Quantum statistical theory of the fermionic quantum Hall effect

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At the even-denominator filling factor $\nu=P/Q$, even Q, a fermionic Quantum Hall (QH) state is normally developed in a pure heterojunction GaAs/AlGaAs at the lowest temperatures. The QH state at $\nu=5/2$ is an exception, and it is known to be bosonic. The fermionic state is formed by the composite (c-) fermions, each with an electron and Q flux quanta (fluxons). The conductivity $\sigma\equiv J/E$, J= current density, E= applied field, becomes a universal constant $(e^2/h)Q^{-1}$ as the temperature approaches zero while the Hall conductivity $\sigma_H \equiv J/E_H$, E_H=Hall field, becomes approximately equal to $(e^2/h)P/Q$. The widths in the resistivity $\rho\equiv\sigma^{-1}$ are symmetric with respect to high and low fields. They are temperature dependent. The resistivity (ρ) curve smoothly approaches a constant as the field is lowered toward zero.

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I. INTRODUCTION

In 1980 von Klitzing *et al.*[1] reported a discovery of the integer Quantum Hall Effect (QHE). In 1982 Tsui *et al.*[2] discovered the fractional QHE. Figure 1 represents the data reported by Tsui[3] for the Hall resistivity $\rho_H\equiv E_H/J$ (E_H= Hall field, J= current density) and the resistivity $\rho\equiv E/J$ (E = applied field) in heterojunction GaAs/AlGaAs at 60 mK. The Quantum Hall (QH) states at the Landau Level (LL) occupation ratio (filling factor) $\nu=P/Q=1,2,\ldots,5$ are visible. Each bosonic QH state with the Hall resistivity plateau (horizontal stretch) is accompanied by zero resistance (superconducting). In 1987 Willett *et al.*[4] discovered the even-denominator QHE at $\nu=5/2$. Their data, Ref. 4, Fig. 2, show that the resistivity has clear dips at 25, 40, and 100 mK. Jiang *et al.*[5] observed similar fermionic QHE at $\nu=1/2,3/2,3/4,\ldots$, where each dip converges to a point. In 1999 Pan *et al.*[6] found that the 5/2 state in high mobility sample shows zero resistance with a visible Hall resistivity plateau at very low electron temperature (~4 mK). This 5/2 state is rather similar to the bosonic QH states at odd-denominator ratio $\nu=P/Q$, odd Q, observed in the same sample GaAs/AlGaAs. Eisenstein *et al.*[7] found that the QHE state at $\nu=5/2$ collapses rapidly as the magnetic field is tilted away from the normal to the plane. This is an anomaly since the states at $\nu=1/2$ and $\nu=3/2$ were found not to collapse.[5] The tilting reduces the diamagnetic effect, rendering the effective g factor to vanish and making spin-mixing more important. We shall treat the 5/2 state collapse in a separate publication. The difference between the fermionic and bosonic QHE was clearly demonstrated in the surface-acoustic wave (SAW) propagation study by Willett *et al.*,[8] where the SAW amplitude deviation is negative (positive) for the fermionic (bosonic) QHE, suggesting different charge carriers present in the system.

The departure point for all theories for the fractional QHE is the Laughlin ground-state wave function.[9] Laughlin[9] and Haldane[10] showed that the qasiparticle (elementary excitation) over the Laughlin ground state at $\nu=P/Q$, odd Q, can have the fractional charge

$$e_b = e/Q \quad \text{for} \quad \text{c-bosons}. \tag{1}$$

This surprising prediction was later confirmed by experiments.[11] The system ground-state does not carry a current. To interpret the experimental data it is convenient to introduce composite (c-) particles (bosons, fermions). The c-boson (fermion), each containing an electron and an odd (even) number of flux quanta (fluxons), were introduced by Zhang *et al.*[12] and others (Jain[13]) for the description of the fractional QHE (Fermi liquid). Originally the c-particle was introduced as a composite of one electron attached with a number of Chern-Simons gauge objects. These objects are neither bosons nor fermions, and hence the statistics of the composite is not clear. The basic particle property (countability) of the fluxons is known as the flux quantization, see Eq. (6). We assume that the fluxon is an elementary fermion with

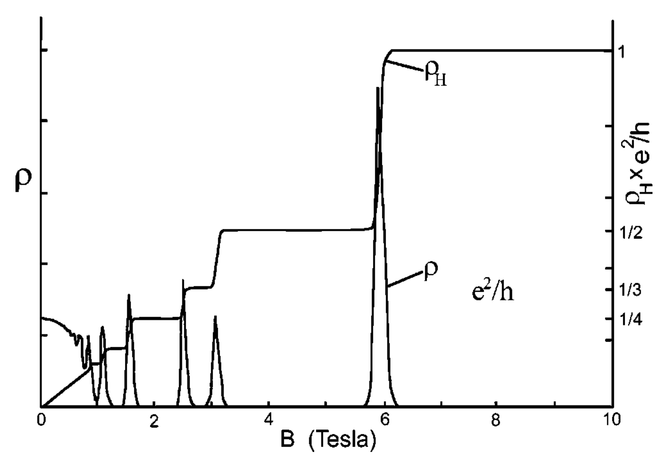

FIG. 1. Observed QHE in GaAs/AlGaAs heterojunction at 60 mK, after Tsui (Ref. 3). The Hall resistivity ρ_H and the resistance ρ are shown as a function of the magnetic field B in tesla.

