

Frequency-dependent third cumulant of current in diffusive conductors

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We calculate the frequency dispersion of the third cumulant of current in diffusive-metal contacts. The cumulant exhibits a dispersion at the inverse time of diffusion across the contact, which is typically much smaller than the inverse RC time. This dispersion is much more pronounced in the case of strong electron-electron scattering than in the case of purely elastic scattering because of a different symmetry of the relevant second-order correlation functions.

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I. INTRODUCTION

Measurements of nonequilibrium noise provide valuable information about the properties of a system, which cannot be extracted from measurements of average quantities. For example, measurements of shot noise give the magnitude of the quasiparticle charge in the case of a tunnel contact and the effective temperature of electrons in the case of a diffusive contact.¹ Recently, Reulet *et al.*² performed first measurements of the third cumulant of current, which may give even more interesting information. For example, this cumulant is very sensitive to the presence of electron-electron scattering in a diffusive contact. Electron-electron scattering changes the shot noise in a diffusive contact only by several percent,^{3,4} but it changes the third cumulant of current almost by an order of magnitude.⁵

Of special interest is the frequency dependence of the third cumulant. Very recently, it was shown that third cumulants of current in a chaotic cavity whose contacts have different transparencies may exhibit a frequency dispersion much more complicated than that of the shot noise. Unlike the conventional shot noise that has a dispersion only at the inverse RC time of the cavity,⁶ the third cumulant of noise may also exhibit a dispersion at the inverse *dwelt time* of an electron on the cavity.⁷ In most cases, this time is much longer than the RC time that describes charge relaxation in the cavity, and therefore the corresponding dispersion takes place at experimentally accessible frequencies. This dispersion is due to slow fluctuations of the distribution function that do not violate electroneutrality and are akin to fluctuations of local temperature. These fluctuations do not directly contribute to the current and therefore are not seen in conventional noise, but they modulate the intensity of noise sources and therefore manifest themselves in higher correlations of current.

Another important example of a system with a long dwell time is a diffusive contact. In this work we investigate the frequency dependence of the third cumulant of a metallic diffusive wire. Like a chaotic cavity, it also has a long dwell time. In addition, the metallic diffusive wire is of interest because its impedance can be easily matched to that required by current experimental detection schemes.² Furthermore, the measuring frequencies in this case are in the range where the frequency dispersion takes place for the system at hand.

The zero-frequency third cumulant of current for a diffusive wire was first calculated by Lee, Levitov, and Yakovets⁸ for noninteracting electrons in the zero-temperature limit. Recently this calculation has been extended to finite temperatures and to the case of strong electron-electron scattering by Gutman and Gefen.⁵ In this paper we calculate the frequency dependence of this quantity both for the case of noninteracting electrons and for the hot-electron regime and show that the latter case is more convenient for the experimental observation of this effect, since the dispersion of the third cumulant is much stronger.

We present a calculation based on the cascaded Boltzmann-Langevin approach.⁹ In the Appendix, we also derive the full generating functional for the frequency-dependent current fluctuations of a metallic wire for both the elastic and hot-electron regimes based on the stochastic path-integral approach to full counting statistics.¹⁰ The third cumulant of current may be expressed in terms of the functional derivatives of this functional.

II. MODEL AND BASIC EQUATIONS

Consider a quasi-one-dimensional diffusive wire of length L and conductivity σ . To explicitly describe its electric environment, the wire is chosen in the shape of a cylinder with a diameter $2r_0$ and is embedded in a perfectly grounded medium, which is separated from the wire by a thin insulating film of thickness δ_0 and with a dielectric constant ϵ_d (see Fig. 1). All dimensions are assumed to be much larger than the elastic mean free path and the screening length in the metal. Since electron-phonon scattering is suppressed at low temperatures, we may also assume that dimensions are much smaller than the electron-phonon scattering length such that

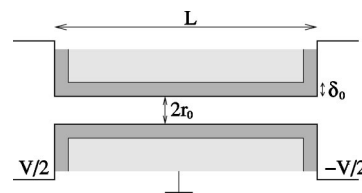


FIG. 1. The geometry of the system considered. White areas show the diffusive metal and dark areas the insulator.

dissipation takes place only in the reservoirs. Electron-phonon scattering renders fluctuations purely Gaussian⁵ and suppresses the dispersion effects to be discussed in this paper. The electrodes are assumed to be perfect conductors, so the third cumulant of current is not affected by the external circuit.^{11–13} We also restrict ourselves to sufficiently high voltages or temperatures; hence the quantum dispersion of this quantity does not show up in the frequency range of interest.¹⁴ We emphasize that despite the particular choice of geometry, our results are valid for any quasi-one-dimensional diffusive contact.

With the above assumptions, the noise of current may be described using the semiclassical Boltzmann-Langevin approach.¹⁵ The frequency dependence of shot noise in diffusive contacts with account taken of electrical screening was calculated in Refs. 16 and 17. To calculate the frequency dependence of the third cumulant of current, we use the cascade extension of this approach.⁹ The key idea of this extension is a large separation between the time scales describing the individual scattering events and the evolution of the distribution function of electrons in the contact. The resulting expressions may be also obtained by considering the corresponding stochastic path integral^{10,18} for the diffusive Boltzmann-Langevin equation (see the Appendix). The cascade expansion corresponds to a systematic expansion of the saddle-point equations of this path integral in powers of the counting field.

