Fluctuations of the Lyapunov exponent in two-dimensional disordered systems

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We report a numerical investigation of the fluctuations of the Lyapunov exponent of a two-dimensional noninteracting disordered system. While the ratio of the mean to the variance of the Lyapunov exponent is not constant, as it is in one dimension, its variation is consistent with the single parameter scaling hypothesis.

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I. INTRODUCTION

The single parameter scaling (SPS) hypothesis[1,2] is the foundation of our understanding of Anderson localization in disordered systems. According to the SPS hypothesis Anderson localization phenomena are governed by a single parameter: The ratio of the system size to the localization length ξ. When applied to the zero temperature conductance g of disordered mesoscopic system the hypothesis implies that the probability distribution $p(g)$ of the conductance obeys[3]

$$p(g) \simeq F(g;L/\xi). \tag{1}$$

Here g is in units of $2e^2/h$ and L is the system size. The SPS hypothesis has been applied to other physically interesting quantities including the localization length[4,5] and Lyapunov exponent spectra[6] of quasi-one-dimensional (1D) systems, as well as the energy level statistics of disordered systems.[7] The probability distribution of all these quantities should have a form similar to Eq. (1) if SPS holds.

Our understanding of scaling is most complete for one-dimensional (1D) systems. There are two properties of 1D systems that distinguish them from higher dimensional systems. First, their electronic eigenstates are, with very few exceptions, always localized. Second, the localization length ξ is comparable to the mean free path so that there is no diffusive regime. It has been shown that, in 1D, the cumulants of $\ln g$ all scale linearly with length L.[8] It follows that $p(g)$ is log-normal when $L \gg \xi$.[9] The log-normal distribution is determined by two parameters, the mean and variance of $\ln g$. Consistency with the SPS hypothesis requires that they be related. For weak disorder a perturbative analysis of the 1D Anderson model reveals[8]

$$\sigma^2_{\ln g} \equiv \langle(\ln g - \langle \ln g\rangle)^2\rangle = 2\langle -\ln g\rangle. \tag{2}$$

Angular brackets mean an average over disorder. For the 1D Anderson model, only weak disorder is relevant since for strong disorder ξ is comparable to the lattice spacing.

Equation (2) holds for many models. The precise conditions for its validity is[10–13]

$$\xi \gg \ell_s, \tag{3}$$

where ℓ_s is a length scale that is related to the integrated density of states. SPS is violated at the boundaries of the original spectrum of the system and for fluctuation states arising due to disorder in the initial band gaps. A violation[8,14] of SPS at the band center of the Anderson model was shown in Ref. 15 to arise for similar reasons.

Single parameter scaling of the conductance distribution [Eq. (1)] has also been verified numerically in the three-dimensional (3D) Anderson model close to the critical point of the Anderson transition.[16,17] The region of validity of the scaling in 3D, however, is not known. One can imagine, however, that an inequality similar to Eq. (3) may be applicable in this case as well.

The situation in two-dimensional (2D) systems is currently very controversial. According to Ref. 2 all states in 2D are localized. At the same time there are a large number of experiments, in which an apparent metal-insulator transition has been observed. (For a recent review see Ref. 18.) The physical meaning of these observation is not yet understood, despite a debate that has already lasted a decade. The validity, or otherwise, of SPS in 2D is, therefore, an important issue. Even for single particle models this issue has not yet been fully resolved. For instance, careful numerical analyses[4,19] of the 2D Anderson model showed excellent agreement with SPS. While other studies[20–23] suggested the existence of power-law localized states and two-parameter scaling. Violations of SPS have also been reported in more recent papers.[24,25]

The example of 1D systems demonstrates that conclusive results regarding scaling properties can only be obtained from studying the distribution functions of relevant quantities. Numerical studies[26,27] in 2D show that $\ln g$ is normally distributed in the regime of strong localization. It follows that single parameter scaling must be manifest in a relation between the average of $\ln g$, and its variance, similar to the 1D equation (2). However, attempts to verify this relation did not reach definite conclusions because of the small system sizes simulated and an approach to Eq. (2) that was too naive.

