Correlation length of the two-dimensional Ising spin glass with Gaussian interactions

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We study the correlation length of the two-dimensional Ising spin glass with a Gaussian distribution of interactions, using an efficient Monte Carlo algorithm, proposed by Houdayer, that allows larger sizes and lower temperatures to be studied than was possible before. We find that the "effective" value of the bulk correlation length exponent ν increases as the temperature is lowered, and, at low temperatures, apparently approaches $-1/\theta$, where $\theta \simeq -0.29$ is the stiffness exponent obtained at zero temperature. This means scaling is satisfied and earlier results at higher temperatures that find a smaller value for ν are affected by corrections to scaling.

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I. INTRODUCTION

There are two main theories to describe the spin-glass state: the droplet theory[1–4] and the replica symmetry breaking theory[5–8] (RSB) of Parisi. According to the droplet picture the lowest energy excitation, or "droplet," of linear size l containing a given site has a characteristic energy of order l^{θ} where $\theta > 0$ is a "stiffness" exponent. Droplets are expected to be compact but with a surface that has a nontrivial fractal dimension d_s, less than the space dimension d. It is further assumed that the same exponent θ describes both droplet and "domain-wall" excitations. In the alternative RSB scenario, the energy of droplets containing a finite fraction of the system does not increase with increasing system size. Furthermore, in RSB the surface of the large-scale, low-energy excitations are expected to fill space and so have the fractal dimension $d_s = d$.

There have been many numerical studies in three[9–13] and four[14,15] dimensions that attempt to determine which of these scenarios, or possibly something else,[16,17] is correct. These calculations are quite limited in the range of sizes that can be studied, although recently[18,19] a larger range has been studied for a one-dimensional model with power law interactions. Another case where a large range of sizes can be studied is the two-dimensional spin glass with short-range interactions, for which there is no spin-glass order at finite temperature, corresponding to $\theta < 0$. It is desirable to understand fully the two-dimensional spin glass, including the nature of corrections to scaling, since this may help in the interpretation of numerical data in higher dimensions.

However, even in $d = 2$, the situation is not completely clearcut. Zero-temperature calculations of the energy of a domain wall[9,20–24] consistently give $\theta \simeq -0.29$, with Ref. 24, for example, quoting $\theta = -0.287 \pm 0.004$. However, some calculations of droplet energies find[25,26] $\theta \simeq -0.47$, while others[27] find results consistent with the domain-wall value. These discrepancies presumably arise because some of the results are affected by corrections to scaling, and, as discussed by Hartmann and Moore,[28] the domain-wall value, $\theta \simeq -0.29$, seems to be the correct asymptotic result.

Although the discrepancy between the estimates for θ from the zero-temperature calculations seems now to be resolved,[28] there is still an apparent conflict between θ and the value of the correlation length exponent $\nu = 2.0 \pm 0.2$ obtained from finite temperature Monte Carlo simulations.[29] Since the spin–glass transition occurs at $T = 0$ in two dimensions, the correlation length ξ diverges as $T \to 0$ as $\xi \sim T^{-\nu}$. According to scaling,[9,10] ν is related to θ by

$$\nu = -\frac{1}{\theta}, \tag{1}$$

which gives $\nu \simeq 3.5$, significantly different from the result $\nu = 2.0 \pm 0.2$ reported in Ref. 29. We investigate this discrepancy here by performing Monte Carlo simulations on the Ising spin glass with Gaussian interactions in two dimensions at larger sizes and lower temperatures than in Ref. 29.

II. MODEL AND OBSERVABLES

The Hamiltonian is

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j, \tag{2}$$

where the sum is over nearest neighbor pairs of sites on a square lattice, the S_i are Ising spins taking values ± 1, and the J_{ij} are Gaussian variables with zero mean and standard deviation unity. The square lattice contains $N = L \times L$ sites with periodic boundary conditions. We use the Monte Carlo algorithm of Houdayer[30] which combines single spin-flip dynamics, parallel tempering,[31] and a type of cluster move, which is significantly more efficient than parallel tempering in two dimensions for large system sizes ($L > 24$). Tests for equilibration are done as in Ref. 32; the parameters used in the simulations are shown in Table I.