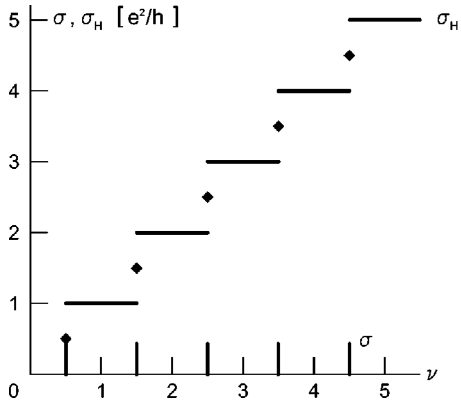

FIG. 2. The conductivity σ and Hall conductivity σ_H as functions of the filling factor $\nu = P/Q$ at 0 K $Q=2$.

zero mass and zero charge, which is supported by the fact that the fluxon, the quantum of the magnetic field **B**, cannot disappear at a sink unlike the bosonic photon, the quantum of the electric field **E**.[14] Fujita and Morabito[15] showed that the center-of-mass (CM) of the composite moves following the Ehrenfest-Oppenheimer-Bethe's (EOB) rule: the composite is fermionic (bosonic) if it contains an odd (even) number of elementary fermions. Hence the quantum statistics of the c-particle is justified. Halperin, Lee and Read reviewed the state of the matter at $\nu = 1/2$ in 1993.[16] In particular they showed that the SAW amplitude sign at $\nu = 1/2$ can be explained with the assumed existence of the c-fermions. They arrived at an expression for the resistivity ρ proportional to the impurity density $n_{\rm imp}$ based on a c-fermionic model. Analyzing the data reported by Tsui,[3] Hajdu *et al.*[17] predicted that the conductivity σ in the integer QHE region is quantized: $\sigma \sim e^2/h$ at the even-denominator filling factor, see Fig. 2.

Laughlin pointed out a remarkable similarity between the QHE and the high-temperature superconductivity (HTSC), both occurring in two-dimensional (2D) systems.[18] The major superconducting properties observed in the HTSC are (a) zero resistance, (b) a sharp phase change at the critical temperature T_c, (c) the energy gap below T_c, (d) the flux quantization, (e) Meissner effect, and (f) Josephson effects. The Josephson effects can be observed in the double-layer QHE systems.[19] All others have been observed in single layer GaAs/AlGaAs. (The Hall resistivity plateau and the Meissner effect are directly connected, see Sec. IV.) Following Bardeen, Cooper, and Schrieffer (BCS),[20] we regard the phonon exchange attraction as the causes of both effects. Starting with a reasonable Hamiltonian, we calculate everything using the standard statistical mechanical methods. Fujita *et al.* developed an electron-fluxon model,[14] in which the electron and fluxons are bound by the phonon excahnge attraction.

Classically speaking, if the magnetic field is applied slowly, the electron can continuously change from the straight line motion at zero field to the circulating motion at a finite B. Quantum mechanically, the change from the momentum state to the Landau state requires a perturbation. We choose the phonon exchange between the electron and the fluxon for the perturbation. For example the c-fermion with two fluxons can be bound as follows: If the B-field is applied adiabatically the energy of the electron does not change but the cyclotron motion always acts so as to reduce the magnetic fields. Hence the total energy of the c-fermion is less than the electron energy and the unperturbed field energy. In other words the c-fermion is stable against the break-up.

In the present work we shall show that a fermionic QH state is developed at $\nu = P/Q$, even Q, in which the conductivity $\sigma \equiv \rho^{-1}$ approaches a universal constant $(e^2/h)Q^{-1}$ as $T \to 0$ and the Hall conductivity $\sigma_H = \rho_H^{-1}$ becomes approximately equal to $(e^2/h)(P/Q)$, thus confirming the prediction by Hajdu *et al.* We also show that the resistivity widths are symmetric with respect to high and low fields. The resistivity curve smoothly approaches a constant as the field is reduced to zero.