The main objective of our paper is to perform a careful analysis of the statistical properties of a 2D disordered system of noninteracting electrons, and verify that they are consistent with SPS. The object of our calculations is the finite length Lyapunov exponent (LE) for a 2D Anderson model with diagonal disorder. (The definition of the LE and the meaning of the qualification *finite length* is given below.) For

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a 2D $L\times L$ system with $L\gg\xi$ the mean of the LE is equal to the inverse of the localization length ξ.

The distribution of conductance has been given special attention in the literature because it is directly accessible in experiments. However, it should be understood that the conductance unavoidably reflects properties not only of the system in question, but also properties of the contacts used to measure it. The Lyapunov exponent, on the other hand, is an intrinsic property of the disordered system, containing information about spatial distribution of the wave functions, which are ultimately responsible for all other properties including conductance.

We find that the distribution function of the LE is approximately normal both when $L\ll\xi$ and when $L\gg\xi$. This contrasts with the conductance which exhibits not only very strong fluctuations but also a significant change in the form of its distribution between the diffusive and localized regimes.

We approach the question of scaling by clarifying the relation between the average and variance of the LE that is implied by the SPS hypothesis in 2D, and checking whether numerical data are consistent with it. We find that the relation between the mean and variance is characterized by a single parameter, namely the ratio of the system size to the localization length. Thus, we provide convincing evidence that the SPS hypothesis is valid in 2D disordered systems of noninteracting electrons.

II. MODEL AND METHOD

A. The transfer matrix for the Anderson model

We simulated the two-dimensional Anderson model with Hamiltonian

$$H=\sum_i \epsilon_i c_i^\dagger c_i-\sum_{\langle i,j\rangle} c_i^\dagger c_j. \tag{4}$$

The first summation is over all sites on an $L_t\times L$ square lattice, i.e., a system of width L_t and length L. The second summation is over all pairs of nearest neighbors. We imposed periodic boundary conditions in the transverse direction and used a "box" distribution of width W for the site energies ϵ_i

$$p(\epsilon_i)=\begin{cases}1/W & |\epsilon_i|\leqslant W/2\\ 0 & |\epsilon_i|>W/2\end{cases}. \tag{5}$$

Lyapunov exponents arise when the time independent Schrödinger equation is expressed as a product of random transfer matrices.[29] We divide the system in the longitudinal direction into L layers. We form vectors Ψ_n of length L_t from the wavefunction amplitudes on each layer. For an arbitrary energy E we derive from the Schrödinger equation the transfer matrix equation

$$\begin{pmatrix}\Psi_{n+1}\\ \Psi_n\end{pmatrix}=M_n\begin{pmatrix}\Psi_n\\ \Psi_{n-1}\end{pmatrix}. \tag{6}$$

The $2L_t\times 2L_t$ transfer matrix M_n relates the wave function amplitudes on layer n and $n-1$ to those on layers n and $n+1$. For Eq. (4), Ψ_n and M_n are real vectors and matrices, respectively, and the transfer matrices are identically and independently distributed random matrices.

B. Definition of Lyapunov exponents

We start with a $2L_t\times 2L_t$ orthogonal matrix Q_0. We perform L transfer matrix multiplications and factor the result into a product of an orthgonal matrix Q, a diagonal matrix D with positive elements, and an upper triangular matrix R with unit diagonal elements

$$M_L\cdots M_1Q_0=QDR. \tag{7}$$

We define $2L_t$ finite length LEs $\gamma_L^{(1)}\cdots\gamma_L^{(2L_t)}$ by

$$\gamma_L^{(n)}=\frac{1}{L}\ln D_n. \tag{8}$$

Here D_n is the nth diagonal element of D. The finite length LEs are random variables that fluctuate as we sample the random potential. For fixed L_t, when $L\to\infty$ the LEs always tend to the same limiting values

$$\lim_{L\to\infty}\gamma_L^{(n)}=\gamma^{(n)}, \tag{9}$$

for (nearly) all samplings of the distribution of transfer matrices and (nearly) any choice of Q_0.[28]

The L_t^{th} LE is the most physically significant: $\gamma^{(L_t)}$ is the inverse of the localization length λ of an electron on an infinite quasi-1D system of width L_t described by (4)[29]

$$\gamma^{(L_t)}=\frac{1}{\lambda}. \tag{10}$$

Therefore, in what follows we focus on $\gamma_L^{(L_t)}$, dropping the superscript and referring to it as *the* LE

$$\gamma_L^{(L_t)}\equiv\gamma_L. \tag{11}$$

In numerical calculations, if only the first m LEs are required, it is sufficient to make Q_0 a $2L_t\times m$ real matrix with orthonormal columns. Depending on the value of m, this can save a considerable amount of computer time. The values for the first m LEs obtained in any particular calculation are independent of whether or not LEs with higher indices are also calculated.