The main focus of our study is the correlation length *of the finite system*[33–36] ξ_L, defined by

$$\xi_L = \frac{1}{2\sin(|\mathbf{k}_{\min}|/2)} \left[\frac{\chi_{\mathrm{SG}}(0)}{\chi_{\mathrm{SG}}(\mathbf{k}_{\min})} - 1 \right]^{1/2}, \tag{3}$$

TABLE I. Parameters of the simulations. N_{samp} is the number of samples, N_{sweep} is the total number of Monte Carlo sweeps for each of the $2N_T$ replicas for a single sample, T_{min} is the lowest temperature simulated, and N_T is the number of temperatures used in the parallel tempering method. Note that for $L \leqslant 16$ standard parallel tempering Monte Carlo was used, whereas for $L \geqslant 32$ the cluster method by Houdayer was applied.

L	N_{samp}	N_{sweep}	T_{min}	N_T
8	10000	2.0×10^5	0.05	20
16	10000	1.0×10^6	0.05	20
32	10000	1.0×10^5	0.05	20
64	1000	1.0×10^6	0.05	40
128	250	1.0×10^6	0.20	63

where $\mathbf{k}_{\text{min}}=(2\pi/L,0,0)$ is the smallest nonzero wave vector, and

$$\chi_{\text{SG}}(\mathbf{k}) = \frac{1}{N}\sum_{i,j}[\langle S_i S_j\rangle^2]_{\text{av}} e^{i\mathbf{k}\cdot(\mathbf{R}_i-\mathbf{R}_j)} \tag{4}$$

is the wave vector-dependent spin-glass susceptibility. In Eq. (4) the angular brackets $\langle\cdots\rangle$ denote a thermal average while the rectangular brackets $[\cdots]_{\text{av}}$ denote an average over the disorder.

Since the ratio ξ_L/L is dimensionless, it satisfies the finite size scaling form[34–36]

$$\xi_L/L = \tilde{X}[L^{1/\nu}T], \tag{5}$$

assuming a zero-temperature transition, where $\tilde{X}$ is a scaling function and $1/\nu=-\theta$. For $L^{1/\nu}T \gg 1$, ξ_L is equal to the bulk (i.e., infinite system size) correlation length ξ_∞, and so

$$\xi_L = \xi_\infty \sim T^{-\nu} \qquad (L \gg \xi_\infty), \tag{6}$$

implying that $\tilde{X}(x)\sim x^{-\nu}$ for $x \gg 1$. In the opposite limit, $x \ll 1$, we expect $\tilde{X}(x)\sim x^{-\lambda}$ where we will estimate λ below.

III. RESULTS

Figure 1 shows data for ξ_L. We see that the data are independent of system size at high T, showing that the bulk behavior has been obtained, and the data "peels off" from this "bulk curve" at temperatures that become successively lower for larger sizes. The bulk results are curved on this log–log plot showing that the asymptotic power law behavior in Eq. (6) has not yet been reached. Rather, the slope of the curve is an "effective exponent," ν_{eff}, which varies with T.

In order to determine the asymptotic value of ν we obtain values for the bulk correlation length at lower T following

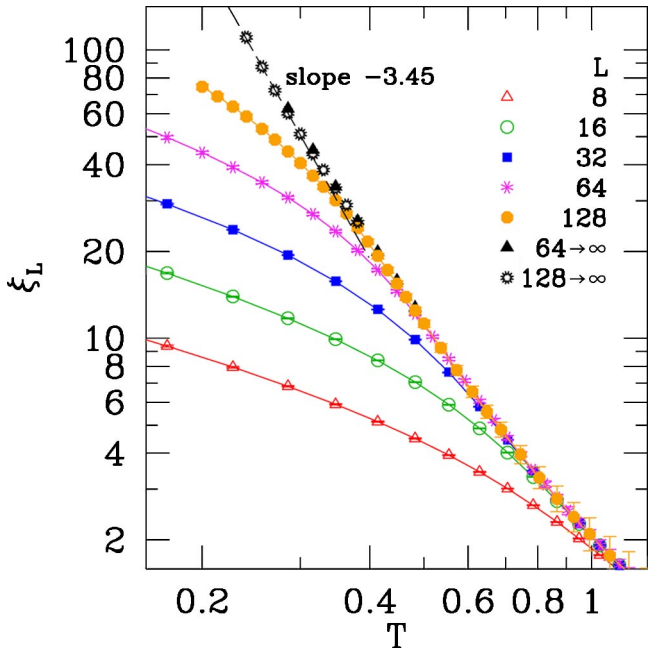

FIG. 1. A log–log plot of the finite size correlation length ξ_L for different system sizes and temperatures. The data labeled $64\to\infty$ and $128\to\infty$ come from an extrapolation to the thermodynamic limit. The slope of the extrapolated data gives $-\nu\simeq-3.45$, which is consistent with $\theta\equiv-1/\nu\simeq-0.287$ found in zero-temperature domain-wall calculations (Ref. 24).