II. THE HAMILTONIAN

Let us take a dilute system of electrons moving in the plane. Applying a magnetic field **B** perpendicular to the plane, each electron will be in the Landau state with the energy given by $E = (N_L + 1/2)\hbar\omega_0$, $\omega_0 \equiv eB/m^*$, $N_L = 0, 1, 2, \ldots$. In this state the electron can be viewed as circulating around the guiding center. The radius of circulation $l \equiv (\hbar/eB)^{1/2}$ for the Landau ground state is about 81 Å at a typical field 10 T (tesla). We now apply a weak electric field **E** in the x-direction. With the scatterers (impurities, phonons) present in the system the guiding centers can jump from place to place preferentially and generate a current in the x-direction.

GaAs forms a zinc blende lattice. We assume that the interface is in the plane (001). The Ga^{3+} ions form a square lattice with the sides directed in $[110]$ and $[1\bar{1}0]$. The "electron" (wave packet) will then move isotropically with an effective mass m_1. The As^{3-} ions also form a square lattice at a different height in [001]. The "holes," each having a positive charge, will move similarly with an effective mass m_2. The electron and the phonon share the same 2D k-space and the same Brillouin zone, and hence they have close affinity. The 2D electron k-vector couples with the 2D phonon k-vector, see below. A longitudinal ionic-lattice wave (phonon) moving in $[110]$ or in $[1\bar{1}0]$ can generate a charge density (current) variations, establishing an interaction between the phonon and the electron (fluxon). We note that the 2D current generates a magnetic moment in the z-direction, which interact with the magnetic flux while the 2D charge density generates a varying electric potential, which affects the electron motion. If one phonon exchange is considered between the electron and the fluxon, a second-order perturbation calculation establishes an effective electron-fluxon interaction[14]

$$V_{\rm ef} \equiv |V_q V_q'| \frac{\hbar\omega_q}{(\epsilon_{|\mathbf{k}+\mathbf{q}|s} - \epsilon_{ks})^2 - \hbar^2\omega_q^2}, \qquad (2)$$

where V_q (V_q') is the electron (fluxon)-phonon interaction strength; the Landau quantum number N_L is omitted; the bold **k** denotes the 2D guiding center momentum and the italic k the magnitude. Briefly the electron emits a phonon of momentum $-\mathbf{q}$ and undergoes a transition in the momentum from $\mathbf{k}$ to $\mathbf{k}+\mathbf{q}$, and subsequently the phonon is absorbed by the fluxon which undergoes a transition from $\mathbf{k}'$ to $\mathbf{k}'-\mathbf{q}$. In the second process the fluxon emits a phonon of momentum $\mathbf{q}$, which is absorbed by the electron which undergoes a transition from $\mathbf{k}$ to $\mathbf{k}+\mathbf{q}$. These two elementary processes contribute to the interaction V_{ef}. The interaction is attractive when the electron states before and after the exchange have the same energy as in the degenerate LL so that $V_{\text{ef}} = -|V_q V'_q|(\hbar\omega_q)^{-1}$. BCS assumed that in spite of the Coulomb interaction among electrons, there exists a sharp Fermi surface for the normal state of a conductor, as described by the Fermi liquid model of Landau. The phonon exchange can generate bound singlet pairs of electrons near the Fermi surface within a distance (energy) equal to Planck's constant $\hbar$ times the Debye frequency ω_D. In our case we assume that the phonon exchange generates a bound c-particle out of the electron and fluxons.

Following BCS,[20] we start with a Hamiltonian H with the phonon variables eliminated:

$$H = \sum_{\mathbf{k}}\sum_{s}\sum_{j} \epsilon_k^{(j)} n_{\mathbf{k}s}^{(j)} - v_0 {\sum_{\mathbf{q}}}' {\sum_{\mathbf{k}}}' {\sum_{\mathbf{k}'}}' \sum_{s} \Big[B_{\mathbf{k}'\mathbf{q}s}^{(1)\dagger} B_{\mathbf{k}\mathbf{q}s}^{(1)} + B_{\mathbf{k}'\mathbf{q}s}^{(1)\dagger} B_{\mathbf{k}\mathbf{q}s}^{(2)\dagger} + B_{\mathbf{k}'\mathbf{q}s}^{(2)} B_{\mathbf{k}\mathbf{q}s}^{(1)} + B_{\mathbf{k}'\mathbf{q}s}^{(2)} B_{\mathbf{k}\mathbf{q}s}^{(2)\dagger} \Big], \tag{3}$$

