\Gamma'_R)[\hat{n},[\hat{n},e^{-i\phi}W_c(t)e^{i\phi}]]/2\}dt$ , in Eq. (19) describe effects of a random phase shift on the measured qubit state, caused by an electron when it tunnels onto or off the SET island. However, these two terms are traceless in the qubit basis and thus do not contribute to the average current

$I_{L/R}(t)=eE[dN_{Lc/Rc}(t)]/dt$ , in Eqs. (27) and (28).

<sup>11</sup>Alternatively, Eq. (30) can be obtained directly by evaluating Eq. (10) in  $|m\rangle|N\rangle$ , and  $|m\rangle|N+1\rangle$  states. Similarly, Eq. (29) can be obtained directly by evaluating the master equation for  $W(t)$  in  $|N\rangle$  and  $|N+1\rangle$  states.

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