The quantity we are going to calculate is the Fourier transform of the third-order current correlation function defined as

$$P_3(\omega_1, \omega_2) = \int d(t_1 - t_2) \int d(t_2 - t_3) \exp[i\omega_1(t_1 - t_3) + i\omega_2(t_2 - t_3)] \langle \delta I(t_1) \delta I(t_2) \delta I(t_3) \rangle. \quad (1)$$

The starting point for our calculations is the stochastic diffusion equation for the fluctuations $\delta f(\varepsilon, \mathbf{r})$ of the distribution function $f(\varepsilon, \mathbf{r})$:

$$\left(\frac{\partial}{\partial t} - D\nabla^2 \right) \delta f - \delta I_{ee} = -e \delta \phi \frac{\partial f}{\partial \varepsilon} - \nabla \delta \mathbf{F}^{imp} - \delta F^{ee}, \quad (2)$$

where D is the diffusion coefficient, δI_{ee} is the linearized electron-electron collision integral, and $\delta \mathbf{F}^{imp}$ and δF^{ee} are random extraneous sources associated with electron-impurity and electron-electron scattering. This equation is obtained from the standard Boltzmann-Langevin equation by defining the electron energy as $\varepsilon = p^2/2m + e\phi(\mathbf{r}, t) - \varepsilon_F$ and isolating the isotropic part of the distribution function in momentum space. The fluctuation of the electric potential $\delta \phi$ that appears in this equation should be calculated self-consistently from the Poisson equation

$$\nabla^2 \delta \phi = -4\pi \delta \rho, \quad (3)$$

where the fluctuation of charge density, $\delta \rho$, is given by

$$\delta \rho = eN_F \left(\int d\varepsilon \delta f(\varepsilon) + e \delta \phi \right) \quad (4)$$

and where N_F is the Fermi density of states. In the case of a quasi-one-dimensional contact, the solution of Eqs. (2)–(4) is of the form¹⁷

$$\delta \phi(x, \omega) = \frac{1}{S_0 \sigma} (\nabla^2 + i\omega RC/L^2)^{-1} \frac{\partial}{\partial x} \int d^2 r_\perp \delta j_x^{ext}(\mathbf{r}), \quad (5)$$

where x is the coordinate along the contact, $\sigma = e^2 N_F D$ is the conductivity of the metal, $S_0 = \pi r_0^2$ is the cross-section area of the contact, $C = L\varepsilon_d r_0 / 2\delta_0$ and $R = L / \pi r_0^2 \sigma$ are the capacitance and the resistance of the contact, and

$$\delta \mathbf{j}^{ext} = eN_F \int d\varepsilon \delta \mathbf{F}^{imp}. \quad (6)$$

A fluctuation of the total current density is given by

$$\delta \mathbf{j} = \delta \mathbf{j}^{ext} - \sigma \nabla \delta \phi, \quad (7)$$

and a fluctuation of the total current at the left end of the contact thus equals

$$\delta I = \sigma \int d^2 r_\perp \left. \frac{\partial \delta \phi(x, \omega)}{\partial x} \right|_{x=-L/2}. \quad (8)$$

Note that we define current fluctuations to the left as possible. Making use of the correlation function of extraneous sources,

$$\begin{aligned} \langle \delta F_\alpha^{imp}(\varepsilon, \mathbf{r}) \delta F_\beta^{imp}(\varepsilon', \mathbf{r}') \rangle_\omega &= 2 \frac{D}{N_F} \delta(\mathbf{r} - \mathbf{r}') \delta(\varepsilon - \varepsilon') \\ &\times \delta_{\alpha\beta} f(\varepsilon, \mathbf{r}) [1 - f(\varepsilon, \mathbf{r})], \end{aligned} \quad (9)$$

one easily obtains the second-order correlation function for the fluctuations of the current as a functional of the distribution function f .¹⁹

Consider now the third cumulant of current. As the direct contribution to this quantity from the third cumulant of extraneous sources is negligibly small in a diffusive metal,⁹ this quantity is dominated by an indirect contribution of the second cumulant of these sources, which results from the modulation of their intensity by fluctuations of the distribution function. It may be written in the form

$$\langle \delta I(t_1) \delta I(t_2) \delta I(t_3) \rangle = \mathcal{P}_{123} \{ \Delta_{123} \}, \quad (10)$$

where

$$\Delta_{123} = \int dt \int d\varepsilon \int d^3 r \frac{\delta \langle \delta I(t_1) \delta I(t_2) \rangle}{\delta f(\varepsilon, \mathbf{r}, t)} \langle \delta f(\varepsilon, \mathbf{r}, t) \delta I(t_3) \rangle \quad (11)$$

and \mathcal{P}_{123} denotes a summation over all inequivalent permutations of indices (123).