where $n_{\mathbf{k}s}^{(j)}$ is the number operator for the "electron" (1) ["hole" (2), fluxon (3)] at momentum $\mathbf{k}$ and spin s with the energy $\epsilon_{ks}^{(j)}$. We represent the "electron" ("hole") number $n_{\mathbf{k}s}^{(j)}$ by $c_{\mathbf{k}s}^{(j)\dagger} c_{\mathbf{k}s}^{(j)}$, where c $(c^\dagger)$ are annihilation (creation) operators satisfying the Fermi anticommutation rules: $\{c_{\mathbf{k}s}^{(i)}, c_{\mathbf{k}'s}^{(j)\dagger}\} \equiv c_{\mathbf{k}s}^{(i)} c_{\mathbf{k}'s'}^{(j)\dagger} + c_{\mathbf{k}'s'}^{(j)\dagger} c_{\mathbf{k}s}^{(i)} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{s,s'}\delta_{i,j}$, $\{c_{\mathbf{k}s}^{(i)}, c_{\mathbf{k}',s'}^{(j)}\} = 0$. We represent the fluxon number $n_{\mathbf{k}s}^{(3)}$ by $a_{\mathbf{k}s}^\dagger a_{\mathbf{k}s}$, with $a(a^\dagger)$, satisfying the anticommutation rules. $B_{\mathbf{k}\mathbf{q}\,s}^{(1)\dagger} \equiv c_{\mathbf{k}+\mathbf{q}/2\ s}^{(1)\dagger} a_{-\mathbf{k}+\mathbf{q}/2-s}^\dagger$, $B_{\mathbf{k}\mathbf{q}\ s}^{(2)} \equiv c_{\mathbf{k}+\mathbf{q}/2s}^{(2)} a_{-\mathbf{k}+\mathbf{q}/2-s}$. The prime on the summation means the restriction: $0 < \epsilon_{ks}^{(j)} < \hbar\omega_D$, ω_D=Debye frequency. If the fluxons are replaced by the conduction electrons ("electrons," "holes") our Hamiltonian H is reduced to the original BCS Hamiltonian, Eq. (24) of Ref. 20. The "electron" and "hole" are generated, depending on the energy contour curvature sign.[21] For example only "electrons" ("holes"), are generated for a circular Fermi surface with the negative (positive) curvature whose inside (outside) is filled with electrons. Since the phonon has no charge, the phonon exchange cannot change the net charge. The pairing interaction terms in Eq. (3) conserve the charge. The term $-v_0 B_{\mathbf{k}'\mathbf{q}s}^{(1)\dagger} B_{\mathbf{k}\mathbf{q}s}^{(1)}$, where $v_0 \equiv V_{\text{ef}}/A$, A= sample area, is the pairing strength, generates a transition in the "electron" states. Similary, the exchange of a phonon generates a transition in the "hole" states, represented by $-v_0 B_{\mathbf{k}'\mathbf{q}s}^{(2)} B_{\mathbf{k}\mathbf{q}s}^{(2)\dagger}$. The phonon exchange can also pair-create and pair-annihilate "electron" ("hole")-fluxon composites, represented by $-v_0 B_{\mathbf{k}'\mathbf{q}s}^{(1)\dagger} B_{\mathbf{k}\mathbf{q}s}^{(2)\dagger}$, $-v_0 B_{\mathbf{k}'\mathbf{q}s}^{(2)} B_{\mathbf{k}\mathbf{q}s}^{(1)}$. At 0 K the system can have equal numbers of $-(+)$c-bosons, "electron" ("hole") composites, generated by $-v_0 B_{\mathbf{k}'\mathbf{q}s}^{(1)\dagger} B_{\mathbf{k}\mathbf{q}s}^{(2)\dagger}$.

III. THE FERMIONIC QUANTUM HALL EFFECT

We consider the state at $\nu = P/Q$, even Q, where a number of c-fermions with Q fluxons are formed. Each c-fermion can be viewed as an electron circulating about Q elementary fluxes. By applying the relativity principle we can also view it as an electron attached with Q fluxons. This is natural since the guiding center coincides with the CM of the c-particle. The CM of the c-fermion can move uninfluenced by the applied magnetic field since all flux lines are attached to the electrons. This justified Jain's effective magnetic field B^*, see below Eq. (27). Halperin, Lee, and Read[16] used a Chern-Simmons gauge field to show that $B^*=0$ in the average. Fluctuations in the gauge field are difficult to treat. We avoided this problem in our model.

We note that our Hamiltonian in Eq. (3) can generate and stabilize the c-particles with an arbitrary number of fluxons. For example a c-fermion with two fluxons is generated by two sets of the ladder diagram bindings, each between the electron and the fluxon. The ladder diagram binding arises as follows. Consider a hydrogen atom. The Hamiltonian contains kinetic energies of the electron and the proton and the attractive Coulomb interaction. If we regard the Coulomb interaction as a perturbation and use a perturbation theory, we can represent the interaction process by an infinite set of ladder diagrams, each ladder step connecting the electron and the proton. The energy eigenvalues of this system is not obtained by using the perturbation theory but they are obtained by directly solving the Schrödinger equation. This example indicates that the binding energy (the negative of the ground-state energy) is calculated by a nonperturbative method.