To avoid numerical difficulties with the transfer matrix multiplication Eq. (7), we performed additional Gram-Schmidt orthogonalization after every 8 transfer matrix multiplications.[5]

The definition, given in Eq. (8), of the LE for *finite length* that we have adopted here is not the only reasonable one. We compare our choice with an obvious alternative in Appendix A.

C. Special considerations for systems of finite length

In this paper we are concerned with the the distribution of γ_L for finite length L rather than with its asymptotic value as $L\to\infty$. Therefore, we have to deal properly with effects related to the finite value of L, effects that were routinely considered unimportant in previous studies.

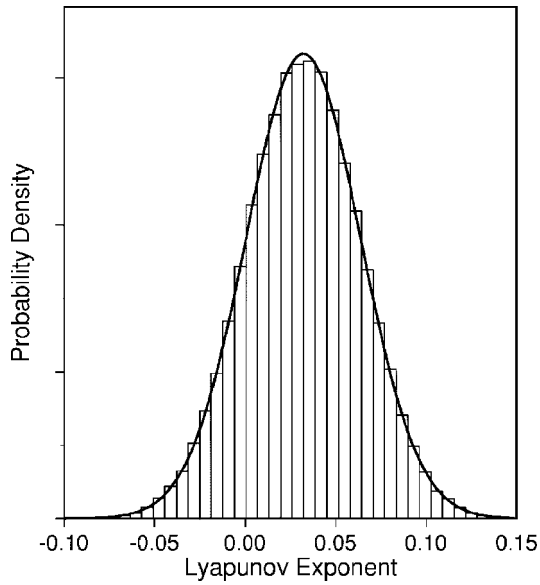

FIG. 1. The distribution of the LE for a 2D $L \times L$ system with $E=1$, $W=5$, and $L=32$. The line is a normal distribution with the mean and variance equal to that of the numerical data. The numerical data have a mean of 0.032, a standard deviation of 0.032, and a skewness of −0.0031. The number of samples is 65 523.

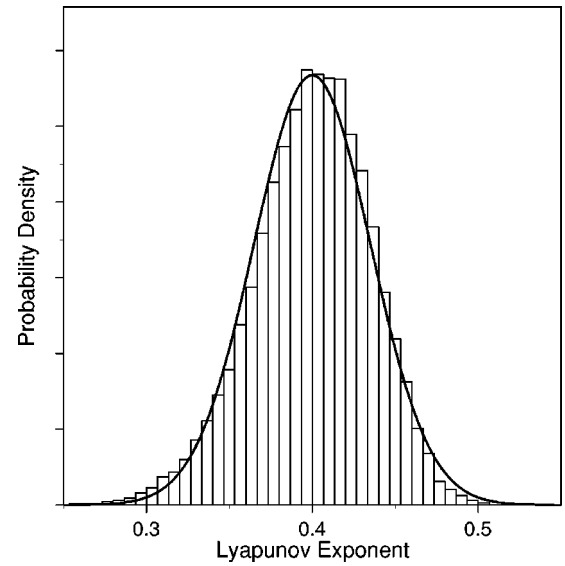

FIG. 2. The distribution of the LE for a 2D $L \times L$ system with $E=1$, $W=14$, and $L=128$. The line is a normal distribution with the mean and variance equal to that of the numerical data. The numerical data have a mean of 0.40, a standard deviation of 0.035, and a skewness of −0.33. The number of samples is 40 000.