Equations (6) and (8) suggest that the second cumulant of current exhibits a dispersion at frequencies of the order of $(RC)^{-1}$. Typically such high frequencies are beyond the experimentally accessible range. Therefore in what follows we

will assume that the frequencies ω_1 , ω_2 , and ω_3 are much smaller than $(RC)^{-1}$. Hence the pileup of charge in the contact may be neglected and the fluctuation of current may be considered as coordinate independent. In this case,

$$\langle \delta I(\omega_1) \delta I(\omega_2) \rangle = 4\pi \delta(\omega_1 + \omega_2) (RL)^{-1} \times \int dx \int d\varepsilon f(\varepsilon, x) [1 - f(\varepsilon, x)], \quad (12)$$

and the only possible dispersion in Eq. (11) is due to the dynamics of a fluctuation δf , so that the expression for the third cumulant assumes the form

$$P_3(\omega_1, \omega_2) = P(\omega_1) + P(\omega_2) + P(-\omega_1 - \omega_2),$$

$$P(\omega) = \frac{2}{RL} \int_{-L/2}^{L/2} dx \int d\varepsilon [1 - 2f(\varepsilon, x)] \langle \delta f(\varepsilon, x) \delta I \rangle_\omega. \quad (13)$$

The quantity $P(\omega)$ has to be calculated in different ways for the case of purely elastic scattering and for the hot-electron regime.

III. PURELY ELASTIC SCATTERING

For purely elastic scattering, δI^{ee} and δF^{ee} in Eq. (2) vanish, and a fluctuation of the distribution function δf may be presented as a sum of a part induced directly by an extraneous source,

$$\delta f_F(\varepsilon, x, \omega) = (\nabla^2 + i\omega/D)^{-1} \frac{\partial F_x^{ext}}{\partial x}, \quad (14)$$

and a part induced by fluctuations of the electric potential,

$$\delta f_\phi(\varepsilon, x, \omega) = -i\omega(\nabla^2 + i\omega/D)^{-1} \left[\frac{\partial f(\varepsilon, x)}{\partial \varepsilon} e \delta \phi(x, \omega) \right]. \quad (15)$$

The existence of the term (15) indicates that the dynamics of charged electrons differs from the dynamics of neutral particles even at frequencies much smaller than $(RC)^{-1}$.

By multiplying these equations with the fluctuation of current (8) and making use of the correlation function (9), we obtain

$$\langle \delta f_F(\varepsilon, x) \delta I \rangle_\omega = -2 \frac{e}{L} (\nabla^2 - i\omega/D)^{-1} \frac{\partial}{\partial x} \{ f(\varepsilon, x) [1 - f(\varepsilon, x)] \} \quad (16)$$

and

$$\langle \delta f_\phi(\varepsilon, x) \delta I \rangle_\omega = i\omega(\nabla^2 - i\omega/D)^{-1} \left[\frac{\partial f(\varepsilon, x)}{\partial \varepsilon} e \langle \delta \phi(x) \delta I \rangle_\omega \right]. \quad (17)$$

At low frequencies, one easily obtains from Eqs. (6) and (8) that

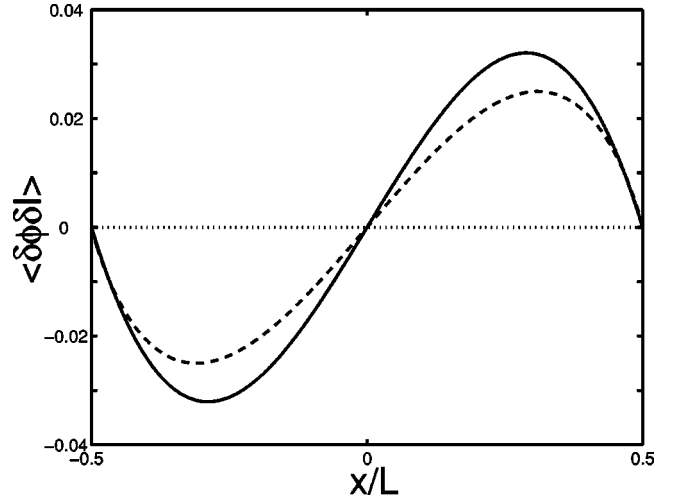


FIG. 2. The coordinate dependence of the zero-frequency correlator $\langle \delta \phi(x) \delta I \rangle_\omega$ normalized to eV for the elastic case (solid line) and the hot-electron regime (dashed line).

$$\langle \delta \phi(x) \delta I \rangle_\omega = \frac{2}{L} (\nabla^2)^{-1} \left[\frac{\partial}{\partial x} \int d\varepsilon f(1-f) \right]. \quad (18)$$

Using the well-known expression for the average distribution function

$$\bar{f}(\varepsilon, x) = \left(\frac{1}{2} + \frac{x}{L} \right) f_0(\varepsilon + eV/2) + \left(\frac{1}{2} - \frac{x}{L} \right) f_0(\varepsilon - eV/2), \quad (19)$$

where f_0 is the equilibrium Fermi distribution and V is the voltage drop across the contact, we obtain

$$\langle \delta \phi(x) \delta I \rangle_\omega = \frac{1}{6L} \left(1 - 4 \frac{x^2}{L^2} \right) \left[eV \coth \left(\frac{eV}{2T} \right) - 2T \right]. \quad (20)$$

The correlator $\langle \delta \phi(x) \delta I \rangle$ vanishes at $V=0$ and is an odd function of x at nonzero V (see Fig. 2). Upon inverting the operator $(\nabla^2 + i\omega/D)$ in Eqs. (16) and (17) and performing the spatial integration in Eq. (13), we arrive at an expression for $P(\omega)$ in terms of $q_\omega = (i\omega/D)^{1/2}$, which is our final goal. Because of its length, we give here only its low-temperature and low-voltage limits

$$P_{el}(\omega) = -\frac{4}{3} \frac{e^2 V}{R} [q_\omega L (q_\omega^2 L^2 + 30) \sinh(q_\omega L) - 8(q_\omega^2 L^2 + 6) \times \cosh(q_\omega L) + 2q_\omega^2 L^2 + 48] / [q_\omega^5 L^5 \sinh(q_\omega L)] \quad (21)$$

and

$$P_{ei}(\omega) = \frac{4}{3} \frac{e^2 V}{R} \frac{2 \cosh(q_\omega L) - q_\omega L \sinh(q_\omega L) - 2}{q_\omega^3 L^3 \sinh(q_\omega L)}. \quad (22)$$