Applying kinetic theory to the guiding-center motion of the c-fermion, we obtain the conductivity

$$\sigma = \frac{(e_f)^2 n}{m^*}\frac{1}{\omega}, \tag{4}$$

where n is the fermion density, e_f the charge (magnitude), and ω the relaxation rate. For high-purity samples at very low temperatures ($\sim$60 mK) the impurity and phonon scatterings are negligible. By energy-time uncertainty principle the c-fermion can spend a short time at an upper LL and come back to the ground LL with a different guiding center, thus causing a guiding center jump. We assume that the relaxation rate is the natural linewidth arising from the LL separation divided by $\hbar$, that is, the cyclotron frequency ω_0,

$$\omega = \omega_0. \tag{5}$$

Using Eqs. (4) and (5) and the flux quantization

$$B = n_\phi (h/e), \quad n_\phi = \text{flux density}, \tag{6}$$

we obtain

$$\sigma = \frac{(e_f)^2 n}{m^*(e_f B/m^*)} = \frac{e_f n}{n_\phi (h/e)} = \frac{e e_f}{h}\frac{n}{n_\phi}. \tag{7}$$

The fluxon number conservation requires that

$$Qn = n_{\phi}. \tag{8}$$

The magnetic focusing experiments by Goldman *et al.*[22] indicate that the charge (magnitude) of the c-fermion with two fluxons is e. We assume for any c-fermion with Q (even) fluxons that

$$e_f = e \quad \text{for c-fermions}. \tag{9}$$

We obtain from the last three equations

$$\sigma = (e^2/h)Q^{-1}. \tag{10}$$

In the Hall effect experimental condition we have

$$E_H = v_d B, \tag{11}$$

where v_d is the drift velocity. Using the standard formula for the current density,

$$J = e_f n v_d, \tag{12}$$

we calculate the Hall resistivity,

$$\rho_H \equiv \frac{E_H}{J} = \frac{v_d B}{e_f n v_d} = \frac{B}{e_f n}. \tag{13}$$

This formula indicates that ρ_H is linear in B. First let us consider the case $Q=2$. Normalizing the field relative to the field $B_{1/2}$ at $\nu=1/2$, we may write $B_\nu \equiv B_{P/2} = B_{1/2}/P$. Using this and Eqs. (6), (9), and (13), we obtain

$$\rho_{H,P/2} = (2/P)(h/e^2). \tag{14}$$

The theory can simply be extended to the c-fermions, each with Q fluxons, at $\nu=P/Q$, even Q. The conductivity σ is given by Eq. (10), and the Hall conductivity $\sigma_H \equiv \rho_H^{-1}$ is

$$\sigma_H = (P/Q)(e^2/h). \tag{15}$$

Our results, Eqs. (10) and (15), are illustrated in Fig. 2, where we choose $Q=2$. This figure is essentially the same as Fig. 2.1 in the book by Hajdu *et al.*[17] Only the factor Q^{-1} for formula (10) is determined explicitly in the present work.

So far we neglected the electron spin. The spin effect is important for the 5/2 problem, which will be discussed separately.

IV. DISCUSSION

Jiang *et al.*[5] oberved that the resistivity ρ at $\nu=1/2$ converges to a point below 1 K, see Ref. 5, Fig. 2. It is highly desirable to experimentally check this property in particular at $\nu=3/2, 7/2$ and also at $\nu=1/4, 3/4$. Jiang *et al.*[5] found that the strength (deviation from the background) of the resistivity minimum changes approximately linearly with the temperature in $2 \sim 10$ K, where almost all features of the fractional QHE disappear. This T-linear behavior should arise from the phonon scattering, which we have neglected in our theory. In fact we see from Eq. (2) that the phonon scattering generates a T-linear relaxation rate ($\omega \propto T$) so that

$$\rho \propto T \tag{16}$$

in agreement with the experiment.

Formula (10) indicates the inverse linear Q-dependence and the P-independence for the conductivity σ. Both behaviors appear to be borne in the experimental observation by Jiang *et al.*,[5] Fig. 1. The resistivity minima at $\nu=1/2$ and $3/2$ are approximately equal, and they are smaller than the minimum at $\nu=3/4$. Further detailed experimental confirmation is required here.

Let us now turn to the Hall resistivity ρ_H. Here the difference between the fermionic and bosonic QHE becomes transparent.