In the asymptotic limit $L \to \infty$, the value of γ_L depends only on the distribution of the transfer matrix $p(M_n)$, and is independent of the choice of the initial matrix Q_0. For finite L, however, the distribution of γ_L depends on Q_0 and L, in addition to $p(M_n)$. The dependence on Q_0 would, if not dealt with, introduce an arbitrary element to our analysis that is undesirable.

To remove the dependence of the distribution of γ_L on Q_0 we used the following observation to our advantage. For (almost) any Q_0, the distribution of the matrix Q approaches an L independent stationary distribution $p_s(Q)$ as L increases. The form of $p_s(Q)$ depends only on $p(M_n)$. By sampling Q_0 from $p_s(Q)$ we obtain a distribution for γ_L that depends only on L and $p(M_n)$.

To generate matrices with the required stationary distribution $p_s(Q)$, we took an arbitrary set of orthonormal vectors, performed N_r transfer matrix multiplications and factored the result according to Eq. (7). To determine how large N_r should be to get a good approximation to $p_s(Q)$, we checked whether or not the Kolmogorov-Smirnov test could distinguish between the distributions of γ_L for $L=1$ obtained with different N_r. The test showed that once $N_r > 100$, the distribution of the LE becomes independent of N_r. Below we set $N_r = 1000$.

III. RESULTS

Since our interest in this paper is in the distribution of the LE in 2D systems we set the width L_t and length L of the system equal

$$L_t = L, \tag{12}$$

i.e., in the remainder of the paper we consider only 2D $L \times L$ systems.

A. Distribution of the LE

We simulated systems with Fermi energy $E=1$, disorder $5 \leqslant W \leqslant 14$ and a range of systems sizes between $L=16$ and $L=512$. The distribution of the LE for two particular cases are shown in Figs. 1 and 2. These are representative of the parameter range we studied. Figure 1 corresponds to the situation $L \ll \xi$, while Fig. 2 corresponds to the situation $L \gg \xi$. In the figure captions we give the values of the mean, variance and skewness for the numerical data, as well as the number of samples simulated.

The skewness is a measure of the symmetry of the distribution. Distributions that are symmetrical about their mean, such as the normal distribution, have a skewness equal to zero. According to Ref. 30, a distribution whose skewness has absolute value greater than unity is considered highly skew. A distribution whose skewness has absolute value less than one half is considered fairly symmetrical. For data sampled from a normal distribution, the skewness is expected to be distributed around zero with a standard deviation of $\sqrt{15/N_s}$ where N_s is the number of samples.[31]

For the data in Fig. 1, the difference of the skewness from zero is not statistically significant. This is consistent with the LE having a normal distribution.

For the data in Fig. 2, the difference of the skewness from zero is statistically significant. What is the physical significance of this deviation? Normally we would expect the scaling hypothesis to apply only when the localization length ξ is much longer than microscopic length scales such as the mean free path, lattice constant, etc. Here these are approximately unity, so this condition corresponds to $\xi \gg 1$. This condition is satisfied for the data in Fig. 1, where $\xi \simeq 180$ (see Table I), but not for the data in Fig. 2, where $\xi \simeq 2.5$. Therefore, we think that the deviation from the normal distribution seen in Fig. 2 is not significant in the context of our study.

TABLE I. The 2D localization length and the scaling function F determined from the scaling of the mean LE. The errors quoted are 95% confidence intervals and are estimated using the Monte Carlo method (Ref. 31).

W	ξ	L/ξ	F
5	178 ± 2	0.1	0.86 ± 0.005
5.5	85 ± 1	0.5	1.51 ± 0.01
6	48 ± 0.5	1	2.11 ± 0.02
6.5	30.2 ± 0.3	2	3.17 ± 0.02
7	20.8 ± 0.2	5	6.16 ± 0.05
8	11.7 ± 0.1	10	11.0 ± 0.1
9	7.69 ± 0.08	20	20.6 ± 0.2
10	5.54 ± 0.06	1000	(1000)
11	4.26 ± 0.04		
12	3.44 ± 0.03		
13	2.88 ± 0.03		
14	2.47 ± 0.02		

In our opinion, the normal distribution is a reasonable approximation to the observed distribution for the range of L/ξ in our simulations. In what follows, we concentrate our attention on the mean and variance of the LE and their scaling.