the method used by Kim.[33] The finite-size scaling expression, Eq. (5), can be written as

$$\frac{\xi_L}{L} = f\left(\frac{\xi_\infty}{L}\right), \tag{7}$$

which can be inverted to give

$$\frac{\xi_\infty}{L} = g\left(\frac{\xi_L}{L}\right), \tag{8}$$

where $g(x)=f^{-1}(x)$. Clearly $g(x)=x$ for $x\to0$. We determine $g(x)$ by fitting to data in the range $0.45<T<1.05$ where we have data for the correlation length in *both* the bulk and finite-size regimes. We consider sizes $16\leqslant L\leqslant128$ for this determination, from which we obtain $g(x)$ in the range $0<x<0.45$. Using $g(x)$ *in this range* we then determine ξ_∞ from Eq. (8) using data for $L=64$ in the range $0.285\leqslant T\leqslant0.482$ and for $L=128$ in the range $0.24\leqslant T\leqslant0.38$. Note that we do not perform any direct extrapolation in this analysis; the function $g(x)$ is determined by fitting and is then used to get ξ_∞ at somewhat lower temperature using only the range of x where it was fitted.

The resulting values of ξ_∞ from the $L=64$ and 128 data are consistent with each other where they overlap, and are shown in Fig. 1. The extrapolated data fit a slope of $-\nu=-3.45$ that corresponds to $\theta\equiv-1/\nu=-0.29$, in good agreement with domain-wall results.[24] At higher temperature, the

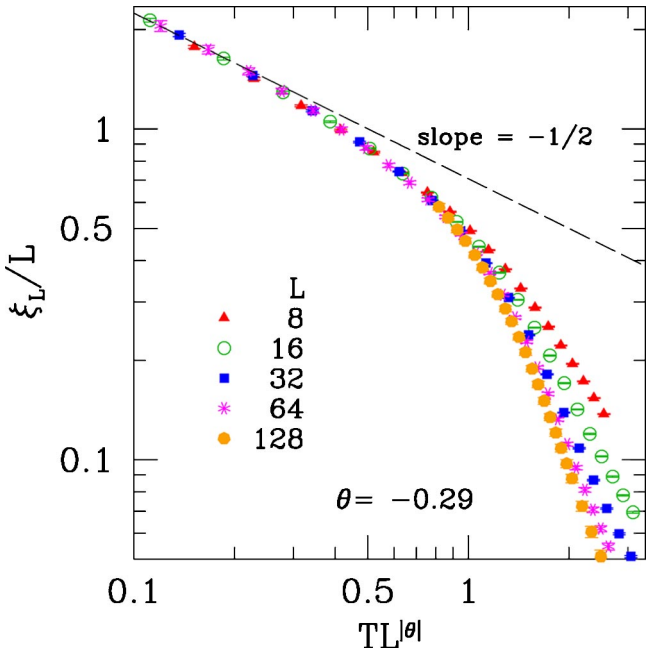

FIG. 2. Scaling plot of the data for the correlation length ξ_L according to Eq. (5), with $\theta=-1/\nu=-0.29$. The dashed line, which fits the data at low T, has slope $-1/2$.