The c-bosons, each with one fluxon, will be called the fundamental (f) c-bosons. Their energies $w_q^{(j)}$ are obtained from[14]

$$w_q^{(j)}\Psi^{(j)}(\mathbf{k},\mathbf{q}) = \epsilon^{(j)}_{|\mathbf{k}+\mathbf{q}|}\Psi^{(j)}(\mathbf{k},\mathbf{q}) - (2\pi\hbar)^{-2}v_1 \int' d^2k' \ \Psi^{(j)} \times(\mathbf{k}',\mathbf{q}), \tag{17}$$

where $\Psi^{(j)}(\mathbf{k},\mathbf{q})$ is the reduced wave function; we neglected the fluxon energy. The energy $w_q^{(j)}$ is negative, which is obtained after an indefinite number of phonon exchanges between the electron and fluxon, each generated by $B^{(j)\dagger}_{\mathbf{k}'\mathbf{q}s}B^{(j)}_{\mathbf{k}\mathbf{q}s}$, called the ladder-binding process. The v_1 represents the attraction strength after the ladder diagram binding. Briefly, start with the equation of motion for $B^{(j)\dagger}_{\mathbf{k}\mathbf{q}s}$. Multiply this equation from the right by the energy-state annihilation operator $\phi_{\mathbf{q}}$ and a density operator ρ. Taking a grand ensemble trace (Tr) and defining $\mathrm{Tr}\{B^{(j)\dagger}_{\mathbf{k}\mathbf{q}s}\phi_{\mathbf{q}}\rho\}$ as the reduced wave function $\Psi^{(j)}(\mathbf{k},\mathbf{q})$, we obtain the left-hand side term $w_q^{(j)}\Psi^{(j)}(\mathbf{k},\mathbf{q})$, the spin-index omitted. The left-hand side can be obtained after evaluating the commutator $[H, B^{(j)\dagger}_{\mathbf{k}\mathbf{q}s}]$ and replacing the bare strength v_0 by the after-the-ladder-diagram strength v_1. For small q, we obtain

$$w_q^{(j)} = w_0 + (2/\pi)v_F^{(j)}q, \quad w_0 = \frac{-\hbar\omega_D}{\exp(v_1 D_0)^{-1} - 1}, \tag{18}$$

where $v_F^{(j)} \equiv (2\epsilon_F/m_j)^{1/2}$ is the Fermi velocity, and $D_0 \equiv D(\epsilon_F)$ the density of states per spin. Note that the energy $w_q^{(j)}$ depends linearly on the momentum q.

The system of free fc-bosons undergoes a Bose-Einstein condensation (BEC) in 2D at the critical temperature[14]

$$k_B T_c = 1.24\ \hbar v_F n_0^{1/2}, \tag{19}$$

where n_0 is the boson density. Briefly Eq. (19) can be obtained from $(2\pi\hbar)^{-2}\int d^2p[\exp(\beta_c cp)-1]^{-1}$, $\beta_c \equiv (k_B T_c)^{-1}$, $c \equiv (2/\pi)v_F$. Note that formula (19) is independent of the pairing strength v_1 unlike the famous BCS formula: $k_B T_c = 1.13\hbar\omega_D \exp(v_1 D_0)^{-1}$. The interboson distance $R_0 \equiv n_0^{-1/2}$ calculated from this expression is $1.24\ \hbar v_F (k_B T_c)^{-1}$. The boson size r_0 calculated from Eq. (18), using the uncertainty relation ($q_{\max} r_0 \sim \hbar$) and $|w_0| \sim k_B T_c$, is $(2/\pi)\hbar v_F (k_B T_c)^{-1}$, which is a few times smaller than R_0. Hence, the bosons do not overlap in space, and the model of free bosons is justified. For GaAs/AlGaAs, $m^* = 0.067 m_c$, m_c = electron mass. For the 2D electron density 10^{11} cm^{-2}, we have $v_F = 1.36 \times 10^6$ cm s^{-1}. Not all electrons are bound with fluxons since the simultaneous generations of ± fc-bosons is required. The minority carrier ("hole") density controls the fc-boson density. For $n_0 = 10^{10}$ cm^{-2}, $T_c = 1.29$ K, which is reasonable. The critical temperature T_c is recognizable by the presence of the Hall resistivity plateau.

The supercurrent is generated by the c-bosons condensed monochromatically at the momentum directed along the sample length. The supercurrent density (magnitude) J, calculated by the rule: (charge e_b) × (c-boson density n_0) × (drift velocity v_d), is

$$J \equiv e_b n_0 v_d = e_b n_0 (2/\pi) |v_F^{(1)} - v_F^{(2)}|. \quad (20)$$

Using Eqs. (6), (11), and (20) we obtain

$$\rho_H \equiv \frac{E_H}{J} = \frac{v_d}{e_b n_0 v_d} n_\phi \left(\frac{h}{e}\right) = \left(\frac{h}{e_b e}\right) \frac{n_\phi}{n_0}. \quad (21)$$