It is also important to bear in mind when looking at Figs. 1 and 2 that the scaling hypothesis is expected to apply to the bulk of the distribution not its tails, i.e., to typical states not necessarily to very rare states. Hence, we use a linear scale for the probability density axis and not a logarithmic scale, which would unduly emphasize the tails of the distribution.

B. The scaling of the mean LE

According to the SPS hypothesis the scaling of both the mean and variance of the LE should be governed by the same length scale, the localization length ξ. A quantitative test of scaling involves checking the consistency of the disorder dependence of ξ obtained independently from the scaling of the mean and variance of the LE. In this section we deal with the scaling of the mean LE.

For the scaling analysis of the mean LE we estimated $\langle\gamma_L\rangle$ to a precision of 0.25% for system sizes $L=16$, 32, 64, 128, 256, and 512. For $W\leqslant 11$ the maximum system size was reduced to 256. We determined $\xi\equiv\xi(W)$ by fitting the variation of the mean LE with L and W to the SPS law

$$\langle\gamma_L\rangle L = F\left(\frac{L}{\xi}\right). \tag{13}$$

When $L\gg\xi$ we suppose that the mean of the LE will tend to the inverse of the 2D localization length ξ, i.e.,

$$\lim_{L\to\infty}\langle\gamma_L\rangle = \frac{1}{\xi}. \tag{14}$$

This is equivalent to

$$F(x)\to x \quad x\gg 1. \tag{15}$$

For numerical reasons we expressed the scaling function in the form

$$\log_{10}\langle\gamma_L\rangle L = f\left(\log_{10}\frac{L}{\xi}\right), \tag{16}$$

and used a spline to interpolate the function f. The values of f, at the values of L/ξ in Table I, were fitting parameters. To ensure the spline interpolation reproduces Eq. (15), we fixed the value of f at $L/\xi=1000$. The corresponding value of F is given in parenthesis in Table I. The remaining fitting parameters were the localization lengths for each disorder. Finally, we used the shape preserving Akima spline to avoid unphysical oscillations of f. We summarize the results in Tables I and II and in Fig. 3. We can see from this figure that the data for different values of disorder W and system size L fall on a

TABLE II. Details of the finite size scaling fits: The number of data N_d, the number of parameters N_p, the value of χ^2 for the best fit and the goodness of fit probability Q.

Statistic and data range	N_p	N_d	χ^2	Q
$\langle\gamma\rangle L$				
$16\leqslant L\leqslant 512$, $5\leqslant W\leqslant 14$	19	63	46.0	0.4
Σ				
$64\leqslant L\leqslant 256$, $5\leqslant W\leqslant 12$	15	30	20.3	0.2

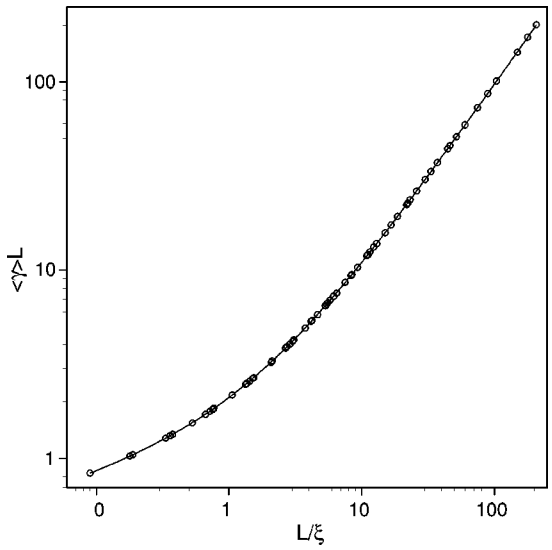

FIG. 3. The finite size scaling fit (line) to the data (circles) for the mean of the Lyapunov exponent. The precision of the numerical data is 0.25%.

common scaling curve when expressed as a function of L/ξ. Moreover, for large L we observe the expected linear dependence, with slope equal to the inverse localization length.