At $\omega=0$ these expressions give $-(1/45)e^2V/R$ and $-(1/9)e^2V/R$, which corresponds to $P_3(0,0) = -(1/15)e^2V/R$ and $P_3(0,0) = -(1/3)e^2V/R$. These zero-frequency results are in agreement with Refs. 5 and 8. At finite frequency, Eqs. (21) and (22) become complex valued

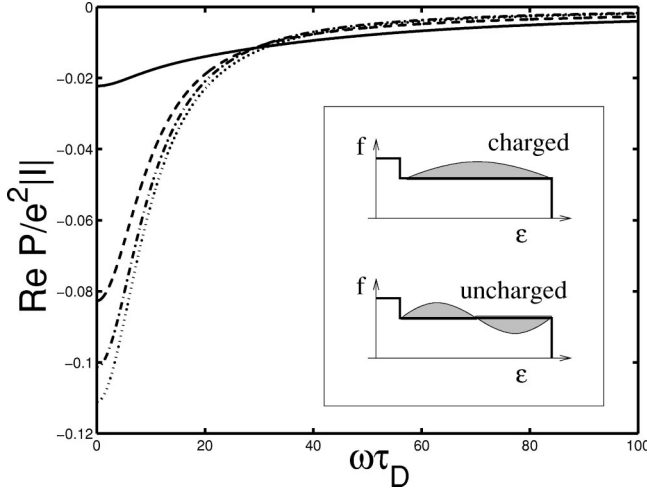


FIG. 3. The real part of the ratio $P(\omega)/(e^2 I)$ vs normalized frequency $\omega\tau_D$ ($\tau_D=L^2/D$) for purely elastic scattering at $eV \gg T$ (solid line), hot-electron regime at $eV \gg T$ (dashed line), purely elastic scattering at $eV \ll T$ (dotted line), and hot-electron regime at $eV \ll T$ (dash-dotted line). Inset: charged and uncharged fluctuations of the distribution function f . Charged fluctuations have a short relaxation time and contribute to fluctuations of current δI . Uncharged fluctuations do not contribute to δI directly but affect the intensity of noise sources. They decay only via slow diffusion and result in the low-frequency dispersion of the third cumulant.

and tend to zero as i/ω at $\omega \rightarrow \infty$. The real and imaginary parts of $P(\omega)$ are shown in Figs. 3 and 4.

IV. HOT-ELECTRON LIMIT

Consider now the limit of strong electron-electron interaction. In this case, the distribution function may be assumed to have a Fermi shape with a coordinate-dependent temperature $T_e(x)$ and electric potential $\phi(x)$:

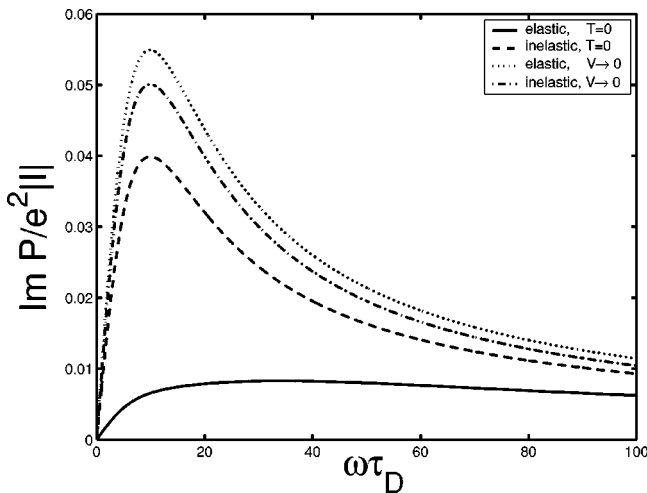


FIG. 4. The imaginary part of the ratio $P(\omega)/(e^2 I)$ vs normalized frequency $\omega\tau_D$ ($\tau_D=L^2/D$) for purely elastic scattering at $eV \gg T$ (solid line), hot-electron regime at $eV \gg T$ (dashed line), purely elastic scattering at $eV \ll T$ (dotted line), and hot-electron regime at $eV \ll T$ (dash-dotted line).

$$f(\varepsilon, x) = \left[1 + \exp\left(\frac{\varepsilon - e\phi(x)}{T_e(x)}\right) \right]^{-1}. \quad (23)$$

If the frequency ω is smaller than the rate of electron-electron collisions, a fluctuation δf can be expressed in terms of fluctuations of these quantities:

$$\delta f(\varepsilon, \mathbf{r}, \omega) = \frac{\partial f(\varepsilon, \mathbf{r})}{\partial \phi} \delta \phi + \frac{\partial f(\varepsilon, \mathbf{r})}{\partial T_e} \delta T_e. \quad (24)$$