First let us consider the integer QHE at $\nu = P$. Since the LL degeneracy (the number of states per LL) $eAB(2\pi\hbar)^{-1}$ is less by the factor P^{-1}, we must consider the lowest P LL's. The BEC occurs at each LL. Hence we have

$$n_0 = n_e / P, \quad (22)$$

where n_e the electron density (constant). The fluxon density n_ϕ per LL is connected with the boson density n_0 by

$$n_\phi = n_0 / P. \quad (23)$$

Second consider the fractional QH state at $\nu = P/Q$, odd Q. This state is formed similarly from the lowest P LL's occupied by the c-fermions, each with $Q-1$ fluxons. Equations (22) and (23) hold in this case. Using Eqs. (1) and (23) we obtain from Eq. (21),

$$\rho_H = \frac{Q}{P} \left(\frac{h}{e^2}\right), \quad (24)$$

as observed in the experiments. In Eq. (20), the drift velocity v_d is given by the unaveraged velocity difference and hence the exact cancellation of the v_d occurs in the calculation of ρ_H in Eqs. (21), giving rise to an extreme accuracy (10^{-8}) for the plateau value.

In the presence of the supercondensate the noncondensed c-boson has an energy gap ϵ_g. Hence the noncondensed c-boson density has the activation energy type exponential temperature dependence:

$$\exp[-\epsilon_g / (k_B T)], \quad (25)$$

which is quite different from the phonon-generated temperature dependence. In the prevalent theories[23] the energy gap for the fractional QHE is identified as the sum of the creation energies of a quasielectron and a quasihole. With this view it is difficult to explain why the activation-energy type temperature dependence shows up in the steady-state quantum transport. Some authors argue that the energy gap ϵ_g for the integer QHE is due to the LL separation$=\hbar\omega_0$. But the separation $\hbar\omega_0$ is much greater than the observed ϵ_g. Besides from the view that the gap ϵ_g equals the LL separation one cannot obtain the activation-type energy dependence.

In contrast the c-fermion with mass m^* moves in all directions in the plane. The drift velocity v_d is the quantity averaged over the angles. Hence the cancellation of v_d from numerator /denominator in Eq. (13) is not exact. After the cancellation the Hall resistivity ρ_H is B-linear and it has the value approximately equal to $(Q/P)(h/e^2)$ at $\nu = P/Q$.

In the present work we calculated the transport coefficients (σ, σ_H) in the traditional way, identifying the carrier charge and density. We note that the current density formula $(J = e^* n v_d)$ and the Hall effect condition $(E_H = v_d B)$ are exact if a single-component current is considered; the drift velocity v_d is a macroscopic quantity. We stress that the Hall resistivity $\rho_H \equiv E_H / J$ needs measurements of the two quantities (E_H, J). In the prevalent theories[23] the Hall resistivity ρ_H is calculated directly through the lowest (L) LL projection or by other methods. This is not a complete solution. If such methods are used to obtain results describing the system state at 0 K, it is difficult to treat the system below and above T_c in a unified manner. At what temperature does the LLL projection fail? This question is difficult to answer. We must separately calculate E_H and J and take the ratio E_H / J. In our theory of the bosonic QHE[14] the Q represents the number of fluxons in the c-boson present and the P the number of the lowest LL's occupied by the parental c-fermions, each with $Q-1$ fluxons. In summary the fermionic (bosonic) QHE arises from the c-fermions (bosons) generated at $\nu = P/Q$, even (odd) Q, see below.

Figure 1 indicates that (a) each of the resistivity maxima at $\nu \equiv P/Q = 3/2, 5/2, \ldots$ is symmetric with respect to high and low fields, (b) the widths do not change much with P and (c) the strength (width) of the (bosonic) integer QH state at $\nu = P$ decrease with the integer P, and (d) the resistivity curve approaches smoothly to a constant as the field is reduced to zero. These features are explained as follows.

Let us consider the condition near 0 K. Only the energy matters. First, take the case at $\nu = 3/2$, where there are c-fermions with two fluxons. The system energy $E_{3/2}$ is given by

$$E_{3/2} = E_0 + C(k_B T)^2, \quad C = \text{constant}, \quad (26)$$

where E_0 is the c-fermion system-ground-state energy; a high Fermi degeneracy is assumed. Following Jain,[13] we introduce the effective magnetic field

$$B^* \equiv B - B_{3/2}, \quad B_{3/2} = \text{field at } \nu = 3/2. \quad (27)$$

If the field is reduced from $B_{3/2}$, the system tends to keep the same number N of the c-fermions by sucking in flux lines, since the bound c-fermion has a negative energy. Thus, the magnetic field becomes inhomogeneous, which generates an extra magnetic energy so that

$$E = E_0 + C(k_B T)^2 + \frac{A}{2\mu_0} (B^*)^2. \quad (28)$$

If the field is raised, the system also tries to keep the same number N by expelling out flux lines. The inhomogeneous fields raise the field energy as represented by the third term in Eq. (28). This explains the high-low field symmetry, behavior (a).