C. Scaling of the fluctuations of the LE

Taking into account that the dimension of γ_L is $1/L$, and that of its variance σ^2 is $1/L^2$, we can define a dimensionless quantity Σ by

$$\Sigma = \frac{\sigma^2 L}{\langle \gamma_L \rangle}. \tag{17}$$

According to the SPS hypothesis the localization length is the only relevant length in the system, so Σ should obey the SPS law

$$\Sigma = F_\Sigma\left(\frac{L}{\xi}\right). \tag{18}$$

In 1D, the linear scaling of the cumulants of $\ln g$, and the relation between the LE and g described in the Appendix, allow us to deduce from Eq. (2) the much more prescriptive statement

$$\Sigma = 1. \tag{19}$$

However, for a 2D $L \times L$ system the cumulants of $\ln g$ do not scale linearly with L, except perhaps in the regime where $L \gg \xi$; a regime which it is more difficult to reach in 2D than in 1D. Therefore, we should not expect that Σ be unity or even constant in our calculations. Confirmation of the SPS hypothesis in 2D consists not in demonstrating that calculated values vary in accord with Eq. (19) but rather in trying to establish Eq. (18).

We have plotted the variation of Σ with system size in Fig. 4, where different lines correspond to different values of disorder, W. These data were analyzed in an analogous way to the mean LE. We expressed the SPS law Eq. (18) in the form

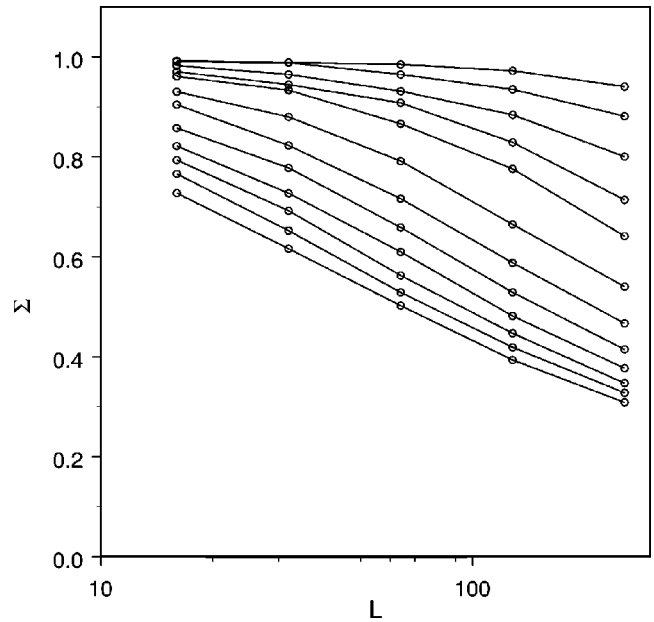

FIG. 4. Σ versus system size. For each point 160 000 samples were simulated, corresponding to a precision of approximately 0.4%. The lines, which connect points corresponding to a common value of the disorder W, are a guide to the eye only.

$$\Sigma = f_\Sigma\left(\log_{10}\frac{L}{\xi}\right), \tag{20}$$

and used an Akima spline interpolation of the function f_Σ. The values of f_Σ, at the values of L/ξ listed in Table III, and the localization lengths for each disorder were fitting parameters. To obtain a reasonable goodness of fit (>0.1) we had to restrict the range of data considered to $5 \leqslant W \leqslant 12$ and $L \geqslant 64$. [There seem to be a more pronounced finite size correction in the data for the variance than in the data for the mean LE. Also, the breakdown of scaling when ξ is comparable to the lattice spacing ($\xi \sim 1$) seems to be evident sooner in the variance of the LE than in the mean LE.]

When fitting data for the mean LE, we were able to determine the absolute value of ξ with the aid of (15). Unfor-

TABLE III. The 2D localization length and the scaling function F determined from the scaling of Σ.