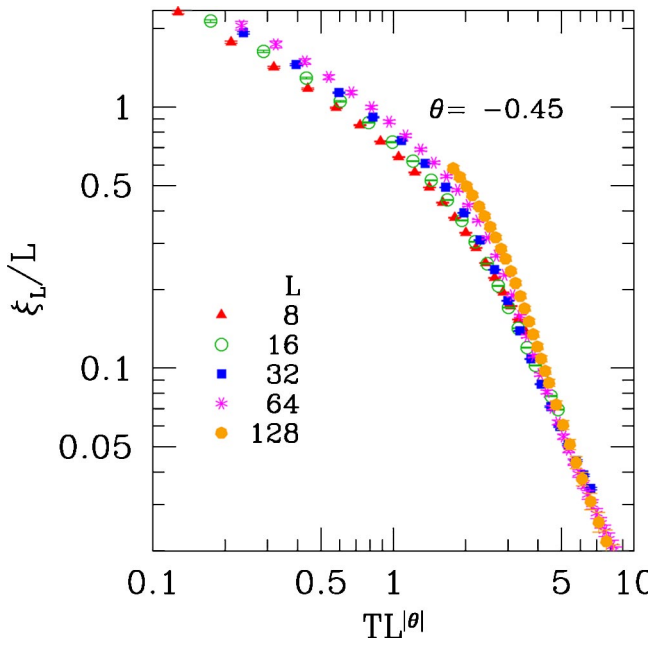

FIG. 3. Scaling plot of the data for the correlation length ξ_L according to Eq. (5), with $\theta=-1/\nu=-0.45$.

slope of the bulk data in Fig. 1 is smaller in magnitude, so we believe that Liang's result, $\nu=2.0\pm0.2$, obtained in the region around $T=1.0$, is only an effective exponent. Neither our results nor the domain-wall results appear to be consistent with an exponential divergence of the correlation length as $T\to0$.

Assuming that the asymptotic value of ν is indeed ≈ 3.45, we estimate from Fig. 1 that one needs to be at or below a temperature of around 0.35, where $\xi_\infty\simeq40$, to see the bulk asymptotic behavior.

Further insight is obtained by scaling the data according to Eq. (5). Figure 2 shows a scaling plot with $\theta=-0.29$, the value expected from zero-temperature domain-wall calculations and our extrapolated data in Fig. 1. We see the data in Fig. 2 scale well at low T but not well at high T. However, the latter point is not surprising in view of Fig. 1, where we see that asymptotic bulk power-law behavior has not yet been reached. Note, though, that the data for the two largest sizes in Fig. 2, $L=64$ and 128, do almost collapse, suggesting that $\theta=-1/\nu=-0.29$ *will* work in the bulk region, $L\gg\xi$, for large enough sizes and low enough temperatures, as we also inferred from Fig. 1.

The dashed line in Fig. 2 has slope $-1/2$, implying from Eq. (5) that $\tilde{X}(x)\sim x^{-\lambda}$ for $x\ll1$ with $\lambda\simeq1/2$. Hence we have

$$\xi_L\sim T^{-1/2}L^{1-1/(2\nu)}\qquad(L\ll\xi). \tag{9}$$

The $T^{-1/2}$ dependence can be understood from Eq. (3) since $\chi_{\rm SG}(0)=L^2$ at $T=0$ (because the ground state is unique), while the fluctuations at nonzero $\mathbf{k}$ are frozen out at $T=0$. It is plausible $\chi_{\rm SG}(\mathbf{k}_{\rm min})\propto T$ at low T from equipartition, and this leads to a $T^{-1/2}$ dependence for ξ_L.

Figure 3 shows a scaling plot for $\theta=-0.45$, which gives the best data collapse in the high-T region. This value is compatible with $\nu=2.0\pm0.2$ found in Ref. 29. Note that the data do not collapse at all in the low-T region, and the collapse becomes worse for larger sizes. Figure 3 shows, again, that an *effective* value of $\nu\approx2$ will fit the data over a range of intermediate temperatures, while in the low-T asymptotic region one has $\theta=-1/\nu\simeq0.29$. We have found that the spin glass susceptibility shows similar behavior. Independent recent results for the spin glass susceptibility and other quantities[37] also find evidence that $\theta\simeq-0.29$ for large sizes.

IV. CONCLUSIONS

To conclude, we have shown that the result $\nu=2.0\pm0.2$ in Ref. 29 is only an effective exponent, and the true value for ν is larger. The data are consistent with scaling holding asymptotically for $\nu=-1/\theta\simeq3.45$. We have strengthened our argument for this conclusion by the extrapolation to $L=\infty$ shown in Fig. 1. Of course it would also be desirable to extend the data to larger sizes, which may be possible in the near future by fine tuning the algorithm.

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