A substitution of Eq. (24) into Eq. (13) and integration over the energy readily gives

$$P_{hot}(\omega) = \frac{2}{RL} \int_{-L/2}^{L/2} dx \langle \delta T_e(x) \delta I \rangle_\omega. \quad (25)$$

To calculate the correlator in Eq. (25), we have to obtain a Langevin-type equation for δT_e . To this end, we multiply Eq. (2) by ε and integrate it over ε , as was done when deriving the equation of heat balance in Refs. 3 and 4. This gives

$$\left(\frac{\partial}{\partial t} - D\nabla^2 \right) \left(\frac{\pi^3}{3} T_e \delta T_e \right) - D\nabla^2 (e^2 \phi \delta \phi) = - \int d\varepsilon \varepsilon \nabla \delta \mathbf{F}^{imp}. \quad (26)$$

Multiplying Eqs. (26) and (8) and averaging the product with the help of Eq. (9) results in an equation for the correlation function $\langle \delta T_e \delta I \rangle_\omega$ of the form

$$\begin{aligned} & \left(\nabla^2 - i\frac{\omega}{D} \right) \left[\frac{\pi^2}{3} T_e \langle \delta T_e(x) \delta I \rangle_\omega \right] \\ & = - \nabla^2 [e^2 \phi \langle \delta \phi(x) \delta I \rangle_\omega] + \frac{2e}{L} \frac{\partial}{\partial x} \int d\varepsilon \varepsilon f(1-f). \end{aligned} \quad (27)$$

The integral over the energy on the right-hand side of Eq. (27) equals $e\phi T_e$, and making use of Eq. (18), one easily obtains the solution of Eq. (27) in a symbolic form

$$\begin{aligned} \langle \delta T_e(x) \delta I \rangle_\omega & = \frac{6}{\pi^2} \frac{e^2}{L} \frac{1}{T_e} (\nabla^2 - i\omega/D)^{-1} \left\{ \frac{\partial(\phi T_e)}{\partial x} \right. \\ & \quad \left. - \nabla^2 \left[\phi (\nabla^2)^{-1} \left(\frac{\partial T_e}{\partial x} \right) \right] \right\}, \end{aligned} \quad (28)$$

where the operators ∇^2 and $\nabla^2 - i\omega/D$ are inverted with zero boundary conditions. According to Ref. 3, the mean potential is given by $\bar{\phi}(x) = Vx/L$ and the mean effective temperature by

$$\bar{T}_e(x) = \left[T^2 + \frac{3}{\pi^2} (eV)^2 \left(\frac{1}{4} - \frac{x^2}{L^2} \right) \right]^{1/2}. \quad (29)$$

As $\bar{\phi}$ and \bar{T}_e are odd and even functions of coordinate, the resulting correlator is an even function of x . In the zero-frequency limit it is given by

$$\begin{aligned} \langle \delta T_e(x) \delta I \rangle_\omega &= \frac{1}{2\pi^2} e^2 V \left(-a^2 - 2 \frac{x^2}{L^2} + \frac{1}{\sqrt{a^2 - 4x^2/L^2}} \right. \\ &\quad \times \left\{ (a^2 - 1)^{3/2} - 3a \frac{x}{L} \arcsin\left(\frac{2x}{La}\right) \right. \\ &\quad \left. \left. + 6 \left[a^2 \arcsin\left(\frac{1}{a}\right) + \sqrt{a^2 - 1} \right] \frac{x^2}{L^2} \right\} \right), \quad (30) \end{aligned}$$

where $a^2 = 1 + (4\pi^2/3)T^2/(eV)^2$. This correlation function vanishes at $V=0$ and is negative at positive voltages—i.e., if the increment of total current through the contact is negative. This fact has a very simple physical meaning: an increase in the total current results in an increase of the Joule heating and hence an increase of the temperature.

An integration of Eq. (28) with respect to x gives the low-frequency third cumulant for arbitrary temperatures and voltages

$$\begin{aligned} P_3(0,0) &= \frac{3}{\pi^2} \frac{e^2 V}{R} \left[\frac{7}{12} - \frac{13}{4} a^2 + \frac{1}{2} (5a^2 - 2) \sqrt{a^2 - 1} \right. \\ &\quad \left. \times \arcsin\left(\frac{1}{a}\right) + \frac{1}{4} a^4 \arcsin^2\left(\frac{1}{a}\right) \right]. \quad (31) \end{aligned}$$

Its limiting values $P_3(0,0) = -(3/\pi^2)e^2V/R$ at $eV \ll T$ and $P_3(0,0) = -(8/\pi^2 - 9/16)e^2V/R$ at $eV \gg T$ coincide with the results of Gutman and Gefen.⁵

In the case of nonzero frequencies, it is possible to obtain analytical results only in the limiting cases of $eV \ll T$ and $eV \gg T$. In the high-temperature limit, one may set $\bar{T}_e(x) = T$ in Eq. (28); hence $\phi(x)$ is the only coordinate-dependent quantity, and the term in curly brackets equals VT/L . Then the diffusion operator is easily inverted and the integration over x gives

$$P_{hor}(\omega) = -\frac{12}{\pi^2} \frac{e^2 V}{R} \frac{1}{q_\omega^2 L^2} \left[1 - \frac{2}{q_\omega L} \tanh\left(\frac{q_\omega L}{2}\right) \right]. \quad (32)$$

In the zero-temperature limit, the term in curly brackets in Eq. (28) may be expanded in a Fourier series in $\cos[(2n+1)\pi x/L]$ and the operator $\nabla^2 - i\omega/D$ is easily inverted. The final result is obtained as a sum of an infinite series

$$P_{hor}(\omega) = \frac{12}{\pi} \frac{e^2 V}{R} \sum_{k=0}^{\infty} \frac{J_0(\pi k + \pi/2) [J_1(\pi k + \pi/2) - (-1)^k]}{(2k+1) [\pi^2 (2k+1)^2 + i\omega L^2/D]}, \quad (33)$$

where J_0 and J_1 are Bessel functions of order 0 and 1.