W	ξ	L/ξ	F_Σ
5	226±38	0.1	0.98±0.02
5.5	96±8	1	0.948±0.005
6	50±2	5	0.804±0.004
6.5	30.4±1	10	0.683±0.004
7	20.9±0.6	20	0.551±0.003
8	11.9±0.3	50	0.403±0.002
9	7.8±0.2		
10	5.6±0.1		
11	4.3±0.1		
12	(3.44)		

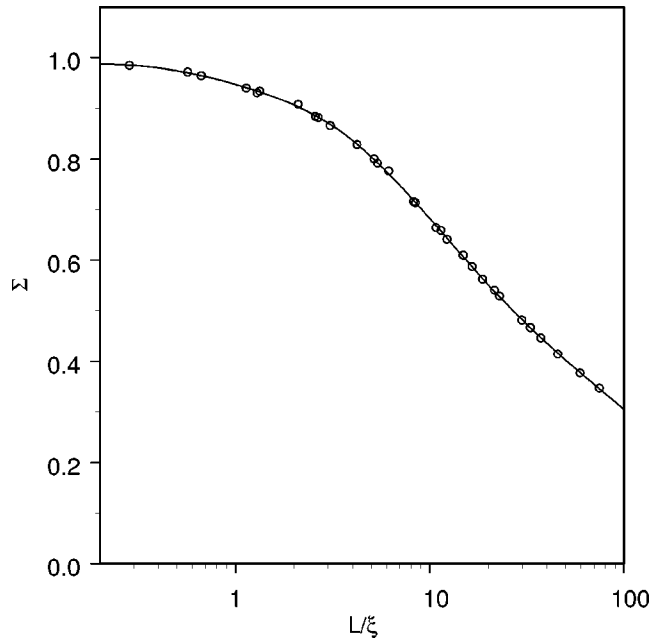

FIG. 5. A visual check that the numerical data for Σ defined in Eq. (17) (points) satisfy the SPS law Eq. (18) (line).

tunately, no similar relation is available for f_Σ and so we cannot fix the absolute scale of ξ by fitting data for Σ alone. Indeed, looking at Fig. 5 we can see that, if we translate both the fit and the data by the same amount parallel to the abscissa, we obtain an equally good fit. To avoid this ambiguity, we set the value of the localization length for $W=12$ to that found for the mean LE. We show the results in Table III. Apart from an over estimate of ξ for $W=5$ and $W=5.5$, the results are consistent with those for the mean LE. In addition, in Fig. 5, we have plotted Σ versus L/ξ, where ξ is estimated from this fit. We see that all the different curves of Fig. 4 collapse on to a single curve, confirming the correctness of Eq. (18). We conclude that the fluctuations of the LE are consistent with the SPS hypothesis.

Looking at Fig. 5, it is plausible that the function Σ will tend to a finite asymptotic value as $L/\xi \to \infty$. If this does occur, the fluctuations of the LE in the 2D asymptotic limit ($L/\xi \to \infty$ with $L_t = L$) decay as $1/\sqrt{L}$. This is similar to the behavior in the quasi-1D limit ($L/\xi \to \infty$ with L_t fixed) where the fluctuations in the LE also decay as $1/\sqrt{L}$. The only difference is that Σ is always unity in quasi-1D, while the asymptotic value of Σ is less than unity in 2D.

IV. CONCLUSION

We have investigated numerically the scaling of the fluctuations of the LE in the 2D Anderson model with diagonal disorder. We found that the distribution of the LE is approximately normal both when $L \ll \xi$ and $L \gg \xi$. We showed that the parameters of the distribution, the mean LE and its variance behave in accordance with the single parameter scaling hypothesis for the energy value considered in our calculations. This value, $E=1$, was chosen to lie far from the boundaries of the initial spectrum, $E=\pm 4$, and from its center $E=0$ in order to avoid anomalies related to the band edge behavior, which were found in 1D systems. We expect that the behavior for any other value of the energy will be similar as long as it is not close to an anomalous region. We found that the manifestation of SPS in numerical studies of 2D systems is different from that of 1D systems. Instead of the simple relation between the mean and variance of the LE given by Eq. (19) and valid for 1D, in 2D one has to analyze the compliance of numerical data with the SPS relation Eq. (18).