At $\nu=1$ there are the fc-boson, each with one fluxon. Hence there is a phase change somewhere between $\nu=3/2$ and $\nu=1$. The c-boson ground-state energy $E_{\rm cb}$ must enter in the arguments. We denote the energy difference by ΔE,

$$\Delta E \equiv E - E_{\rm cb} = C(k_B T)^2 + \frac{A}{2\mu_0}(B^*)^2. \quad (29)$$

We assume that the c-fermion system is stable if

$$\Delta E < E_1 = \text{constant}. \quad (30)$$

At the center $\nu=3/2$, $B^*=0$. There is a critical temperature T_c defined by

$$C(k_B T_c)^2 = E_1. \quad (31)$$

Below T_c the c-fermions system is stable. The critical field B_c is

$$B_c = \sqrt{\frac{2\mu_0 C}{A} k_B^2 (T_c^2 - T^2)}. \quad (32)$$

This B_c may be regarded as the half-width. Equation (32) indicates that the width B_c becomes a constant as $T\to 0$. We can simply extend the theory to the case $\nu=P/2$. The Fermi energy of the system remains the same irrespective of the field. Then Eq. (32) must hold irrespective of the integer P, explaining behavior (b).

The BEC occurs at each LL, and therefore the c-boson density n_0 is less for high P, see Eqs. (19) and (22), and the strength becomes weaker as P increases, explaining behavior (c). This weakening of the c-boson minima and the constancy of the c-fermion widths make a smooth curve over large P, explaining behavior (d). The same theory explains the smooth wide minimum observed at $\nu=1/2$. Here the phonon exchange attraction played an important role. The bound c-fermion needs an attraction, and cannot be derived from the repulsive Coulomb interaction, the starting Hamiltonian in the prevalent theories.[23] Jain's effective magnetic field's can only be justified with the concept of the bound c-fermion. Jain's unification theory[13] gives the location of the bosonic QHE states but does not explain the states' strengths.

The Halperin-Lee-Read theory[16] based on the c-fermion model, each c-fermion composed of an electron and an even number of Cheren-Simons gauge objects, generates a finite conductivity linear in the impurity density $n_{\rm imp}$, which vanishes in the limit $n_{\rm imp}\to 0$. Our formula (10) remains finite at 0 K.

Before closing we briefly discuss a connection between our theory and the Laughlin wave function. The ground-state wave function for any quantum particle can be represented by a positive near-constant everywhere except at the sample boundary. The state in which all c-bosons with (odd) Q fluxons occupy the same state is the many-boson ground state at $\nu=1/Q$. If this state is viewed in terms of the N electrons in the system, the Laughlin wave function for the $1/Q$ state emerges,

$$\Psi_Q(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N) = \Pi_{i<j}(z_i - z_j)^Q \exp\Big(-\sum_i |z_i|^2\Big),$$
$$z \equiv (x - iy)/l_{\rm B}, \quad l_{\rm B} \equiv (\hbar/eB)^{1/2}. \quad (33)$$

This highly correlated electron state can be developed by the phonon exchange and/or the repulsive Coulomb interaction. As in the HTSC the phonon exchange is more relevant here since this can generate an attractive interaction needed to form the bound c-particle. The ground-state wave function can carry no current. The wave functions $\exp(ip_n x/\hbar)$, $p_n \equiv 2\pi\hbar n/L$ can carry currents, where L = sample length and $n=\pm 1, \pm 2, \ldots,$ and a periodic boundary condition is assumed. Since L is macroscopic, the momentum p_n is small and so is the associated energy. If all ± c-bosons occupy a single p_n, the supercurrent density J is given by Eq. (20). At any other p's there will be an energy gap ϵ_g and hence the supercurrent is stable against the applied electric field.

The supercurrent is stable against the applied magnetic field B due to the Meissner effect as explained below. Let us consider the condition near $\nu=1$ below T_c. We introduce the effective magnetic field in the form (27) with B_1=field at $\nu=1$. If the field is raised from B_1, the system tries to stay in the superconducting state by expelling out the extra flux lines (Meissner effect). The magnetic fields become inhomogeneous, which generates an extra field energy given by $(A/2\mu_0)B^{*2}$. If the field is reduced from B_1, the system also tries to maintain in the same superconducting state by sucking in the flux lines. The inhomogeneous fields outside generates the extra field energy. In either case the superconducting state is sustained, generating the Hall resistivity plateau for small effective fields B^*. For a sufficiently high effective field, the superconducting state is broken and the resistivity becomes finite.

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