V. DISCUSSION

Our results for the noninteracting regime [Eqs. (21) and (22)] and for the hot-electron regime [Eqs. (32) and (33)] are displayed in Figs. 3 and 4. It is clearly seen that both real and imaginary parts of the third cumulant have the most pronounced dispersion in the case of a high temperature or for strong electron-electron scattering—i.e., when the local distribution function has a nearly Fermian shape. This unexpected result is in a sharp contrast with the dispersion of

quantum noise,²⁰ which results from sharp singularities in the energy dependence of the distribution function.

Mathematically, the different shape of the frequency dependence for purely elastic scattering and interacting electrons at high voltages can be explained as follows. In the case of hot electrons, both $\delta\langle\delta I^2\rangle/\delta T_e(x)$ and $\langle\delta T_e(x)\delta I\rangle_\omega$ are even functions of the coordinate x (measured from the middle of the contact). On the contrary, for purely elastic scattering at $T=0$ both $\delta\langle\delta I^2\rangle/\delta f(\varepsilon, x)$ and $\langle\delta f(\varepsilon, x)\delta I\rangle_\omega$ are odd functions of x . The functions acted upon by the inverse diffusion operator $(\nabla^2 - i\omega/D)^{-1}$ in Eqs. (28) and (16) are also even and odd, respectively. At low frequencies, the inverse diffusion operator is essentially nonlocal in space and applying it results in an effective averaging of the argument on the scale of the order of L . In the elastic case, this averaging involves both negative and positive values, and this is why the elastic third cumulant is suppressed at low frequencies as compared to the hot-electron value. However, at high frequencies, the inverse diffusion operator becomes almost local in space; therefore, there is no averaging of negative and positive values and both cumulants become of nearly the same magnitude. This absence of spatial averaging partially compensates for the increasing frequency and makes the frequency dependence of the “elastic” cumulant more flat. Therefore the different shape of the frequency dependence in the elastic and hot-electron limits may be traced back to the different symmetry of relevant second-order correlation functions.

To show that the low-frequency dispersion in diffusive wires is experimentally accessible, we need to compare the bandwidth of 1 GHz for the measurement of the third cumulant² to typical diffusion times across diffusive wires. We use the experimental parameters of Ref. 21 (diffusion constant $D=40$ cm²/s, contact width $W=40$ nm, sheet resistance $R_\square=9\Omega$) and assume for the length of the wire $L=1$ μ m, for the dielectric constant $\varepsilon_d=1$, and for the thickness of the dielectric layer $\delta_0=10$ nm. We then find for the RC time $RC=10^{-13}$ s and for the diffusion time $L^2/D=2.5 \times 10^{-10}$ s. These are very prudent estimates. We emphasize that noise in the hot-electron regime has been measured in wires as long as $L=100$ μ m.^{22,23} For such wires, the low-frequency dispersion thus lies clearly inside the experimental bandwidth.

VI. SUMMARY

In summary, we have shown that diffusive contacts exhibit a nontrivial internal dynamics even at frequencies much smaller than the inverse charge-relaxation time. Though this dynamics is not affected by the electric environment of the contact, it differs from the dynamics of charge-neutral particles and manifests itself as a low-frequency dispersion of the third cumulant of current. In view of the fact that both the dynamic conductance and shot noise of metallic conductors depend only on the RC time, this frequency dispersion of the third cumulant on the scale of the dwell time is a very interesting result.

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APPENDIX: STOCHASTIC PATH-INTEGRAL REPRESENTATION

In this appendix, we derive a stochastic path-integral representation^{10,18} for the full current statistics of diffusive conductors. In contrast to the cascade rules⁹ applied in the main body of the paper, such a representation is not restricted to the calculation of specific cumulants, but allows for the calculation of the full probability distribution of frequency-dependent fluctuations of current. Here, the stochastic path-integral representation serves to verify the calculation of the third-order current correlation function. We give first a detailed derivation of the case without electron-electron scattering. For simplicity, we restrict ourselves to quasi-one-dimensional wires and consider only slow dynamics due to the longitudinal diffusion modes. In this case, the kinetic transport equation for the fluctuating contribution $\delta f(\epsilon, x, t)$ to the electron occupation function $f = \bar{f}(\epsilon, x) + \delta f(\epsilon, x, t)$ is given by the one-dimensional equivalent of Eq. (2). The mean distribution function \bar{f} is defined in Eq. (19). Since we restricted ourselves to the regime of slow diffusive dynamics, we may assume charge neutrality $\partial(\delta j)/\partial x = 0$ where the fluctuations of the electrical current density are given by Eqs. (6) and (7). Finally, we make use of the fact⁹ that the extraneous sources of noise are Gaussian random variables δ correlated in time, space, and energy and described by Eq. (9).