The fact that we verified SPS both when $L \ll \xi$ and $L \gg \xi$ is significant because it contradicts the conclusions of Refs. 24 and 25, where a behavior inconsistent with SPS was found. While a complete elucidation of the sources of this disagreement is beyond the scope of this paper, we can suggest some possibilities that might be worth pursuing in future work. First, it is possible that the logarithmic increase of the localization length seen in Ref. 24 might be reconcilable with the SPS hypothesis, in much the same way as we have reconciled the system size and disorder dependence of the ratio Σ with SPS here. Second, the authors of Refs. 24 and 25 analyzed the spatial properties of eigenfunctions. In 1D the relationship between lengths that characterize transport and wave functions is well established.[32] In 2D there may be aspects of this relationship that have not yet been properly understood. Third, in Refs. 24 and 25 wave functions corresponding to $E=0$ were studied. In 1D this is a special spectral point, at which SPS is violated.[14,15] It seems reasonable to suggest that $E=0$ is also a special point for 2D where SPS should not be expected.

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APPENDIX A: ALTERNATIVE DEFINITION OF THE LE FOR FINITE LENGTH

The definition of the finite length LE (which we have used in this work) is not the only reasonable one. In this appendix we will describe an alternative and compare with the definition described in the main text of this paper. Given a transfer matrix M

$$M = \prod_{n=1}^{L} M_n, \tag{A1}$$

we can define a matrix Ω by

$$\Omega = \ln M^\dagger M. \tag{A2}$$

The eigenvalue spectrum of Ω is composed of pairs of opposite sign $\{+\nu^{(n)}, -\nu^{(n)} : n = 1 \cdots L_t\}$. From these eigenvalues we could define the LE in an alternative way as

$$\gamma_L^{(n)} = \frac{\nu^{(n)}}{2L}. \tag{A3}$$

In the limit that $L \to \infty$ at fixed L_t, the random variables defined by Eq. (A3) always tend to be the same limiting values

for (nearly) all samplings of the distribution of transfer matrices. These values are the same as those obtained with Eq. (8) in the same limit. For finite L the values of Eqs. (8) and (A3) are different. We summarizes the main characteristics of each definition below.

For the definition Eq. (8) in the main text:

P1 The LE are not the eigenvalues of a matrix. The indices of the LE refer to the order in which they are obtained from the Gram-Schmidt procedure. In general, this is *not* in a strictly decreasing order.

P2 Though the sum of the all LEs is always zero, for finite L and for a single sample, the LE do not occur in pairs of opposite sign. This symmetry is restored after taking the limit $L\to\infty$ for a single sample, or after averaging over an ensemble of samples. For a single sample we have found that the symmetry also appears when Q_0 is sampled from the stationary distribution $p_s(Q_0)$ described in Sec. II C.

P3 For fixed L_t and Q_0 sampled from $p_s(Q_0)$, the mean of the LEs are independent of L. (Note that in the main text we consider scaling with $L_t=L$, so this property is not applicable there.)

P4 The LE have a simple geometrical interpretation in terms of the exponential rate of increase of lengths, areas, volumes, etc.

For the definition (A3):

P1 The LEs are related to the eigenvalues of a matrix and hence there is no prescribed ordering for them. It is conventional to put the LEs in decreasing order and the index in the definition (A3) usually refers to this order.

P2 For all L, the LE occur in pairs of opposite sign. This exact symmetry is exhibited not just after averaging over an ensemble of samples but also by a single sample.

P3 In this definition there is no analogue of Q_0 and hence no analogue of property P3 for Eq. (8).

P4 There is no simple geometric interpretation except in the asymptotic limit.

SPS can be investigated using either definition. The quantity defined by Eq. (8) has the advantage that its distribution is normal, while at the same time retaining a straightforward relationship to the decay of the wave function in the disordered system.

APPENDIX B: RELATION OF LE TO CONDUCTANCE IN 1D

For a strictly 1D system whose length is much longer than the localization length the transmission coefficient t for the transmission of electrons through the disordered sample decays as

$$-\ln|t| = \ln D_1 + O(L^0)(L \gg \xi). \tag{B1}$$

The $O(L^0)$ term is a fluctuating term that depends on the nature of the leads attached to the sample when defining the scattering problem. Using the Landauer formula to relate the transmission and the conductance we have

$$-\ln g = 2\gamma_L L + O(L^0)(L \gg \xi). \tag{B2}$$

From this we deduce that equation Eq. (2) is equivalent to Eq. (19) when $L \gg \xi$.

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