In order to construct a path-integral representation of this Boltzmann-Langevin equation, we define the probability functional P which gives the probability to find a certain realization of extraneous currents $\delta F^{imp}(\epsilon, x, t)$. This functional may be written as a path integral

$$P[\delta F^{imp}] = \int \mathcal{D}\eta \exp \left\{ \int dt dx d\epsilon [-i\eta \delta F^{imp} + H(\eta)] \right\}, \quad (\text{A1})$$

taking Fourier transforms of the generating function

$$H = -\frac{D}{N_F} f(1-f) \eta^2. \quad (\text{A2})$$

Since the Fourier transforms are independently taken for each point in space, time, and energy, Eq. (A1) indeed characterizes white noise. The electron occupation function $f = \bar{f} + \delta f$ is considered to be a slowly changing variable which modulates the instantaneous noise intensity. Its evolution is determined by the kinetic equation (2) which we represent as a δ functional expressed by a path integral:

$$\begin{aligned} \delta \left[\delta \dot{f} - D \delta f'' + e \frac{\partial f}{\partial \epsilon} \delta \dot{\phi} + (\delta F^{imp})' \right] \\ = \int \mathcal{D}\lambda \exp \left\{ i \int dt dx d\epsilon \lambda \left(\delta \dot{f} - D \delta f'' + e \frac{\partial f}{\partial \epsilon} \delta \dot{\phi} \right. \right. \\ \left. \left. + (\delta F^{imp})' \right) \right\}, \end{aligned} \quad (\text{A3})$$

where the prime stands for $\partial/\partial x$. The dynamics of the potential fluctuations $\delta\phi$ can be expressed by a second δ functional which enforces charge neutrality:

$$\begin{aligned} \delta \left[\sigma \delta \phi'' - e N_F \int d\epsilon (\delta F^{imp})' \right] \\ = \int \mathcal{D}\xi \exp \left\{ i \int dt dx \xi \left(\sigma \delta \phi'' - e N_F \int d\epsilon (\delta F^{imp})' \right) \right\}. \end{aligned} \quad (\text{A4})$$

The fields λ and ξ can be understood as Lagrange multipliers. Combining Eqs. (A1), (A3), and (A4), we construct the probability P_t to find a certain realization of extraneous currents δF^{imp} under the constraint of current conservation and charge neutrality:

$$P_t[\delta F^{imp}] = \int \mathcal{D}\delta\phi \mathcal{D}\delta f \delta[\dots] \delta[\dots] P[\delta F^{imp}]. \quad (\text{A5})$$

We are now in a position to calculate the generating functional $S[i\chi]$ of current fluctuations $\delta I = \delta j(x = -L/2)$ at the left contact $x = -L/2$:

$$e^{S_{el}[i\chi]} = \int \mathcal{D}\delta F^{imp} P_t[\delta F^{imp}] \exp \left\{ i \int dt \chi \delta I \right\}. \quad (\text{A6})$$

This equation may be considerably simplified. In a first step, we can integrate out the extraneous currents δF^{imp} as well as the field η introduced in Eq. (A1). We are then left with four functional integrations over δf , $\delta\phi$, λ , and ξ . In a second step, we evaluate these integrations in the saddle-point approximation. As the diffusive conductor is essentially semiclassical, the corrections to the saddle-point action are small.¹⁰ After rescaling $\lambda \mapsto e N_F \lambda$, we are left with the generating functional

$$\begin{aligned} S_{el}[\chi, \lambda, \xi, \delta f, \delta\phi] = \int dt dx \left\{ \sigma \xi \delta \phi'' + \int d\epsilon \left[\frac{\sigma}{e} \lambda \delta f'' + \sigma f(1 \right. \right. \\ \left. \left. - f)(\lambda' + \xi')^2 - e N_F \lambda \left(\delta \dot{f} + e \frac{\partial f}{\partial \epsilon} \delta \dot{\phi} \right) \right] \right\}, \end{aligned} \quad (\text{A7})$$

which has to be evaluated at the saddle point given by

$$\frac{\delta S_{el}}{\delta \lambda} = 0, \quad \frac{\delta S_{el}}{\delta \xi} = 0,$$

$$\frac{\delta S_{el}}{\delta(\delta f)} = 0, \quad \frac{\delta S_{el}}{\delta(\delta\phi)} = 0. \quad (\text{A8})$$

Note that we performed a complex continuation $i\lambda \mapsto \lambda$, $i\xi \mapsto \xi$, and $i\chi \mapsto \chi$. We are therefore left with purely real quantities. The saddle-point equations are supplemented with boundary conditions: the three fields δf , $\delta\phi$, and λ vanish at both boundaries. The external counting field χ is incorporated into the boundary conditions for ξ :

$$\xi(-L/2) = \chi, \quad \xi(L/2) = 0. \quad (\text{A9})$$

The frequency-dependent third cumulant under consideration in this paper is obtained from the third functional derivative:

$$\langle \delta I(\omega_1) \delta I(\omega_2) \delta I(\omega_3) \rangle = \left. \frac{\delta^3 S_{el}}{\delta\chi(\omega_1) \delta\chi(\omega_2) \delta\chi(\omega_3)} \right|_{\chi=0}. \quad (\text{A10})$$

We calculate this cumulant by a systematic expansion of the action (A7), saddle-point equations (A8), and fields:

$$\delta f = \delta f_1 + \delta f_2 + \dots, \quad \delta\phi = \delta\phi_1 + \dots, \quad \dots, \quad (\text{A11})$$

in orders of the external field χ . It can be straightforwardly shown by inserting the saddle-point equations (A8) back into the action (A7) that the third-order contribution to the action has the form

$$S_{el,3}[\chi] = \sigma \int dt \int dx \int d\varepsilon (1 - 2\bar{f}) \delta f_1(\xi'_1)^2. \quad (\text{A12})$$

It remains to solve the saddle-point equations to first order in χ . For the Lagrange multipliers we find

$$\lambda'_1 + \xi'_1 = 0, \quad \text{thus } \lambda_1 = 0, \quad \xi_1 = -\chi(x/L - 1/2). \quad (\text{A13})$$

Their dynamics is trivial, since they follow instantaneously the external field χ . The interesting dispersion effect stems from the saddle-point equation for the occupation function which takes the form of an inhomogeneous diffusion equation

$$\begin{aligned} \delta \dot{f}_1 - D \delta f_1'' &= -2eD[f_0(1-f_0)\xi'_1]' \\ &\quad - 2eD \frac{\partial f_0}{\partial \varepsilon} \int d\varepsilon [f_0(1-f_0)\xi'_1]'. \end{aligned} \quad (\text{A14})$$

This diffusion equation has a very appealing interpretation: When we integrate the equation over energy, we find that the two source terms on the right-hand side cancel. The left-hand side becomes a homogeneous diffusion equation for the fluctuations of the charge density, $\delta\rho_1 = e \int d\varepsilon \delta f_1$, which has the trivial solution $\delta\rho_1 = 0$. This is nothing than the charge neutrality which we demanded in the beginning of this section. The second source term which is due to variations of the electrostatic potential thus compensates the first term in such a way that all fluctuations of the occupation function δf_1 are charge neutral. We decompose the total variation $\delta f_1 = \delta f_1^{fF} + \delta f_1^{\phi}$ into a contribution δf_1^{fF} due to free fluctuations of the occupation function and a contribution δf_1^{ϕ} due to potential

fluctuations. Using the identity $\delta\phi_1 = D\delta\phi_1'$, we solve the diffusion equation by inverting the diffusion operator and find

$$\delta f_1^{fF}(\omega) = \langle \delta f_F(\varepsilon, x) \delta I \rangle_\omega \frac{\chi(\omega)}{D}, \quad (\text{A15})$$

$$\delta f_1^{\phi}(\omega) = \langle \delta f_\phi(\varepsilon, x) \delta I \rangle_\omega \frac{\chi(\omega)}{D}. \quad (\text{A16})$$

The two correlators are defined in Eqs. (16) and (17), respectively. They can now be inserted into Eq. (A12) to obtain the third-order correlation function (13). We thus derived the cascade rules applied in the main body of this paper from the stochastic path-integral formalism.

The derivation of an action describing the hot-electron regime requires only a minor additional effort. Here we cite directly the result which has been derived by one of the authors for the zero-frequency limit in a different context:²⁴

$$\begin{aligned} S_{hot} &= \int dt \int dx \left\{ -N_F \lambda T_e \dot{T}_e + \sigma(\xi' \quad \lambda') \hat{A} \begin{pmatrix} \xi' \\ \lambda' \end{pmatrix} \right. \\ &\quad \left. - \sigma(\xi' \quad \lambda') \hat{B} \begin{pmatrix} \phi' \\ T_e' \end{pmatrix} \right\}. \end{aligned} \quad (\text{A17})$$

In this action, we introduced the local electron temperature $T_e(x, t) = \bar{T}_e(x) + \delta T_e(x, t)$ and the local electrostatic potential $\phi(x, t) = \bar{\phi}(x) + \delta\phi(x, t)$. The boundary conditions for these fields are the potentials and temperatures of the left and right reservoirs. As for the case of noninteracting electrons, the Lagrange multiplier ξ ensures charge neutrality and obeys the boundary condition (A9). The field λ is linked to the conservation of energy current and is zero at the boundaries. The matrix \hat{A} describes the local noise created by the extraneous sources of noise δF^{imp} :

$$\hat{A} = T_e \begin{pmatrix} 1 & \phi \\ \phi & (\phi^2 + \pi^2 T_e^2 / 3e^2) \end{pmatrix}. \quad (\text{A18})$$

The second matrix \hat{B} is the linear response tensor

$$\hat{B} = \begin{pmatrix} 1 & 0 \\ \phi & \pi^2 T_e / 3e^2 \end{pmatrix}. \quad (\text{A19})$$

In complete analogy to the derivation of Eq. (A12), we may again collect all third-order terms which contribute to the action (A17) and find

$$S_{hot,3}[\chi] = \sigma \int dt \int dx \delta T_{e,1}(\xi'_1)^2, \quad (\text{A20})$$

where the variation $\delta T_{e,1}$ can be identified with

$$\delta T_{e,1} = \langle \delta T_e(x) \delta I \rangle_\omega \quad (\text{A21})$$

[see Eq. (28)]. The total-third order contribution $S_{hot,3}[\chi]$ therefore corresponds exactly to Eq. (25).

The main results of this appendix are the dynamic generating functionals (A7) and (A17). These functionals permit the calculation of the full probability distribution of frequency-dependent fluctuations of current.

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